Algebras of singular integral operators with kernels controlled by multiple norms

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This is a report on joint work with

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Calderón-Zygmund kernels

A Calderón-Zygmund kernel on $\mathbb{R}^n$ associated with the dilations

$$\lambda_{\mathbf{a}}(\mathbf{x}) = \lambda_{\mathbf{a}}(x_1, \dots, x_n) = (\lambda^{a_1} x_1, \dots, \lambda^{a_n} x_n)$$

is a tempered distribution K on $\mathbb{R}^n$ with the following properties.

1. K is given away from the origin by integration against a smooth function.
2. For every multi-index $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and all $\mathbf{x} \neq 0$

$$\left|\partial^{\boldsymbol{\alpha}} K(\mathbf{x})\right| \leq C_{\boldsymbol{\alpha}} \left(|x_1|^{\frac{1}{a_1}} + \cdots + |x_n|^{\frac{1}{a_n}}\right)^{-Q_{\mathbf{a}} - \sum_{j=1}^{n} a_j \alpha_j}$$

   where $Q_{\mathbf{a}} = a_1 + \cdots + a_n$ is the homogeneous dimension.

3. K satisfies appropriate cancellation conditions so that in particular the Fourier transform $\widehat{K} = m$ is a bounded function.

K will often have compact support or have rapid decay outside the unit ball.

Homogeneous nilpotent Lie groups

G denotes a homogeneous nilpotent Lie group with underlying manifold $\mathbb{R}^n$ and automorphic dilations

$$\lambda_{\mathbf{d}}(x_1, \dots, x_n) = \big(\lambda^{d_1} x_1, \dots, \lambda^{d_n} x_n\big)$$

with $d_1 \leq d_2 \leq \cdots \leq d_n$. For $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in G$

$$\mathbf{x} \cdot \mathbf{y} = \big(x_1 + y_1, x_2 + y_2 + M_2(\mathbf{x}, \mathbf{y}), \dots, x_n + y_n + M_n(\mathbf{x}, \mathbf{y})\big)$$

where $M_j(\mathbf{x}, \mathbf{y})$ is a polynomial vanishing if $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$ and

$$M_j\big(\lambda_{\mathbf{d}}(\mathbf{x}), \lambda_{\mathbf{d}}(\mathbf{y})\big) = \lambda^{d_j} M_j(\mathbf{x}, \mathbf{y}).$$

Convolution on G is given by

$$f * g(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{x} \cdot \mathbf{y}^{-1}) g(\mathbf{y})\, d\mathbf{y} = \int_{\mathbb{R}^n} f(\mathbf{y}) g(\mathbf{y}^{-1} \cdot \mathbf{x})\, d\mathbf{y}.$$

Example: The Heisenberg group

The m-dimensional Heisenberg group $\mathbb{H}^m$ has underlying space $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ with automorphic dilations

$$\lambda_{1,1,2}(\mathbf{x}, \mathbf{y}, t) = \big(\lambda \mathbf{x}, \lambda \mathbf{y}, \lambda^2 t\big)$$

and group multiplication

$$\begin{aligned}(\mathbf{x}_1, \mathbf{y}_1, t_1) \cdot (\mathbf{x}_2, \mathbf{y}_2, t_2) \\ = \big(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}_1 + \mathbf{y}_2, t_1 + t_2 - 2\langle \mathbf{x}_1, \mathbf{y}_2 \rangle + 2\langle \mathbf{x}_2, \mathbf{y}_1 \rangle\big)\end{aligned}$$

where $\langle \mathbf{x}, \mathbf{y} \rangle$ is the Euclidean inner product. If

$$X_j = \partial_{x_j} + 2y_j \partial_t, \qquad Y_j = \partial_{y_j} - 2x_j \partial_t, \qquad T = \partial_t,$$

then

$$\{X_1, \dots, X_n, Y_1, \dots, Y_n, T\}$$

is a basis for the left-invariant vector fields on $\mathbb{H}^n$.

The problem

Consider two different dilation structures on G given by

$$\lambda_{\mathbf{a}}(x_1, \dots, x_n) = (\lambda^{a_1 d_1} x_1, \dots, \lambda^{a_n d_n} x_n) \quad \text{with} \quad a_1 \geq a_2 \geq \cdots \geq a_n,$$
$$\lambda_{\mathbf{b}}(x_1, \dots, x_n) = (\lambda^{b_1 d_1} x_1, \dots, \lambda^{b_n d_n} x_n) \quad \text{with} \quad b_1 \geq b_2 \geq \cdots \geq b_n.$$

Let $K_{\mathbf{a}}$ and $K_{\mathbf{b}}$ be Calderón-Zygmund kernels on $\mathbb{R}^n$ with compact support associated with these dilations. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ set

$$T_{\mathbf{a}}[\varphi] = \varphi * K_a \quad \text{and} \quad T_{\mathbf{b}}[\varphi] = \varphi * K_{\mathbf{b}}.$$

- **What are the mapping properties of the operator $T_{\mathbf{a}} \circ T_{\mathbf{b}}$ on various function spaces?**
- **What can be said about size and cancellation properties of the Schwartz kernel of $T_{\mathbf{a}} \circ T_{\mathbf{b}}$?**
- **Are there reasonably small algebras of convolution operators containing Calderón-Zygmund kernels associated to both dilations?**

