Cauchy Integrals and Cauchy-Szegö Projections

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The goal: In $\mathbb{C}^n$, to study the L^p properties of Cauchy integrals and the Cauchy-Szegö projection under optimal regularity assumptions on the boundary (and related geometric restrictions).

Joint work with Loredona Lanzani

- Advances in Math. 264 (2014)
- Duke Math. J. (to appear)
- in preparation

Outline:

I: $n = 1$, a quick review

II: $n > 1$, recent results

III: some counter-examples

I — Two theorems for Cauchy integrals in $\mathbb{C}^1$ - the "gold standard"

Theorem A

$$\mathbf{C}(f)(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}d\zeta, \qquad z \in \Omega$$

$$\mathcal{C}(f)(z) = \lim_{z'\to z, z'\in\Omega} \mathbf{C}(f)(z), \qquad z \in \partial\Omega$$

Then

$$||\mathcal{C}(f)||_{L^p(\partial\Omega)} \lesssim_p ||f||_{L^p(\Omega)}, \qquad 1 < p < \infty$$

when $\partial\Omega$ is Lipschitz.

- Calderón 1977, Coifman, McIntosh, and Meyer 1982.
- David 1989 when Ahlfors-regular.
- $T(1)$, $T(b)$ theorems, David, Journé, and Semmes 1984, 1985.

Let $S =$ Cauchy-Szegö projection: orthogonal projection of $L^2(\partial\Omega)$ to $H^2(\partial\Omega)$.

Theorem B S is bounded on L^p

- If $\partial\Omega$ is Lipschitz, then for $p_0 < p < p_0'$, with $p_0' = p_o'(M)$ and $p_0' > 4$.
- If $\partial\Omega$ is of class C^1, then for $1 < p < \infty$

Calderón 1977, Kenig 1980, Pommerenke 1992, ..., Lanzani and S. 2004.

Two approaches to results of this kind.

First, by conformal mappings: Let

$$\Phi : \mathbb{D} \to \Omega$$

$$\tau(f)(e^{\prime\theta}) = (\Phi'(e^{\prime\theta}))^{\frac{1}{2}}(f \circ \Phi)(e^{\prime\theta}).$$

Then

$$S = \tau^{-1} S_0 \tau$$

where S_0 is the Cauchy-Szegö projection for $\mathbb{D}$.
Hence the question reduces to: for which p is

$$|\Phi'(e^{I\theta})|^{1-\frac{p}{2}} \in A_p?$$

Second approach:

(I) $$\mathcal{C} = S(I - A)$$

with
$A = \mathcal{C}^* - \mathcal{C}$.

Then if A is "small" we can "invert" $I - A$ to estimate S in terms of $\mathcal{C}$.

For example, if $\partial\Omega$ is a class of C^1, then A is compact, and $I - A$ can be inverted by the Fredholm alternative.

II – The case of $\mathbb{C}^n, n > 1$.

Some challenges that present themselvse:

1. Infinitely many different "Cauchy integrals"
2. Pseudo-convexity of Ω must play a role so minimal smoothness of $\partial\Omega$ should be "near" C^2, not C^1, as when $n = 1$.
3. Conformal equivalance of domains breaks down.
4. No choice of the Cauchy integral seems to work with the identity (I), unless $\partial\Omega$ is smooth.
5. Fefferman's asymptotic formula for the Cauchy-Szegö kernel holds only if $\partial\Omega$ is sufficiently smooth.

The Cauchy-Leray intergral (1956)

There is a class of domains Ω, which have a natural (i.e. unique, cannonical) Cauchy integral attached to them. Suppose Ω is convex, and $\partial\Omega$ is of class C^2.

Let $\rho(z)$ be a defining function of Ω, $\Omega = \{\rho(z) < 0\}$, while $\rho \in C^{(2)}$, and $\nabla\rho \neq 0, z \in \partial\Omega$.

Conside the denominator:

$$\Delta(w,z) = \langle \partial\rho(w), w - z\rangle = \sum_{j=1}^{n} \frac{\partial\rho(w)}{\partial w_j}(w_j - z_j).$$

Then

$$C_L(f)(z) = \int_{\partial\Omega} \frac{f(w)}{(\Delta(w,z))^n} d\lambda(w), z \in \Omega$$

where $d\lambda(w) = \frac{1}{(2\pi i)^n}\partial\rho \wedge (\bar{\partial}\partial\rho)^{n-1} =$ "Leray-Levi" measure.

- Note that $\Delta(w,z) \neq 0$ if $w \in \partial\Omega$ and $z \in \Omega$, since Ω is convex.
- The Cauchy-Fanttapié formalism shows that $C_L(f)(z) = f(z)$ if f is holomorphic in Ω and coninuous in $\bar{\Omega}$.

Notions of “convexity” of Ω relevant to complex analysis. (Here $n > 1$.)

