Spectral multipliers on 2-step groups: Topological versus homogeneous dimension

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Conference in Harmonic Analysis in Honor of Michael Christ
May 16-20, 2016, Madison

I. Spectral multipliers

General Setup:

$$\begin{aligned}
\mathcal{L} &\geq 0 \quad \text{self-adjoint op. on } \mathcal{H} = L^2(X, d\mu) \\
\mathcal{L} &= \int_0^\infty \lambda \, dE_\lambda \\
F(\mathcal{L}) &= \int_0^\infty F(\lambda) \, dE_\lambda \quad (F \in \mathcal{B} \text{ Borel measurable}) \\
\|F\|_{L^2_{s,\mathrm{sloc}}} &:= \sup_{t \geq 0} \|\chi_1 F(t \, \cdot)\|_{L^2_s} ,
\end{aligned}$$

where $\chi_1 \in C_0^\infty(\mathbb{R})$, $\quad \operatorname{supp} \chi_1 \subset [1, 2]$, $\quad \chi_1 \neq 0$. Let $\varsigma(\mathcal{L})$ be the infimum of the $s > 1/2$ such that

$$\exists C > 0 : \forall F \in \mathcal{B} : \|F(\mathcal{L})\|_{L^1 \to L^{1,\infty}} \leq C \, \|F\|_{L^2_{s,\mathrm{sloc}}} . \tag{1.1}$$

II. Mikhlin-Hörmander multipliers:

If $\varsigma(\mathcal{L}) < \infty$, then every F satisfying $\|F\|_{L^2_{s,\mathrm{sloc}}} < \infty$ for some $s > \varsigma(\mathcal{L})$ is a Mikhlin-Hörmander multiplier for $\mathcal{L}$, i.e., $F(\mathcal{L})$ extends from $L^p \cap L^2(X)$ to a bounded operator on $L^p(X), 1 < p < \infty$, and is of weak-type $(1,1)$.

Classical.

- For $\mathcal{L} := -\Delta$ on $\mathbb{R}^d$ we have $\varsigma(\mathcal{L}) = d/2$. [Mikhlin, Hörmander]
- Analogous result holds for elliptic ΨDO's on compact Riemannian d-manifolds [Seeger/Sogge '89]

  (This makes use of ΨDO-calculus for elliptic operators and FIO-representation of solutions to associates wave equations)

III. Spectral multipliers for sub-Laplacians: some classical results

G connected Lie group with Lie algebra $\mathfrak{g}$ and right-invariant Haar measure dx; view $X \in \mathfrak{g}$ as left-invariant differential operator

$$Xf(g) := \frac{d}{dt}\Big|_{t=0} f(g\exp(tX)).$$

Assume that $X_1, \dots, X_m$ generate $\mathfrak{g}$ as a Lie algebra. Then these vector fields satisfy Hörmander's condition, and thus

$$\mathcal{L} = -\sum_{j=1}^{m} X_j^2 \quad \text{is hypoelliptic}$$

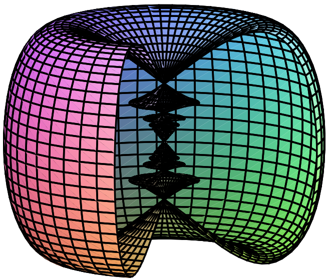
[Hörmander, Rothschild/Stein, Nagel/Stein/Wainger, ...]. Moreover, $\mathcal{L}$ is essentially s.a. on $C_0^\infty \subset L^2(G, dx)$[Nelson/Stinespring]

If $m < d := \dim G$, then $\mathcal{L}$ is not elliptic!

PROBLEMS: a) No good ΨDO- calculus or FIO-calculus for associated wave equations available!

b) Complicated underlying sub-Riemannian geometry!

Singular support of a wave emenating from a single point in the Heisenberg group



Selected results: A) Groups of polynomial volume growth

– The first multiplier theorem for stratified groups G was due to Hulanicki/Stein $\sim'$ 81

Theorem (Christ; Mauceri /Meda '91)

Let $\mathcal{L}$ be a homogeneous sub-Laplacian on a stratified group G of homogeneous dimension Q. Then $\varsigma(\mathcal{L}) \leq Q/2$.