Background and Motivation

- Let $\Omega \subset \mathbb{C}^{n+1}$ be a strictly pseudoconvex domain and let

$$\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} : L^2_{(0,1)}(\Omega) \to L^2_{(0,1)}(\Omega).$$

Given $f \in \mathcal{C}^\infty_{(0,1)}(\overline{\Omega})$, the $\bar{\partial}$-Neumann problem is to solve

$$\begin{aligned} \Box u &= f \quad \text{on } \Omega \\ u \lrcorner \bar{\partial}\rho &= 0 \quad \text{on } \partial\Omega \\ \bar{\partial}u \lrcorner \bar{\partial}\rho &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Greiner and Stein (1977) constructed a parametrix for this boundary value problem which involves the composition of two kinds of operators:

  - operators of elliptic type coming from Poisson integrals and the Green's function for $\Box$ (which is essentially the Laplacian);
  - operators of non-isotropic type coming from inverting the sub-Laplacian $\mathcal{L} = -\sum_{j=1}^n (X_j^2 + Y_j^2)$ on the Heisenberg group $\mathbb{H}^n$.

- Phong and Stein (1982) studied compositions of convolution operators on $\mathbb{R}^d \times \mathbb{R}$

$$T_E[\varphi] = \varphi * K_E \quad \text{and} \quad T_H[\varphi] = \varphi * K_H$$

where $*$ is either Euclidean or Heisenberg convolution, and K_E and K_H are Calderón-Zygmund kernels on $\mathbb{R}^d \times \mathbb{R}$ associated with the two different dilations:

$$\lambda_E(\mathbf{x}, t) = (\lambda \mathbf{x}, \lambda t) \quad \text{and} \quad \lambda_H(\mathbf{x}, t) = (\lambda \mathbf{x}, \lambda^2 t).$$

The local (near 0) and the global (near infinity) behavior of compositions $T_E \circ T_H$ or $T_H \circ T_E$ are different. Phong and Stein established:

  - necessary and sufficient conditions for the compositions of such operators to be of weak-type (1,1);
  - boundedness of the composition on non-isotropic Lipschitz spaces.

- On the Heisenberg group $\mathbb{H}^n$, $T = \partial_t$ and the sub-Laplacian is

$$\mathcal{L} = -\sum_{j=1}^{n}(X_j^2 + Y_j^2).$$

Müller, Ricci, and Stein (1995) studied general functions of $\mathcal{L}$ and T, and in particular established Marcinkiewicz-type theorems for multiplier operators $m(\mathcal{L}, T)$ where

$$\left|\partial_\xi^\alpha \partial_\tau^\beta m(\xi,\tau)\right| \lesssim |\xi|^{-\alpha}|\tau|^{-\beta}.$$

  - Such multiplier operators are bounded on $L^p(\mathbb{H}^n)$ for $1 < p < \infty$.
  - The corresponding kernels satisfied differential inequalities

$$\left|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}} \partial_t^\beta K(\mathbf{z},t)\right| \lesssim |\mathbf{z}|^{-2n-|\boldsymbol{\alpha}|}\left(|\mathbf{z}|^2 + |t|\right)^{-1-\beta}.$$

This last estimate reflects the multi-parameter structure but is stronger than a product estimate.

- A flag kernel associated with the dilations

$$\lambda_{\mathbf{a}}(\mathbf{x}) = (\lambda^{a_1} x_1, \dots, \lambda^{a_n} x_n)$$

is a tempered distribution K with appropriate cancellation and singularities on an increasing sequence of subspaces:

$$\begin{aligned}\left|\partial^{\boldsymbol{\alpha}} K(\mathbf{x})\right| &\lesssim \left(|x_1|^{\frac{1}{a_1}}\right)^{-a_1(1+\alpha_1)} \left(|x_1|^{\frac{1}{a_1}} + |x_2|^{\frac{1}{a_2}}\right)^{-a_2(1+\alpha_2)} \cdots \\ &= \prod_{j=1}^{n} \left(|x_1|^{\frac{1}{a_1}} + |x_2|^{\frac{1}{a_2}} + \cdots + |x_{j-1}|^{\frac{1}{a_{j-1}}} + |x_j|^{\frac{1}{a_j}}\right)^{-a_j(1+\alpha_j)}.\end{aligned}$$

  - N., Ricci, Stein (2001) studied such operators when they arise in the study of the Bergman projection in tube domains over polyhedral cones and in solving $\Box_b$ on certain quadratic submanifolds having the structure of step-2 nilpotent Lie groups.
  - N., Ricci, Stein, Wainger (2012) and Głowacki (2010), (2013) extended this theory to homogeneous nilpotent Lie groups G of higher step, established boundedness in $L^p(G)$ for $1 < p < \infty$, and showed that such operators form an algebra under convolution.