Three versions, in increasing order of generality:

1. Strong convexity
2. Strong $\mathbb{C}-$ linear convexity
3. Strong pseudo-convexity

Note that (2) is equivalent (when $\partial\Omega$ is of class C^2) with:
$2'$ Distance $(T_z^{\mathbb{C}}, w) \geq C|z-w|^2$, if $z, w \in \partial\Omega$

Theorem 1
Suppose $\partial\Omega$ is of class $C^{1,1}$ and Ω is strongly $\mathbb{C}-$ linearly convex, as in $2'$. Then C_L extends to a bounded linear operator on $L^p(\partial\Omega), 1 < p < \infty$.

Counter example, (D. Barrett, L. Lanzani, 2009): cannot replace $C^{1,1}$ by $C^{2-\epsilon}$.

Theorem 2

Suppose $\partial\Omega$ is of class C^2 and Ω is strongly pseudo-convex. Then the Cauchy-Szegö projection S extends to a bounded operator on $L^p(\partial\Omega), 1 < p < \infty$.

Note:

S is the orthogonal projection of $L^2(bD)$ to $H^2(bD)$, and $L^2 = L^2(\partial\Omega, d\sigma)$, where $d\sigma$ is the induced Lebesgue measure. A similar result holds for the orthogonal projection S_ω of $L^2(\partial\Omega, \omega d\sigma)$, whenever ω is a continuous strictly positive density on $\partial\Omega$.

The special case $\omega d\sigma = d\lambda =$ Leray-Levi measure is key in proving the general result.

Two issues in the proof of Theorem 1

- The "definition of C_L (and the resulting proofs) raise the following "restriction" problem:

Suppose F is a $C^{1,1}$ function on $\mathbb{R}^n$, and M is a submanifold of $\mathbb{R}^n$.

Question: Does the restriction $\left.\frac{\partial^2 F}{\partial x_j, \partial x_k}\right|_M$ make sense?

Answer: It can be defined as an $L^\infty(M)$ function if the derivatives are "tangential."

One version: For x, and $x+h$ in M, and for $a.e., x \in M$

$$F(x+h) = F(x) + \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(x) h_j$$

$$+ \sum a_{jk}(x) h_j h_k + \circ(|h|^2),$$

as $h \to 0$, where $a_{jk} \in L^\infty(M)$.

Another version:

Suppose $F \in C^{1,1}$ on $\mathbb{C}^n, M = \partial\Omega$. Then there exists a two-form on M (called $\bar\partial\partial F$), with $L^\infty(M)$ coefficients so that

$$\int_M \phi \wedge \bar\partial\partial F = \int_M d\phi \wedge \bar\partial F$$

for all test $2n-3$ forms ϕ on M.

Second issue for C_L : the "cancellation conditions" needed to apply $T(1)-$type theorems.

Here there is an identity, holding for $n > 1$, (but not for $n = 1$):

$$C_L(f)(z) = \frac{1}{(n-1)(2\pi i)^n} \int_{\partial\Omega} \frac{1}{(\Delta(w,z))^{n-1}} df \wedge (\partial\bar\partial\rho)^{n-1}$$

$+$ negligible terms in f.

We turn to the Cauchy-Szegö projection.

To exploit the idea of the identity (I) and follow what worked when $\partial\Omega$ was smooth and strongly psuedo-convex, we begin by constructing an appropriate Cauchy integral in this setting.

Replace the denominator
$\Delta(w,z) = \langle \partial\rho(w), w-z\rangle$ by

$$\Delta'(w,z) = \Delta(w,z) + \frac{1}{2}\sum \frac{\partial^2\rho(w)}{\partial w_j \partial w_k}(w_j - z_j)(w_k - z_k)$$

$=$ (the "Levi polynomial"): in particular,

$|\Delta'(w,z)| \geq C|w-z|^2,$ for z near w; $z, w \in \partial\Omega$.

This allows one to construct a Cauchy integral C like C_L —using the Cauchy-Fantappié formalism, (Δ' replacing Δ), (Henkin, Ramirez 1969).
This has the additional property that $C^* - C$ is small (Kerzman and S. (1978)).

All this when $\partial\Omega$ is smooth.

Now we turn to our situation, when $\partial\Omega$ (i.e. ρ), is merely of class C^2. Then we have the essential difficulty that the denominator $\Delta'(w,z)$ is only continuous in w, and not smoother. Hence all known methods for proving L^2 (or L^p) estimates, such as by $T(1)$ techniques, fail.

A first try in overcoming this difficulty is to replace $\Delta'(w,z)$ by $\Delta_\epsilon(w,z)$ with $\Delta_\epsilon(w,z) = \langle \partial\rho(w), w-z\rangle +$

$\frac{1}{2}\sum_{j,k} \tau_{jk}(w)(w_j - z_j)(w_k - z_k)$,

where $|\tau_{jk}(w) - \frac{\partial^2\rho(w)}{\partial w_j \partial w_k}| < \epsilon$ and $\tau_{jk}(w)$ of class C^1.

With this we can construct a Cauchy integral C_ϵ that has L^p estimates.

Unfortunately, in general, $||C_\epsilon||_{L^p \to L^p} \to \infty$, as $\epsilon \to 0$.

How do we get around this quandry?