– Generalization to groups of polynomial growth: Alexopoulos '94

B) Groups of exponential volume growth

Theorem (Hebisch; Guilini/Mauceri '91)

C^k- functional calculus for certain (sub-) Laplacians on some solvable Lie groups of exponential growth

Theorem (Christ, M. '96)

Identification of a solvable $4-$ dim. group of exponential growth and a distinguished Laplacian which is of holomorphic L^p-type for $p \neq 2$.

– In the setting of non-compact symmetric spaces, a related result had been familiar: Clerc/Stein '74

– Extensions to wide classes of Lie groups: Ludwig/ M. /Souaifi '00 – –08

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IV. Sub-Laplacians on 2-step stratified groups

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_1 \oplus \mathfrak{g}_2 \simeq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \\ G &= \mathfrak{g} \text{ as underlying manifold} \\ \exp &: \mathfrak{g} \to G \text{ the identity map} \\ [\mathfrak{g}_1, \mathfrak{g}_1] &= \mathfrak{g}_2, \quad [\mathfrak{g}, \mathfrak{g}_2] = 0 \end{aligned}$$

Let $X_1, \ldots, X_{d_1}$ be a basis of $\mathfrak{g}_1$ and $\mathcal{L} = -\sum_{j=1}^{d_1} X_j^2$.

$d := d_1 + d_2$ be the topological dimension of G,

$Q := d_1 + 2d_2$ be the homogeneous dimension of G.

The latter is natural in view of the automorphic dilations $\delta_r, r > 0$, given by $\delta_r|_{\mathfrak{g}_1} = r \,\mathrm{id}_{\mathfrak{g}_1}$, $\delta_r|_{\mathfrak{g}_1} = r^2 \,\mathrm{id}_{\mathfrak{g}_1}$.

V. Topological versus homogeneous dimension

Theorem (M. /Stein; Hebisch '93)

For sub-Laplacians on Heisenberg (-type) groups, we have $\varsigma(\mathcal{L}) \leq d/2$. *This is sharp for the Heisenberg group* $\mathbb{H}_1$.

Theorem (M./Stein '99; M./Seeger'14)

Let $1 < p < \infty$, $\gamma(p) := (d-1)|1/p - 1/2|$. *Then for* $-\infty < t < \infty$, *the operators* $(1 + t^2\mathcal{L})^{-\gamma(p)/2} e^{it\sqrt{\mathcal{L}}}$ *on Heisenberg type groups extend to bounded operators on* $L^p(G)$. *Moreover, for* $p = 1$ *and time* t *fixed, the corresponding operator is bounded on a suitable local Hardy space.*

Theorem (Martini/M., $\geq'$ 12)

For various further classes of 2-step groups, including all groups of dimensions $\dim \mathfrak{g} \leq 7$ *(in particular the free 2-step group in 3 generators) and all groups with* $\dim \mathfrak{g}_2 \leq 2$, *we have* $\varsigma(\mathcal{L}) \leq d/2$.

Recent result

Theorem (Martini /M.)

Let $\mathcal{L}$ be a homogeneous sub-Laplacian on a 2-step stratified group G of topological dimension d and homogeneous dimension Q. Then

$$d/2 \le \varsigma(\mathcal{L}) < Q/2.$$

1. The proof of the sufficiency of the condition $\|F\|_{L^2_{\alpha,\mathrm{sloc}}} < \infty$ for some $\alpha < Q/2$ makes use of ideas and (rather technical) methods from previous work by the authors, in combination with elementary estimates for algebraic functions.
2. For the talk, I concentrate on the necessary condition. To the best of our knowledge, the only case where the necessity of $\varsigma(\mathcal{L}) \ge d/2$ had been proved before was for the 3-dimensional Heisenberg group $\mathbb{H}_1$.