A Step 3 Example

Let K_1 and K_2 be Calderón-Zygmund kernels with compact support associated with the dilations

$$\lambda_1(\xi, \eta, \tau) = (\lambda\xi, \lambda\eta, \lambda\tau) \quad \text{and} \quad \lambda_2(\xi, \eta, \tau) = (\lambda\xi, \lambda^2\eta, \lambda^3\tau).$$

Thus

$$\left|\partial_x^\alpha \partial_y^\beta \partial^{\gamma_z} K_1(x, y, z)\right| \lesssim (|x| + |y| + |z|)^{-\alpha-\beta-\gamma},$$

$$\left|\partial_x^\alpha \partial_y^\beta \partial^{\gamma_z} K_2(x, y, z)\right| \lesssim \left(|x| + |y|^{\frac{1}{2}} + |z|^{\frac{1}{3}}\right)^{-\alpha-2\beta-3\gamma}.$$

The Fourier transforms $m_1 = \widehat{K_1}$ and $m_2 = \widehat{K_2}$ are smooth bounded functions satisfying the differential inequalities

$$\left|\partial^{\alpha_\xi} \partial_\eta^\beta \partial_\tau^\gamma m_1(\xi, \eta, \tau)\right| \lesssim \left(1 + |\xi| + |\eta| + |\tau|\right)^{-\alpha-\beta-\gamma},$$

$$\left|\partial^{\alpha_\xi} \partial_\eta^\beta \partial_\tau^\gamma m_2(\xi, \eta, \tau)\right| \lesssim \left(1 + |\xi| + |\eta|^{\frac{1}{2}} + |\tau|^{\frac{1}{3}}\right)^{-\alpha-2\beta-3\gamma}.$$

Using Euclidean convolution, let $T_j[\varphi] = \varphi * K_j$ so $T_1 \circ T_2$ is convolution with a distribution K and

$$\widehat{K}(\xi, \eta, \tau) = m(\xi, \eta, \tau) = m_1(\xi, \eta, \tau) m_2(\xi, \eta, \tau).$$

To estimate derivatives of m note that

$$\begin{aligned}
\partial_\xi \text{ gains } (1 + |\xi| + |\eta| + |\tau|)^{-1} &\text{ or } (1 + |\xi| + |\eta|^{\frac{1}{2}} + |\tau|^{\frac{1}{3}})^{-1}, \\
\partial_\eta \text{ gains } (1 + |\xi| + |\eta| + |\tau|)^{-1} &\text{ or } (1 + |\xi|^2 + |\eta| + |\tau|^{\frac{2}{3}})^{-1}, \\
\partial_\tau \text{ gains } (1 + |\xi| + |\eta| + |\tau|)^{-1} &\text{ or } (1 + |\xi|^3 + |\eta|^{\frac{3}{2}} + |\tau|)^{-1}.
\end{aligned}$$

Using just the product formula, the best estimates for m are

$$\begin{aligned}
\left|\partial_\xi^\alpha \partial_\eta^\beta \partial_\tau^\gamma m(\xi, \eta, \tau)\right| \lesssim \left(1 + |\xi| + |\eta|^{\frac{1}{2}} + |\tau|^{\frac{1}{3}}\right)^{-\alpha} \\
\left(1 + |\xi| + |\eta| + |\tau|^{\frac{2}{3}}\right)^{-\beta} \\
\left(1 + |\xi| + |\eta| + |\tau|\right)^{-\gamma}.
\end{aligned}$$

How should one think about such estimates?

One possibility is to use the theory of flag kernels and multipliers. If

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \partial_\tau^\gamma m(\xi,\eta,\tau)\right| \lesssim \left(1+|\xi|+|\eta|^{\frac{1}{2}}+|\tau|^{\frac{1}{3}}\right)^{-\alpha} \left(1+|\xi|+|\eta|+|\tau|^{\frac{2}{3}}\right)^{-\beta} \left(1+|\xi|+|\eta|+|\tau|\right)^{-\gamma}$$

then

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \partial_\tau^\gamma m(\xi,\eta,\tau)\right| \lesssim \begin{cases} |\xi|^{-\alpha}(|\xi|+|\eta|)^{-\beta}(|\xi|+|\eta|+|\tau|)^{-\gamma}, \\ \\ (|\xi|+|\eta|^{\frac{1}{2}}+|\tau|^{\frac{1}{3}})^{-\alpha}(|\eta|+|\tau|^{\frac{2}{3}})^{-\beta}|\tau|^{-\gamma}. \end{cases}$$

Thus m is a flag multiplier for two opposite flags

$$(0) \subset \{(\xi,0,0)\} \subset \{(\xi,\eta,0)\} \subset \{(\xi,\eta,\tau)\} = \mathbb{R}^3,$$
$$(0) \subset \{(0,0,\tau)\} \subset \{(0,\eta,\tau)\} \subset \{(\xi,\eta,\tau)\} = \mathbb{R}^3.$$

The distribution $K = K_1 * K_2$ is thus a flag kernel satisfying

$$\left|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma K(x,y,z)\right| \lesssim \begin{cases} |x|^{-1-\alpha}(|x|+|y|)^{-1-\beta}(|x|+|y|+|z|)^{-1-\gamma}, \\ \\ (|x|+|y|^{\frac{1}{2}}+|z|^{\frac{1}{3}})^{-1-\alpha}(|y|+|z|^{\frac{2}{3}})^{-1-\beta}|z|^{-1-\gamma}. \end{cases}$$

for the two opposite flags

$$(0) \subset \{(x,0,0)\} \subset \{(x,y,0)\} \subset \{(x,y,z)\} = \mathbb{R}^3,$$
$$(0) \subset \{(0,0,z)\} \subset \{(0,y,z)\} \subset \{(x,y,z)\} = \mathbb{R}^3.$$

