$\ell^p(\mathbb{Z}^d)$ boundedness for discrete operators of Radon type: maximal and variational estimates

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Maximal Radon transform

The maximal Radon transform is defined for $x \in \mathbb{R}^d$ by setting

$$\mathcal{M}_*^{\mathcal{P}} f(x) = \sup_{t>0} \left|\mathcal{M}_t^{\mathcal{P}} f(x)\right|,$$

where

$$\mathcal{M}_t^{\mathcal{P}} f(x) = \frac{1}{|B_t|} \int_{B_t} f(x - \mathcal{P}(y)) dy,$$

$B_t = \{y \in \mathbb{R}^k : |y| < t\}$ and

$$\mathcal{P}(y) = (\mathcal{P}_1(y), \ldots, \mathcal{P}_d(y))$$

is a polynomial mapping, i.e. $\mathcal{P}_j(y)$ is a real-valued polynomial on $\mathbb{R}^k$.

- It is very well known that for every $p > 1$ there is a $C_p > 0$ such that

$$\|\mathcal{M}_*^{\mathcal{P}} f\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$$

for any $f \in L^p(\mathbb{R}^d)$.

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The Hardy–Littlewood maximal function

If $k = d = 1$ and $\mathcal{P}(y) = y$ then $\mathcal{M}_*^{\mathcal{P}}$ coincides with the Hardy–Littlewood maximal function $\mathcal{M}_* f(x) = \sup_{t>0} |\mathcal{M}_t f(x)|$, where

$$\mathcal{M}_t f(x) = \frac{1}{2t} \int_{-t}^{t} f(x-y) dy.$$

Their $L^p(\mathbb{R})$ with $p > 1$ and weak type $(1,1)$ bounds are useful in proving the Lebesgue differentiation theorem, i.e. for every $f \in L^p(\mathbb{R})$ with $p \geq 1$ we have

$$\lim_{t \to 0} \mathcal{M}_t f(x) = f(x)$$

almost everywhere on $\mathbb{R}$.

The discrete counterpart of $\mathcal{M}_*$ can be defined as $\sup_{N \in \mathbb{N}} |M_N f(x)|$, where

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Birkhoff's ergodic theorem

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. Let $T : X \to X$ be an invertible measure preserving transformation. Classical Birkhoff's Theorem states that for any $f \in L^p(X, \mu)$ with $p \geq 1$, the averages

$$A_N f(x) = \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$$

converge μ-almost everywhere on X.

For the proof one shows

- L^p boundedness (p>1) of a maximal function

  $$\|\sup_{N \in \mathbb{N}} |A_N f|\|_{L^p} \lesssim \|f\|_{L^p},$$

  or weak type (1, 1) estimates at the endpoint. By the Calderón transference principle it follows from the corresponding estimates for the discrete Hardy–Littlewood maximal function.
- the convergence for a dense class.

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Convergence for a dense class

$$A_N f(x) = \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$$

- $\mathcal{I} = \left\{ f \in L^2 : f(Tx) = f(x) \right\}$. If $f \in \mathcal{I}$ then

$$A_N f = f,$$

  μ-almost everywhere.
- $\mathcal{B} = \left\{ g(Tx) - g(x) : g \in L^2 \cap L^\infty \right\}$. If $f \in \mathcal{B}$ then

$$|A_N f(x)| = \frac{1}{N} \left| \sum_{n=0}^{N-1} g(T^{n+1}x) - g(T^n x) \right| = \frac{1}{N} |g(T^N x) - g(x)|.$$

- $\mathcal{I} \oplus \mathcal{B}$ is dense in L^2.

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- $\mathcal{I} \oplus \mathcal{B}$ is dense in L^2.

Variational seminorm

For any complex-valued functions $(a_t : t > 0)$ and $r \geq 1$ the variational seminorm is

$$V_r(a_t : t > 0) = \sup_{\substack{t_0 < t_1 < \ldots < t_J \\ t_j > 0}} \left(\sum_{j=0}^{J-1} |a_{t_{j+1}} - a_{t_j}|^r \right)^{1/r}.$$

Observe that

- $V_r(a_t : t > 0) < \infty$ implies $(a_t : t > 0)$ is a Cauchy sequence.
- Moreover, we have

$$\sup_{t>0} |a_t| \leq V_r(a_t : t > 0) + |a_{t_0}|$$

where t_0 is an arbitrary element of $(0, \infty)$.