VI. Some ideas of the proof that $\varsigma(\mathcal{L}) \geq d/2$ is necessary

1. In [M. /Stein] the multipliers $F_\alpha(\lambda) = \lambda^{i\alpha}$ were used in order to establish lower bounds on the Heisenberg group on $\mathbb{H}_1$. However, for more general 2-step groups, even Heisenberg groups, this approach seems to become hopeless.
2. Instead, we use the following multipliers: For $\chi \in C_0^\infty(\mathbb{R}), \chi \geq 0$, and $t > 0$, define $m_t \in \mathcal{S}$ by

$$m_t(\lambda) = \int_{\mathbb{R}} \chi(s)\, e^{i(t-s)\lambda}\, ds = e^{it\lambda}\, \hat{\chi}(\lambda), \tag{2.1}$$

Then

$$\|m_t\|_{L^2_{\alpha,\mathrm{sloc}}} \leq C_{\alpha,\chi}\, (1 + |t|)^\alpha. \tag{2.2}$$

3. If F is a spectral multiplier, denote by $K_F = F(\mathcal{L})\delta_0$ its convolution kernel, i.e.,

$$F(\mathcal{L})\varphi = \varphi * K_F, \qquad \varphi \in \mathcal{S}(G).$$

A. Proof in the Euclidean case $\mathbb{R}^d$

On $\mathbb{R}^d$, the idea of the proof becomes very easy; in fact, some of the arguments to follow are even unreasonably overcomplicated (for some good reason):

Let $\mathcal{L} = -\Delta$, and consider the Schrödinger propagators for $t > 0$:

$$p_t(x) = e^{it\mathcal{L}}\delta_0(x)\Big(= c_d t^{-d/2} e^{-i\frac{|x|^2}{4t}}\Big),$$

i.e.,

$$\widehat{p_t}(\xi) = e^{it|\xi|^2}.$$

Write $K_t := K_{m_t}$. Then by (2.1) and Fourier inversion, for $t \gg 1$

$$K_t(x) = c_d \iint \chi(s) e^{i(t-s)|\xi|^2} e^{i\xi\cdot x}\, d\xi ds.$$

Next, writing $x = 2ty$, with $|y| \sim 1$, we get

$$K_t(2ty) = c_d \iint \chi(s) e^{it\big((1-\frac{s}{t})|\xi|^2 + 2\xi\cdot y\big)}\, d\xi ds.$$

Applying stationary phase to the integral in $\xi \in \mathbb{R}^d$, we obtain

$$K_t(2ty) = c_d t^{-d/2} e^{-it|y|^2} \int (1-\frac{s}{t})^{-d/2} \chi(s) e^{-i\frac{s}{1-s/t}|y|^2}\, ds,$$

and since $t \gg 1$, $|s| \lesssim 1$ and $|y| \sim 1$, we find that

$$\Big| \int (1-\frac{s}{t})^{-d/2} \chi(s) e^{-i\frac{s}{1-s/t}|y|^2}\, ds \Big| \sim |\hat{\chi}(|y|^2)| \sim 1,$$

if χ is chosen properly. We thus see that

$$|K_t(2ty)| \geq C t^{-d/2} \qquad \forall |y| \sim 1. \tag{2.3}$$

This implies that

$$\|K_t\|_1 \gtrsim \int_{|y|\sim 1} |K_t(2ty)| t^d\, dy \gtrsim t^{d/2},$$

and comparing with (2.2), i.e., $\|m_t\|_{L^2_{\alpha,\mathrm{sloc}}} \leq C_{\alpha,\chi}\, t^{\alpha}$, "morally" this implies that necessarily $\alpha \geq d/2$, if we assume that $F(\mathcal{L})$ is of weak-type (1,1) whenever $\|F\|_{L^2_{\alpha,\mathrm{sloc}}} < \infty$.

B. Proof for 2-step groups

Recall that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \simeq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, and let for $f \in L^1(G)$ and $\mu \in \mathfrak{g}_2^*$

$$f^\mu(x) = \int_{\mathfrak{g}_2} f(x,u)\, e^{-i\langle \mu, u\rangle}\, du, \qquad x \in \mathfrak{g}_1.$$

Define the skew-symmetric endomorphism J_μ on $\mathfrak{g}_1$ by

$$\langle J_\mu x, x'\rangle = \mu([x,x']) \qquad \forall x, x' \in \mathfrak{g}_1,$$

and

$$\mathrm{T}(z) = \frac{z}{\tan z}, \qquad \mathrm{S}(z) = \frac{z}{\sin z}, \qquad z \in \mathbb{C} \setminus \{k\pi \,:\, 0 \neq k \in \mathbb{Z}\}.$$

The key is the following Mehler-type formula:

Theorem (Gaveau, Hulanicki, Cygan, M./Ricci,. . .)