In the 2-step case on $\mathbb{R}^n \times \mathbb{R}$, the differential inequalities

$$\left|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_t^{\beta} K(\mathbf{x},t)\right| \lesssim \big(|\mathbf{x}| + |t|\big)^{-n-|\boldsymbol{\alpha}|}\big(|\mathbf{x}|^2 + |t|\big)^{-1-\beta}$$

are equivalent to the pair of flag inequalities

$$\left|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_t^{\beta} K(\mathbf{x},t)\right| \lesssim \begin{cases} |\mathbf{x}|^{-n-|\boldsymbol{\alpha}|}(|\mathbf{x}|^2 + |t|)^{-1-\beta} & \text{and} \\ \\ (|\mathbf{x}| + |t|)^{-n-|\boldsymbol{\alpha}|}|t|^{-1-\beta} \end{cases}$$

However in step 3, the two-flag kernel inequalities from the last slide,

$$\left|\partial_x^{\alpha}\partial_y^{\beta}\partial_z^{\gamma} K(x,y,z)\right| \lesssim \begin{cases} |x|^{-1-\alpha}(|x|+|y|)^{-1-\beta}(|x|+|y|+|z|)^{-1-\gamma} \\ \\ (|x|+|y|^{\frac{1}{2}}+|z|^{\frac{1}{3}})^{-1-\alpha}(|y|+|z|^{\frac{2}{3}})^{-1-\beta}|z|^{-1-\gamma} \end{cases},$$

seem to give no information when $x = z = 0$.

These flag estimates, together with the flag cancellation conditions, do imply that K is singular only at the origin.

A second possible way of thinking about the inequalities

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \partial_\tau^\gamma m(\xi,\eta,\tau)\right| \lesssim \left(1+|\xi|+|\eta|^{\frac{1}{2}}+|\tau|^{\frac{1}{3}}\right)^{-\alpha}$$
$$\left(1+|\xi|+|\eta|+|\tau|^{\frac{2}{3}}\right)^{-\beta}$$
$$\left(1+|\xi|+|\eta|+|\tau|\right)^{-\gamma}$$

is to observe that the ξ, η, and τ derivatives are controlled by different norms and hence different families of dilations:

$$\begin{aligned}
\partial_\xi &\quad\longleftrightarrow\quad \widehat{N}_1(\xi,\eta,\tau)=|\xi|+|\eta|^{\frac{1}{2}}+|\tau|^{\frac{1}{3}} \\
\partial_\eta &\quad\longleftrightarrow\quad \widehat{N}_2(\xi,\eta,\tau)=|\xi|+|\eta|+|\tau|^{\frac{2}{3}} \\
\partial_\eta &\quad\longleftrightarrow\quad \widehat{N}_3(\xi,\eta,\tau)=|\xi|+|\eta|+|\tau|
\end{aligned}$$

The class $\mathcal{P}(\mathbf{E})$

We introduce a class $\mathcal{P}(\mathbf{E})$ of distributions on $\mathbb{R}^n$ singular only at the origin and depending on an $n \times n$ matrix $\mathbf{E}$. We study:

A. Properties of distributions $K \in \mathcal{P}(\mathbf{E})$
   a. Significance of the rank of $\mathbf{E}$
   b. Characterization using the Fourier transform
   c. Marked partitions
   d. Characterizations via dyadic decompositions
   e. Two-flag kernels

B. Convolution operators $T_K\varphi = \varphi * K$ on a homogeneous nilpotent groups
   a. Continuity on $L^p(G)$
   b. $\mathcal{P}(\mathbf{E})$ is an algebra
   c. Composition of Calderón-Zygmund kernels with different homogeneities

Let

$$\mathbf{E} = \begin{bmatrix} 1 & e(1,2) & e(1,3) & \cdots & e(1,n) \\ e(2,1) & 1 & e(2,3) & \cdots & e(2,n) \\ e(3,1) & e(3,2) & 1 & \cdots & e(2,n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e(n,1) & e(n,2) & e(n,3) & \cdots & 1 \end{bmatrix}$$

be an $n \times n$ matrix satisfying the basic hypotheses:

$$\begin{aligned} e(j,k) &> 0 && \text{for all } 1 \leq j,k \leq n, \\ e(j,j) &= 1 && \text{for all } 1 \leq j \leq n, \\ e(j,l) &\leq e(j,k)e(k,l) && \text{for all } 1 \leq j,k,l \leq n. \end{aligned}$$

Recall that the automorphic dilations on the group G are given by

$$\lambda_{\mathbf{d}}(x_1, \ldots, x_n) = (\lambda^{d_1} x_1, \ldots, \lambda^{d_n} x_n).$$

For $1 \leq j \leq n$ let

$$N_j(x_1, \ldots, x_n) = |x_1|^{e(j,1)/d_1} + \cdots + |x_n|^{e(j,n)/d_n},$$

Each N_j is a homogeneous norm for a family of dilations on $\mathbb{R}^n$:

$$N_j\big(\lambda^{\frac{d_1}{e(j,1)}} x_1, \ldots, \lambda^{\frac{d_n}{e(j,n)}} x_n\big) = \lambda\, N(x_1, \ldots, x_n).$$

$\mathcal{P}(\mathbf{E})$ is a class of tempered distributions, depending on the matrix $\mathbf{E}$ and smooth away from the origin on $\mathbb{R}^n$, defined in terms of differential inequalities and cancellation conditions.