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Variational estimates in the continuous setup

Let $B_t = \{y \in \mathbb{R}^k : |y| < t\}$ and recall that

$$\mathcal{M}_t^{\mathcal{P}} f(x) = \frac{1}{|B_t|} \int_{B_t} f(x - \mathcal{P}(y)) dy,$$

where $\mathcal{P} : \mathbb{R}^k \to \mathbb{R}^d$ is a polynomial mapping.

$$V_r(\mathcal{M}_t^{\mathcal{P}} f(x) : t > 0) = \sup_{\substack{t_0 < t_1 < \ldots < t_J \\ t_j > 0}} \left(\sum_{j=0}^{J-1} |\mathcal{M}_{t_{j+1}}^{\mathcal{P}} f(x) - \mathcal{M}_{t_j}^{\mathcal{P}} f(x)|^r \right)^{1/r}.$$

Theorem (Jones, Seeger and Wright)

For every $p \in (1, \infty)$ and $r \in (2, \infty)$ there is $C_p > 0$ such that for all $f \in L^p(\mathbb{R}^d)$

$$\left\| V_r\left(\mathcal{M}_t^{\mathcal{P}} f : t > 0\right) \right\|_{L^p} \leq C_p \frac{r}{r-2} \|f\|_{L^p}.$$

Moreover, the constant C_p is independent of coefficients of the polynomial mapping $\mathcal{P}$.

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Bourgain's ergodic theorem

In the mid 1980's Bourgain extended Birkhoff's ergodic theorem and showed that for every $f \in L^p(X,\mu)$ with $p > 1$ there is a function $f^* \in L^p(X,\mu)$ such that

$$\lim_{N\to\infty} A_N f(x) = f^*(x)$$

μ-almost everywhere on X for the averages

$$A_N^P f(x) = \frac{1}{N}\sum_{n=1}^{N} f(T^{P(n)}x)$$

defined along any integer-valued polynomial P.

Pointwise convergence

Although, for Birkhoff's averaging operator, it was not very difficult to find a dense class of functions (say on $L^2(X,\mu)$) for which pointwise convergence holds, for Bourgain's averaging operator

$$A_N^P f(x) = \frac{1}{N} \sum_{n=1}^{N} f(T^{P(n)}x)$$

along the polynomials P of degree > 1, it is a hard problem. Even for $P(n) = n^2$, since $(n+1)^2 - n^2 = 2n+1$.

For overcoming the lack of dense class, Bourgain showed

- L^p boundedness of the maximal function,
- Given a lacunary sequence $(N_j : j \in \mathbb{N})$, for each $J > 0$ there is $C > 0$ such that

$$\Big(\sum_{j=0}^{J} \Big\| \sup_{N \in [N_j, N_{j+1})} \big| A_N^P f - A_{N_j}^P f \big| \Big\|_{L^2}^2 \Big)^{1/2} \leq C J^c \|f\|_{L^2}$$

for some $c < 1/2$.

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Oscillation seminorm vs variation seminorm

Given a lacunary sequence $(N_j : j \in \mathbb{N})$, the oscillation seminorm for a sequence $\big(a_n(x) : n \in \mathbb{N}\big)$ of complex-valued functions is defined by

$$O_J\big(a_n(x) : n \in \mathbb{N}\big) = \Big(\sum_{j=1}^{J} \sup_{N_j \le n < N_{j+1}} \big|a_n(x) - a_{N_j}(x)\big|^2\Big)^{1/2}.$$

Bourgain's oscillation inequality

There are constants $C > 0$ and $c < 1/2$ such that for all $J \in \mathbb{N}$

$$\big\|O_J\big(A_N^P f : N \in \mathbb{N}\big)\big\|_{L^2} \le CJ^c \|f\|_{L^2}.$$

For any $r > 2$ by Hölder's inequality we have

$$O_J\big(a_n(x) : n \in \mathbb{N}\big) \le J^{1/2-1/r} V_r\big(a_n(x) : n \in \mathbb{N}\big).$$

where

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Bourgain's strategy

- In the discrete settings Bourgain used the circle method of Hardy and Littlewood to provide $\ell^p(\mathbb{Z})$ estimates. The method arising from analytic number theory which allows us to obtain the asymptotic formula for the number of solutions in the Waring problem

$$n_1^k + \ldots + n_d^k = N.$$

- Bourgain's method was a breakthrough shedding new light on various discrete analogues in harmonic analysis, but his $\ell^p(\mathbb{Z})$ theory does not fall into the Littlewood–Paley paradigm.