For all $t \in \mathbb{R}$ we have $e^{it\mathcal{L}} f = f *_G p_t$, where for every $\mu \in \mathfrak{g}_2^*$, $x \in \mathfrak{g}_1$,

$$p_t^\mu(x) = \frac{1}{(4\pi i t)^{d_1/2}} \det{}^{1/2}\, \mathrm{S}(itJ_\mu) \exp\left(\frac{i}{4t} \langle (\mathrm{T}(itJ_\mu)x, x\rangle\right). \qquad (2.4)$$

This allows to proceed in a similar way as on $\mathbb{R}^d$, with isotropic dilation $x = 2ty$ replaced by automorphic dilation $(x, u) = (2ty, t^2 v)$, where then $|(y, v)| \sim 1$. Matters eventually boil down to proving that we may apply stationary phase to the integral in μ arising from Fourier inversion in $\mathfrak{g}_2$. More precisely, we need to prove (recall that $\mathrm{T}(z) = z/\tan z$):

Theorem

Let

$$\Phi(y, v, \mu) = -\langle \mathrm{T}(iJ_\mu)y, y\rangle + \langle \mu, v\rangle.$$

There exist $y_0 \in \mathfrak{g}_1$, $v_0 \in \mathfrak{g}_2$, $\mu_0 \in \mathfrak{g}_2^*$ *such that*

$$|\mu_0| < 1, \qquad \nabla_\mu \Phi(y_0, v_0, \mu_0) = 0, \qquad \det \nabla_\mu^2 \Phi(y_0, v_0, \mu_0) \neq 0. \quad (2.5)$$

Analysis of the phase Φ

Note that

$$\mathrm{T}(z) = 1 - \sum_{k>0} b_k z^{2k},$$

$$\text{with} \quad b_k = (-1)^{k-1} 2^{2k} B_{2k}/(2k)! = 2\pi^{-2k}\zeta(2k),$$

the B_{2k} are Bernoulli numbers, and ζ is the Riemann zeta function. Thus

$$\langle \mathrm{T}(iJ_\mu)y, y\rangle = |y|^2 - \sum_{k>0} b_k |J_\mu^k y|^2$$

hence

$$\Phi(y, v, \mu) = -|y|^2 + \Phi_0(y, \mu) + \langle v, \mu\rangle,$$

where

$$\Phi_0(y, \mu) = \sum_{k>0} b_k |J_\mu^k y|^2,$$

Thus, by choosing v_0 appropriately, matters reduce to showing that there exist $y_0 \in \mathfrak{g}_1$ and $\mu_0 \in \mathfrak{g}_2^*$ such that $|\mu_0| < 1$ and $\nabla_\mu^2 \Phi_0(y_0, \mu_0)$ is non-degenerate.

Identify $\mathfrak{g}_2^*$ with

$$V = \{J_\mu : \mu \in \mathfrak{g}_2^*\} \subset \mathfrak{so}(\mathfrak{g}_1),$$

and consider Φ_0 as a function $\mathfrak{g}_1 \times V \to \mathbb{R}$. Let $V_{\rm gen}$ be the homogeneous Zariski-open subset of V whose elements have a maximal number of distinct eigenvalues among the elements of V.

Lemma (Key Lemma)

Let $S \in V_{\rm gen}$ and let $e \in \mathfrak{g}_1$ be such that the orthogonal projection of e on each eigenspace of S^2 is nonzero. Let N be the number of distinct eigenvalues of S^2. For all $T \in V$, if

$$TS^j e = 0 \qquad \text{for } j = 0, \dots, 2N-1, \tag{2.6}$$

then $T = 0$.