Differential inequalities:

If $K \in \mathcal{P}(\mathbf{E})$ then away from the origin K is given by a smooth function and for each multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$

$$\left|\partial^{\boldsymbol{\alpha}} K(x_1, \ldots, x_n)\right| \leq C_{\boldsymbol{\alpha}} \prod_{j=1}^{n} N_j(x_1, \ldots, x_n)^{-d_j(1+\alpha_j)}$$

where

$$N_j(x_1, \ldots, x_n) = |x_1|^{e(j,1)/d_1} + \cdots + |x_n|^{e(j,n)/d_n}.$$

Note that

$$N_j\left(\lambda^{\frac{d_1}{e(j,1)}} x_1, \ldots, \lambda^{\frac{d_n}{e(j,n)}} x_n\right) = \lambda\, N(x_1, \ldots, x_n)$$

so N_j is a homogeneous norm for the family of dilations on $\mathbb{R}^n$ given by

$$\lambda \cdot_j (x_1, \ldots, x_n) = \left(\lambda^{\frac{d_1}{e(j,1)}} x_1, \ldots, \lambda^{\frac{d_n}{e(j,n)}} x_n\right).$$

Derivatives of K with respect to the variable x_j are controlled by the norm N_j.

Cancellation Conditions:

These are defined in terms of the action of the distribution K on dilates of normalized bump functions. Let $0 \leq m \leq n-1$ and let ψ be any normalized bump function of $n-m$ variables. There are two requirements:

- If $R = (R_{m+1}, \ldots, R_n)$ and each $R_j > 0$ set

  $$\psi_R(x_{m+1}, \ldots, x_n) = \psi(R_{m+1}x_{m+1}, \ldots, R_n x_n).$$

  Define a tempered distribution $K_R^{\#}$ on $\mathbb{R}^m$ by setting

  $$\langle K_R^{\#}, \varphi \rangle = \langle K, \varphi \otimes \psi_R \rangle \qquad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^m).$$

  Away from the origin $K_R^{\#}$ is given by a smooth function and

  $$\left|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m} K_R^{\#}(x_1, \ldots, x_m)\right| \leq C_{\boldsymbol{\alpha}} \prod_{j=1}^{m} N_j(x_1, \ldots, x_m, 0, \ldots, 0)^{-a_j(1+\alpha_j)}$$

  with $C_{\boldsymbol{\alpha}}$ independent of ψ and R.

- The same holds for any permutation of the variables $x_1, \ldots, x_n$.

The rank of $\mathbf{E}$

Let $\mathbf{E}$ be an $n \times n$ satisfying the basic hypotheses.

Lemma

(a) *Rank* $(\mathbf{E}) = 1$ *if and only if there is a dilation structure on* $\mathbb{R}^n$ *for which* $\mathcal{P}(\mathbf{E})$ *is the space of Calderón-Zygmund kernels.*

(b) *If rank* $(\mathbf{E}) > 1$ *and* $K \in \mathcal{P}(\mathbf{E})$ *then* K *is integrable at infinity.*

(c) *Suppose the rank of* $\mathbf{E}$ *is* m*. If* $K \in \mathcal{P}(\mathbf{E})$ *then for* $\lambda \geq 1$

$$\left|\left\{\mathbf{x} \in \mathbb{R}^n : |K(\mathbf{x})| > \lambda\right\}\right| \lesssim \lambda^{-1} \log(\lambda)^{m-1}.$$

Moreover there exists $K \in \mathcal{P}(\mathbf{E})$ *so that*

$$\left|\left\{\mathbf{x} \in \mathbb{R}^n : |K(\mathbf{x})| > \lambda\right\}\right| \gtrsim \lambda^{-1} \log(\lambda)^{m-1}.$$

Characterization via the Fourier transform

For $\operatorname{rank}(\mathbf{E}) > 1$ only the local behavior is important. Set

$$\mathcal{P}_0(\mathbf{E}) = \Big\{ K \in \mathcal{P}(\mathbf{E}) : K \text{ is rapidly decreasing outside the unit ball} \Big\}.$$

Such distributions can be characterized by the behavior of their Fourier transform $\widehat{K}$. Recall that

$$N_j(\mathbf{x}) = |x_1|^{e(j,1)/d_1} + \cdots + |x_n|^{e(j,n)/d_n}, \qquad 1 \le j \le n.$$

The dual norms are then given by

$$\widehat{N}_j(\boldsymbol{\xi}) = |\xi_1|^{1/e(1,j)d_1} + \cdots + |\xi_n|^{1/e(n,j)d_n}, \qquad 1 \le j \le n.$$

Put

$$\mathcal{M}_\infty(\mathbf{E}) = \Big\{ m \in \mathcal{C}^\infty(\mathbb{R}^n) : \big|\partial^{\boldsymbol{\alpha}} m(\boldsymbol{\xi})\big| \le C_{\boldsymbol{\alpha}} \prod_{j=1}^{n} \big(1 + \widehat{N}_j(\boldsymbol{\xi})\big)^{-\alpha_j d_j} \Big\}.$$

Theorem

$K \in \mathcal{P}_0(\mathbf{E})$ if and only if $\widehat{K} = m \in \mathcal{M}_\infty(\mathbf{E})$.

