EXTREMAL DISCRETE MEASURES FOR RIESZ POTENTIALS

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May 19, 2016

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Let $A \subset \mathbb{R}^p$ be a compact d-dimensional set, and $s > 0$.

N-th Polarization (Chebyshev) constant:

$$\text{Define}\quad \mathcal{P}_s(A;N) := \sup P_s(A;\omega_N) := \sup \inf_{y\in A} \sum_{x_j\in\omega_N} \frac{1}{|y-x_j|^s},$$

where the supremum is taken over all N-point sets $\omega_N \subset A$.

Compare to the minimal discrete energy:

$$\mathcal{E}_s(A;N) := \min E_s(\omega_N) = \min \sum_{i\neq j} \frac{1}{|x_i-x_j|^s}$$

Three cases should be considered separately:

- $s < d$: the "continuous" problem $\sup_\mu \inf_{y\in A} \int_A \frac{d\mu(x)}{|y-x|^s}$ is non-trivial;
- $s = d$: the transitional case;
- $s > d$, when the kernel $\frac{1}{|x-y|^s}$ is very singular.

s<d

- The "dicrete" problem tends to the "continuous" problem; i.e., $\mathcal{P}_s(A;N)/N \to T_s(A)$;
- If $A = \mathbb{S}^1$, then ω_N consists of N equally spaced points;
- If $A = \mathbb{S}^2$ and $N = 4$, then any maximizing configuration ω_N forms a regular simplex inscribed in $\mathbb{S}^2$;
- If $A = \mathbb{S}^d$ and $\mu_N := \frac{1}{N}\sum_{x_j \in \omega_N} \delta_{x_j}$, then measures μ_N weakly tend to the equillibrium (that is, surface) measure on $\mathbb{S}^d$. But if $A = \mathbb{B}_d$ and $s < d - 2$, then $\omega_N = \{0, 0, \ldots, 0\}$.

s>d

- There exists a constant $\sigma_{s,d} > 0$, such that

$$\lim_N \frac{\mathcal{P}_s([0,1]^d;N)}{N^{s/d}} = \sigma_{s,d}.$$

As a (non-immediate) corollary of the cube case, if $A \subset \mathbb{R}^p$ has positive Lebesgue measure, and ∂A has zero Lebesgue measure, then

$$\lim_N \frac{\mathcal{P}_s(A; N)}{N^{s/p}} = \frac{\sigma_{s,p}}{m_p(A)^{s/p}}, \quad s > p.$$

Moreover,

$$\frac{1}{N} \sum_{x_j \in \omega_N} \delta_{x_j} \to^* m_p(\cdot \cap A)/m_p(A).$$

The idea is to approximate A from the inside by cubes and use a tricky semi-additivity. The difficulty one has to overcome is that the quantity $\mathcal{P}_s(A; N)$ is not monotone in A.

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$$\lim_N \frac{\mathcal{P}_s(A; N)}{N^{s/d}} = \frac{\sigma_{s,d}}{H_d(A)^{s/d}}.$$

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THANK YOU!