Fix $S \in V_{\mathrm{gen}}$ and $e \in \mathfrak{g}_1$ as in the lemma. Put

$$V_j = \{T \in V \,:\, TS^l e = 0 \text{ for } l = 0, \dots, j-1\}, \quad j \in \mathbb{N}.$$

Then by the lemma

$$V = V_0 \supseteq V_1 \supseteq \cdots V_{r-1} \supseteq V_r = \{0\},$$

where we choose $r \leq 2N$ minimal. Decompose $V_j = W_j \oplus V_{j+1}$. Then

$$V = W_0 \oplus \cdots \oplus W_{r-1}. \tag{2.7}$$

In addition, for all nonzero $T \in W_j$, $TS^l e = 0$ for $l < j$ but $TS^j e \neq 0$, and in particular the map $W_j \ni T \mapsto TS^j e \in \mathfrak{g}_1$ is injective.

Let $\Phi_{00}(\mu) := \Phi_0(e, \mu)$ and define, for all sufficiently small $\epsilon > 0$, the bilinear form $H(\epsilon) : V \times V \to \mathbb{R}$ by

$$H(\epsilon) = \frac{1}{2} \nabla^2 \Phi_{00}(\epsilon S).$$

Let $H_{ij}(\epsilon)$ be the restriction of $H(\epsilon)$ to $W_i \times W_j$ for all $i, j = 0, \dots, r-1$. Identify bilinear forms with their representing matrices. Then $H(\epsilon)$ is a block matrix with respect to the decomposition (2.7) of V, with blocks $H_{ij}(\epsilon)$.

Lemma

For all $i, j = 0, \dots, r-1$, and all small $\epsilon \in \mathbb{R}$, $A \in W_i$ and $B \in W_j$,

$$H_{ij}(\epsilon)(A, B) = \begin{cases} O(\epsilon^{i+j+1}) & \text{if } i+j \text{ is odd,} \\ (-1)^{(i-j)/2} b_{1+(i+j)/2} \epsilon^{i+j} \langle AS^i e, BS^j e \rangle + O(\epsilon^{i+j+2}) & \text{if } i+j \text{ is even.} \end{cases}$$

Proof. Recall that $\Phi_0(y,\mu) = \sum_{k>0} b_k |J_\mu^k y|^2$, i.e.,

$$\nabla^2\Phi_{00}(\epsilon S) = \sum_{k>0} b_k \nabla^2\Phi_k(\epsilon S),$$

where $\Phi_k(T) = |T^k e|^2$. Moreover the Hessian $\nabla^2\Phi_k(\epsilon S)(A,B)$ is the bilinear part in (A,B) of the Maclaurin expansion of $\Phi_k(\epsilon S + A + B)$ with respect to (A,B).

Let $k > 0$. In the expansion of $|(\epsilon S + A + B)^k e|^2$, the bilinear part in (A,B) is

$$2(-1)^k \epsilon^{2k-2} \sum_{\alpha+\beta+\gamma=2k-2} \langle S^\alpha A S^\beta B S^\gamma e, e\rangle. \tag{2.8}$$

If we assume that $A \in W_i$ and $B \in W_j$, then the sum can be restricted to the indices α, β, γ such that $\alpha \geq i$ and $\gamma \geq j$, because the other summands vanish. In particular the entire sum vanishes unless $2k-2 \geq i+j$. Hence, if $i+j$ is odd, then (2.8) vanishes unless $2k-2 \geq i+j+1$, and in particular (2.8) is $O(\epsilon^{i+j+1})$ for all k. Similar argument for $i+j$ even.

Q.E.D.

Lemma

For all sufficiently small $\epsilon \neq 0$, $H(\epsilon)$ is positive definite.

Proof. Scaling essentially by suitable powers of ϵ in the blocks W_j allows to reduce to a matrix

$$\tilde{H}_{ij}(\epsilon)(A,B) = \begin{cases} O(\epsilon) & \text{if } i+j \text{ is odd,} \\ b_{1+(i+j)/2}\langle AS^i e, BS^j e\rangle + O(\epsilon^2) & \text{if } i+j \text{ is even,} \end{cases}$$

hence even to

$$\tilde{H}_{ij}(0)(A,B) = \begin{cases} 0 & \text{if } i+j \text{ is odd,} \\ b_{1+(i+j)/2}\langle AS^i e, BS^j e\rangle & \text{if } i+j \text{ is even.} \end{cases}$$