Marked Partitions

The analysis of distributions $K \in \mathcal{P}_0(\mathbf{E})$ relies on a partition of the unit ball $\mathbb{B}(1) \subset \mathbb{R}^n$ into regions where one summand is dominant in each norm $N_j(\mathbf{x})$. Similarly the analysis of $m \in \mathcal{M}_\infty(\mathbf{E})$ depends on a partition of $\mathbb{R}^n \setminus \mathbb{B}(1)$ into regions where one term in each dual norm $\widehat{N}_j(\boldsymbol{\xi})$ is dominant.

For example, on the unit ball, the principal region S_0 is the set where for $1 \leq j \leq n$ the term $|x_j|^{e(j,j)/d_j} = |x_j|^{1/d_j}$ is dominant in $N_j(\mathbf{x}) = \sum_{k=1}^n |x_k|^{e(j,k)/d_k}$:

$$S_0 = \Big\{\mathbf{x} \in \mathbb{B}(1) : |x_k|^{e(j,k)/d_k} \leq |x_j|^{1/d_j} \text{ for all } j, k\Big\}.$$

Note that if $\mathbf{x} \in S_0$ then $N_j(\mathbf{x}) \approx |x_j|^{1/d_j}$. On this region, the differential inequalities for $K \in \mathcal{P}_0(\mathbf{E})$ simplify:

$$\big|\partial^{\boldsymbol{\alpha}} K(x_1, \ldots, x_n)\big| \lesssim \prod_{j=1}^n N_j(x_1, \ldots, x_n)^{-d_j(1+\alpha_j)} \approx \prod_{j=1}^n |x_j|^{-(1+\alpha_j)}.$$

On the subset S_0 the differential inequalities for K are exactly the same as the differential inequalities for a product kernel.

The following simple observation is a key to the study of other regions.

Lemma

Suppose $\mathbf{x} \in \mathbb{B}(1)$ *and suppose the term* $|x_k|^{e(j,k)/d_k}$ *is dominant in the norm* $N_j(\mathbf{x})$*. Then* $|x_k|^{e(k,k)/d_k} = |x_k|^{1/d_k}$ *is dominant in* $N_k(\mathbf{x})$*.*

Proof.

Since $|x_k|^{e(j,k)/d_k}$ is dominant in the norm $N_j(\mathbf{x})$,

$$|x_k|^{e(j,k)/d_k} \geq |x_l|^{e(l,k)/d_l}, \qquad 1 \leq l \leq n.$$

According to the basic hypothesis,

$$e(l,k) \leq e(l,j)e(j,k)$$

for any $1 \leq j,k,l \leq n$. Then since $|x_l| \leq 1$,

$$\begin{aligned} |x_k|^{1/d_k} &\geq |x_l|^{e(l,k)/e(j,k)d_l} \\ &\geq |x_l|^{e(l,j)e(j,k)/e(j,k)d_l} \\ &= |x_l|^{e(l,j)/d_l}. \end{aligned}$$

$\square$

A marked partition S of $\{1, \ldots, n\}$ is a collection of disjoint subsets $I_1, \ldots, I_s \subset \{1, \ldots, n\}$ whose union is all of $\{1, \ldots, n\}$, together with a 'marked' element $k_r \in I_r$, $1 \leq r \leq s$. Write

$$S = \big((I_1, k_1), \ldots, (I_s, k_s)\big).$$

The decomposition of $\mathbb{B}(1)$ is parameterized by marked partitions. For $\mathbf{x} \in \mathbb{B}(1)$ outside a set of measure zero, there is exactly one dominant term in each norm $N_j(\mathbf{x})$. Let $k_1, \ldots, k_s$ be the distinct integers which arise as subscripts of these dominant terms. Let

$$I_r = \Big\{ j \in \{1, \ldots, n\} : |x_{k_r}|^{e(j,k_r)/d_{k_r}} \text{ is dominant in } N_j(\mathbf{x}) \Big\},\ 1 \leq r \leq s.$$

The subsets $I_1, \ldots, I_s$ are disjoint, $I_1 \cup \cdots \cup I_s = \{1, \ldots, n\}$, and by the Lemma, $k_r \in I_r$. Thus $S = \big((I_1, k_1), \ldots, (I_s, k_s)\big)$ is a marked partition.

Characterization of $\mathcal{P}(\mathbf{E})$ via dyadic decompositions

Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

- φ has strong cancellation if $\int_{\mathbb{R}} \varphi(x_1, \ldots, x_k, \ldots, x_n)\, dx_k = 0$ for $1 \leq k \leq n$.
- $[\varphi]_I(\mathbf{x}) = 2^{-\sum_{k=1}^n i_k} \varphi(2^{-i_1}x_1, \ldots, 2^{-i_n}x_n)$ if $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$.