For each $\epsilon > 0$ we **truncate** the kernel of C_ϵ to an appropriately narrow neighborhood of the origin, obtaining

$$C_\epsilon = \tilde{C}_\epsilon + R_\epsilon.$$

While $\tilde{C}_\epsilon$ is no longer a Cauchy integral, we have

- $||(\tilde{C}_\epsilon)^* - \tilde{C}_\epsilon||_{L^p \to L^p} \lesssim_p \epsilon^{\frac{1}{2}}$
- R_ϵ maps $L^1(\partial\Omega)$ to $L^\infty(\partial\Omega)$ (while $||R_\epsilon|| \to \infty$, as $\epsilon \to 0$.)

Now we use (I) to get

$$C_\epsilon = S(I - C_\epsilon^* + C_\epsilon),$$

and thus

$$C_\epsilon + SR_\epsilon^* - SR_\epsilon = S(I - (\tilde{C}_\epsilon)^* + \tilde{C}_\epsilon)$$

Take $p \leq 2$. The left side is bounded on L^p

For the right-side use a Neumann series to invert $I - (\tilde{C}_\epsilon)^* + \tilde{C}_\epsilon$ for ϵ sufficiently small, since

$$||(\tilde{C}_\epsilon)^* - \tilde{C}_\epsilon||_{L^p \to L^p} \lesssim_p \epsilon^{\frac{1}{2}}.$$

III Some counter-examples

For the Cauchy-Leray integral.

Assertion:

There exists a (simple!) bounded domain Ω so that

- $\partial\Omega \in C^\infty$ (in fact is real-analytic)
- Ω is convex (in fact, *strictly* convex)
- Ω is *strongly* pseudo-convex

However, CL_Ω is not bounded on L^p for any p.

In $\mathbb{C}^2$, with $z_j = x_j + iy_j, \quad j = 1, 2$, take

$$\Omega = \{|z_2|^2 + x_1^2 + y_1^4 < 1\}$$

or more generally

$$\{|z_2|^2 + x_1^2 + y_1^{2k} < 1\}, k > 1, k \text{ an integer.}$$

Model results:

Consider the domains in $\mathbb{C}^2$:
$\Omega^{(a)} = \{\mathrm{Im}\ z_2 > \frac{1}{4}|z_1|^2\}$
$\Omega^{(b)} = \{\mathrm{Im}\ z_2 > \frac{1}{2}x_1^2\}$

These two are biholomorphically equivalent
$(z_1 \to z_1, z_2 \to z_2 \pm iz_1^2)$

But the Cauchy-Leray denominators behave differently:
$|\Delta^{(a)}(0,z)|^2 \approx x_2^2 + |z_1|^4$

$|\Delta^{(b)}(0,z)|^2 \approx x_2^2 + x_1^4.$

So $\Delta^{(a)}(0,z)$ vanishes only when
$(x_1, y_1, x_2) = 0.$

But $\Delta^{(b)}(0,z)$ vanishes when $x_1 = x_2 = 0$, all
y_1.

Can construct a skewed "bump" χ_δ, so that

$$||\chi_\delta||_{L^p} \leq \delta^{4/p},\ \text{while}\ ||(CL_b(\chi_\delta)||_{L^p} \geq \delta^{\frac{3}{p}}.$$

Next let,
$\Omega_1 = \{|z_2 - i|^2 + x_1^2 + y_1^4 < 1\}$

and $\tau_\lambda(z_1, z_2) = (\lambda z_1, \lambda^2 z_2), \lambda > 0$

$\Omega_\lambda = \tau_\lambda(\Omega_1), \tau_\lambda(f)(z_1, z_2) = f(\frac{z_1}{\lambda}, \frac{z_2}{\lambda^2})$

Now $\Omega_\lambda = \{|\frac{z_2}{\lambda^2} - i|^2 + \frac{x_1^2}{\lambda^2} + \frac{y_1^4}{\lambda^4} < 1\}$
$= \{2 \text{ Im } z_2 > x_1^2 + \frac{y_2^2}{\lambda} + \frac{y_1^4}{\lambda^2}\}$

Let $\lambda \to \infty$, then
$\Omega_\lambda \to \Omega^{(b)} = \{2 \text{ Im } z_2 > x_1^2\}$.

However, (and this is a little trickier) one can show that

$||CL_{\Omega_1}(f)||_p \leq A||f||_p \Rightarrow ||CL^*_{\Omega_\lambda}(f)||_{L^p} \leq A||f||_{L^p} \Rightarrow$

$||CL_b(f)||_{L^p} \leq A||f||_{L^p}$

Second example.

For each $p \neq 2$, there exists a bounded C^∞ psuedo-convex domain W so that the Cauchy-Szegö projection is not bounded on $L^p(\partial W)$.

W is a "worm domain"

$W = \{|z_2 - ie^{ih(z_1)}|^2 + k(|z_1|) < 1\}$
where $k(t) \geq 0, k(t) = 0$ in a non-empty interval.
$h(z_1) =$ is a suitable function of $|z_1|$.

Brief history of worm domains:

- K. Diederich and J. Fornaess, 1977
- D. Barrett, 1992
- M. Christ, 1996

- E. Straube, 1993
- S. Krantz, M. Peloso, 2008