Since the linear map

$$V = W_0 \oplus \cdots \oplus W_{r-1} \ni (T_0, \ldots, T_{r-1}) \mapsto (T_j S^j e)_{j=0}^{r-1} \in \mathfrak{g}_1^r$$

is injective, we can consider $\tilde{H}(0)$ as the restriction to a suitable subspace of the bilinear form

$K : \mathfrak{g}_1^r \times \mathfrak{g}_1^r \to \mathbb{R}$, given by

$$K((v_0,\dots,v_{r-1}),(w_0,\dots,w_{r-1})) := \sum_{i,j=0}^{r-1} c_{ij}\, \langle v_i, w_j \rangle,$$

where

$$c_{ij} := \begin{cases} 0 & \text{if } i+j \text{ is odd,} \\ b_{1+(i+j)/2} & \text{if } i+j \text{ is even.} \end{cases}$$

So it is sufficient to show that K is positive definite. But, $K = C \otimes I_{d_1}$, where C is the matrix $C = (c_{ij})_{i,j=0}^{r-1}$ so it suffice to prove that C is positive definite. By Sylvester's criterion, this can be reduced to showing that matrices of the form

$$Z_{m,s} = \begin{pmatrix} b_{m+1} & b_{m+2} & \cdots & b_{m+s} \\ b_{m+2} & b_{m+3} & \cdots & b_{m+s+1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m+s} & b_{m+s+1} & \cdots & b_{m+2s-1} \end{pmatrix}$$

have positive determinant for all m, s.

Determinants involving Bernoulli numbers

Such matrices have been studied since long time and explicit formulas for some of them can be found in the literature.
However, positivity of $Z_{m,s}$ can easily be seen by means of properties of the Riemann zeta function ζ:
Since $b_k = 2\,\pi^{-2k}\zeta(2k)$, we have

$$\det Z_{m,s} = c(s)\det\begin{pmatrix} \zeta(2m+2) & \zeta(2m+4) & \cdots & \zeta(2m+2s) \\ \zeta(2m+4) & \zeta(2m+6) & \cdots & \zeta(2m+2s+2) \\ \vdots & \vdots & \ddots & \vdots \\ \zeta(2m+2s) & \zeta(2m+2s+2) & \cdots & \zeta(2m+4s-2) \end{pmatrix}.$$

Let $\mathfrak{S}_s$ denote the permutation group of $\{1,\ldots,s\}$, and $\varepsilon(\sigma)$ the signature of $\sigma \in \mathfrak{S}_s$.

Then the last determinant can be rewritten as

$$\sum_{\sigma\in\mathfrak{S}_s} \varepsilon(\sigma) \prod_{i=1}^{s} \zeta(2(i+\sigma(i)+m-1))$$

$$= \frac{1}{s!} \sum_{\sigma,\tau\in\mathfrak{S}_s} \varepsilon(\sigma)\,\varepsilon(\tau) \prod_{i=1}^{s} \zeta(2(\sigma(i)+\tau(i)+m-1))$$

$$= \frac{1}{s!} \sum_{\sigma,\tau\in\mathfrak{S}_s} \varepsilon(\sigma)\,\varepsilon(\tau) \sum_{k_1=1}^{\infty} \cdots \sum_{k_s=1}^{\infty} k_1^{-2(\sigma(1)+\tau(1)+m-1)} \cdots k_s^{-2(\sigma(s)+\tau(s)+m-1)}$$

$$= \frac{1}{s!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_s=1}^{\infty} (k_1 \cdots k_s)^{-2(2s+m-1)} \left(\sum_{\sigma\in\mathfrak{S}_s} \varepsilon(\sigma)\, k_1^{2(s-\sigma(1))} \cdots k_s^{2(s-\sigma(s))} \right)^2$$

$$= \frac{1}{s!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_s=1}^{\infty} (k_1 \cdots k_s)^{-2(2s+m-1)} \prod_{1\le i<j\le s} (k_i^2 - k_j^2)^2 > 0.$$

where in the last passage the Vandermonde determinant formula was used.

Q.E.D.

Open problem:

Is $\varsigma(\mathcal{L}) = d/2$ for every sub-Laplacian on any (2-step) stratified group?