Set

$$\Gamma_{\mathbb{Z}}(\mathbf{E}) = \Big\{(i_1, \ldots, i_n) \in \mathbb{Z}^n : e(j,k)i_k \leq i_j < 0 \text{ for all } 1 \leq j, k \leq n\Big\}.$$

Lemma

Let $\{\varphi^I : I \in \Gamma(\mathbf{E})\} \subset \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a family of normalized bump functions. Assume that each φ^I has strong cancellation. Then

$$K(\mathbf{x}) = \sum_{I \in \Gamma(\mathbf{E})} [\varphi^I]_I(\mathbf{x})$$

converges in the sense of distributions to an element of $\mathcal{P}_0(\mathbf{E})$.

The formulation of the converse assertion uses the decomposition of $\mathbb{R}^n$ based on marked partitions. Let $S = \big((I_1, k_1), \dots, (I_s, k_s)\big)$ be a marked partition. Write

$$\mathbb{R}^n = \mathbb{R}^{C_1} \oplus \cdots \oplus \mathbb{R}^{C_s} \quad \text{and} \quad (x_1, \dots, x_n) = (\bar{x}_1, \dots, \bar{x}_s)$$

where $\mathbb{R}^{C_r}$ is the set of variables $x_j \in \mathbb{R}^n$ with $j \in I_r$. Then

- There is an $s \times s$ matrix $\mathbf{E}_S$ satisfying the basic hypotheses.
- There is a space of distributions $\mathcal{P}(\mathbf{E}_S) \subset \mathcal{P}(\mathbf{E})$.
- There is a cone

$$\Gamma_{\mathbb{Z}}(\mathbf{E}_S) = \Big\{(i_1, \dots, i_n) \in \mathbb{Z}^n : e_S(j,k) i_k \leq i_j < 0\Big\}.$$

If $K \in \mathcal{P}(\mathbf{E})$ then

$$K = \psi_0 + \sum_S K_S$$

where $\psi_0 \in \mathcal{S}(\mathbb{R}^n)$ and $K_S \in \mathcal{P}(\mathbf{E}_S)$. Moreover

$$K_S(\mathbf{x}) = \sum_{I \in \Gamma(\mathbf{E}_S)} [\varphi^I]_I(\mathbf{x}).$$

The Basic Hypotheses

If the principal region is not empty, then there exists $(x_1, \ldots, x_n)$ with each $|x_j| < 1$ so that

$$|x_j|^{\frac{1}{d_j}} \geq |x_k|^{\frac{e(j,k)}{d_k}} \geq \left(|x_\ell|^{\frac{e(k,\ell)}{d_\ell}}\right)^{e(j,k)} = |x_\ell|^{\frac{d(j,k)e(k,\ell)}{d_\ell}}. \tag{1}$$

But since $\mathbf{x} \in S_0$,

$$|x_j| \geq |x_\ell|^{e(j,\ell)}. \tag{2}$$

Unless the inequalities defining S_0 are 'self-improving', inequality (2) should imply (1), and so

$$|x_\ell|^{e(j,\ell)} \geq |x_\ell|^{e(j,k)e(k,\ell)}.$$

Since $|x_\ell| \leq 1$ it follows that

$$e(j,\ell) \leq e(j,k)e(k,\ell).$$

The basic hypotheses also arise as follows. Let $\mathbf{F} = \{f(j,k)\}$ be any $n \times n$ matrix with positive entries, and let

$$\Gamma(\mathbf{F}) = \Big\{(t_1, \ldots, t_n) \in \mathbb{R}^n : f(j,k)t_k \le t_j < 0 \text{ for all } 1 \le j,k \le n\Big\}.$$

Lemma

If $\Gamma(\mathbf{F}) \neq \emptyset$ *there exists a unique* $n \times n$ *matrix* $\mathbf{E} = \{e(j,k)\}$ *with positive entries such that*

$$\Gamma(\mathbf{E}) = \Gamma(\mathbf{F})$$

and such that the entries of $\mathbf{E}$ *satisfy*

$$\begin{aligned} e(j,j) &= 1 & 1 \le j \le n, \\ e(j,l) &\le e(j,k)e(k,l) & 1 \le j,k,l \le n. \end{aligned}$$

Moreover, $e(j,k) \le f(j,k)$.

Two-flag kernels

Consider a distribution K which is a flag kernel relative to two opposite flags with dilations

$$\lambda_{\mathbf{a}}(\mathbf{x}) = (\lambda^{a_1} x_1, \dots, \lambda^{a_n} x_n),$$
$$\lambda_{\mathbf{b}}(\mathbf{x}) = (\lambda^{b_1} x_1, \dots, \lambda^{b_n} x_n).$$

Then K satisfies appropriate cancellation conditions and the following differential inequalities:

$$\left|\partial^{\boldsymbol{\alpha}} K(\mathbf{x})\right| \lesssim \begin{cases} \prod_{j=1}^{n} \left(|x_1|^{a_j/a_1} + |x_2|^{a_j/a_2} + \cdots + |x_{j-1}|^{a_j/a_{j-1}} + |x_j|\right)^{-1-\alpha_j} \\ \\ \prod_{j=1}^{n} \left(|x_j| + |x_{j+1}|^{b_j/b_{j+1}} + \cdots + |x_{n-1}|^{b_j/b_{n-1}} + |x_n|^{b_j/b_n}\right)^{-1-\alpha_j} \end{cases}$$

Note that the differential inequalities give no information if $x_1 = x_n = 0$ and $n \geq 3$.

Theorem

Suppose that $\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \cdots \leq \frac{a_n}{b_n}$ *with at least one strict inequality.*

(a) *The function* K *is integrable at infinity, and we can write* $K = K_0 + K_\infty$ *where* $K_\infty \in L^1(\mathbb{R}^N) \cap \mathcal{C}^\infty(\mathbb{R}^N)$, *and* K_0 *is a two-flag kernel supported in* $\mathbb{B}(1)$.

(b) *The kernel* K_0 *belongs to the class* $\mathcal{P}_0(\mathbf{E})$ *associated to the matrix*

$$\mathbf{E} = \begin{bmatrix} 1 & b_1/b_2 & b_1/b_3 & \cdots & b_1/b_{n-1} & b_1/b_n \\ a_2/a_1 & 1 & b_2/b_3 & \cdots & b_2/b_{n-1} & b_2/b_n \\ a_3/a_1 & a_3/a_2 & 1 & \cdots & b_3/b_{n-1} & b_3/b_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1}/a_1 & a_{n-1}/a_2 & a_{n-1}/a_3 & \cdots & 1 & b_{n-1}/b_n \\ a_n/a_1 & a_n/a_2 & a_n/a_3 & \cdots & a_n/a_{n-1} & 1 \end{bmatrix}.$$

Convolution on groups

When studying convolution on general homogeneous nilpotent Lie groups, we need to impose an additional condition on the matrix $\mathbf{E} = \{e(j,k)\}$ called double monotonicity: each row is weakly increasing from left to right and each column is weakly decreasing from top to bottom. Explicitly

$$e(j,k) \le e(j,k+1) \quad \text{and} \quad e(j,k) \ge e(j+1,k).$$

Lemma

Suppose $\mathbf{E}$ *is doubly monotone, and let* $I = (i_1, \ldots, i_n) \in \Gamma_{\mathbb{Z}}(\mathbf{E})$, $J = (j_1, \ldots, j_n) \in \Gamma_{\mathbb{Z}}(\mathbf{E})$. *If* φ, ψ *are normalized bump functions, then*

$$[\varphi]_I * [\psi]_J = [\theta]_K$$

where

(a) θ *is a normalized bump function;*

(b) $K = (k_1, \ldots, k_n) \in \Gamma(\mathbf{E})$ *and* $k_m = \max\{i_m, j_m\}$.

Convolution operators are bounded on L^p

Theorem

Let $G \cong \mathbb{R}^n$ be a homogeneous nilpotent Lie group and let $\mathbf{E}$ be a doubly monotone matrix. If $K \in \mathcal{P}_0(\mathbf{E})$ then the operator $T_K\varphi = \varphi * K$, defined initially on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, extends uniquely to a bounded operator on $L^p(G)$ for $1 < p < \infty$.

In fact, every kernel $K \in \mathcal{P}(\mathbf{E})$ is a flag kernel on an appropriate flag.

$\mathcal{P}(\mathbf{E})$ is an algebra under convolution

Theorem

Let $G \cong \mathbb{R}^n$ be a homogeneous nilpotent Lie group and let $\mathbf{E}$ be a doubly monotone matrix. If $K, \mathcal{L} \in \mathcal{P}_0(\mathbf{E})$ then there exists $\mathcal{M} \in \mathcal{P}_0(\mathbf{E})$ such that $T_K \circ T_{\mathcal{L}} = T_{\mathcal{M}}$.

The proof is quite technical and uses the dyadic decomposition of kernels. If S and T are marked partitions, one studies

$$\sum_{I \in \Gamma(\mathbf{E}_S)} [\varphi^I]_I * \sum_{J \in \Gamma(\mathbf{E}_T} [\psi^J]_J.$$

Composition of Calderón-Zygmund kernels

Let $K_{\mathbf{a}}, K_{\mathbf{b}}$ be Calderón-Zygmund kernels associated with the dilations

$$\lambda_{\mathbf{a}}(\mathbf{x}) = \left(\lambda^{a_1}x_1, \ldots, \lambda^{a_n}x_n\right) \qquad \lambda_{\mathbf{b}}(\mathbf{x}) = \left(\lambda^{b_1}x_1, \ldots, \lambda^{b_n}x_n\right).$$

Assume that $a_k \geq a_{k+1}$ and $b_k \geq b_{k+1}$.

Theorem

Let $\mathbf{E} = \{e(j,k)\}$ *where*

$$e(j,k) = \max\left\{\frac{a_j}{a_k}, \frac{b_j}{b_k}\right\}.$$

The operator $T_{K_1} \circ T_{K_2}$ *is given by convolution with a tempered distribution* L *with* $L \in \mathcal{P}(\mathbf{E})$*.*

Final Remarks:

1. For appropriate $\mathbf{a}$ and $\mathbf{b}$ the rank of $\mathbf{E}$ can be as large as n.
2. However $L = \sum_{j=1}^{n-1} L_j$ where $L_j \in \mathcal{P}(\mathbf{E}_j)$ and $\mathbf{E}_j$ has rank 2.