- Is it possible to build up an appropriate Littlewood–Paley theory in the discrete setup which would allow us to deal with $\ell^p(\mathbb{Z})$ boundedness of discrete operators of Radon type?

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- Bourgain's method was a breakthrough shedding new light on various discrete analogues in harmonic analysis, but his $\ell^p(\mathbb{Z})$ theory does not fall into the Littlewood–Paley paradigm.
- Is it possible to build up an appropriate Littlewood–Paley theory in the discrete setup which would allow us to deal with $\ell^p(\mathbb{Z})$ boundedness of discrete operators of Radon type?

Variational estimates in the discrete setup

Let $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_d) : \mathbb{Z}^k \to \mathbb{Z}^d$ be a polynomial mapping. Define Radon averages

$$M_N^{\mathcal{P}} f(x) = N^{-k} \sum_{y \in [1,N]^k \cap \mathbb{N}^k} f\big(x - \mathcal{P}(y)\big).$$

Theorem (M., E.M. Stein and B. Trojan)

For every $p \in (1, \infty)$ and $r \in (2, \infty)$ there is $C_p > 0$ such that for all $f \in \ell^p\big(\mathbb{Z}^d\big)$

$$\big\| V_r\big(M_N^{\mathcal{P}} f : N \in \mathbb{N}\big) \big\|_{\ell^p} \leq C_p \frac{r}{r-2} \|f\|_{\ell^p}.$$

Moreover, the constant C_p is independent of coefficients of the polynomial mapping $\mathcal{P}$.

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Variational estimates for truncated Radon transform

Suppose that $K \in \mathcal{C}^1\big(\mathbb{R}^k \setminus \{0\}\big)$ is a Calderón–Zygmund kernel obeying

$$|y|^k |K(y)| + |y|^{k+1} |\nabla K(y)| \leq 1$$

for all $y \in \mathbb{R}^k \setminus \{0\}$ and a cancellation condition

$$\int_{\lambda_1 \leq |y| \leq \lambda_2} K(y) \mathrm{d}y = 0$$

for all $\lambda_1 < \lambda_2$. Define truncated Radon transform

$$T_N^{\mathcal{P}} f(x) = \sum_{y \in \mathbb{B}_N \setminus \{0\}} f\big(x - \mathcal{P}(y)\big) K(y)$$

where $\mathbb{B}_N = \{x \in \mathbb{R}^k : |x| \leq N\} \cap \mathbb{Z}^k$.

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Let $p \in (1, \infty)$, then for every $f \in \ell^p\big(\mathbb{Z}^d\big)$, the discrete Radon transform

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Known partial results for discrete variational estimates

- The estimates of r-variations for the one dimensional Bourgain's averaging operator were provided by Krause for all $p \in (1, \infty)$ and $r > \max\{p, p'\}$.
- Not long afterwards Zorin-Kranich obtained r-variational estimates for all $r > 2$ and $p > 1$ satisfying
$$\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{1}{2(D+1)}$$
where D denotes the degree of the polynomial $\mathcal{P}$.
- Their proofs were based on variational estimates of famous Bourgain's logarithmic lemma provided by Nazarov, Oberlin and Thiele. That was the critical building block in their arguments.
- Although, logarithmic lemma gives very nice $\ell^2(\mathbb{Z})$ theory in Bourgain's maximal theorem, it is very inefficient in $\ell^p(\mathbb{Z})$ theory. The reason, loosely speaking, is that it produces a polynomial growth of norm for $p \neq 2$.

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Discrete Littlewood–Paley theory

We propose completely different approach to attack r-variations. Instead of Bourgain's logarithmic lemma we established a discrete counterpart of the Littlewood–Paley theory by introducing the following family of projections

$$\Delta_{n,s}(\xi) = \sum_{a/q \in \mathscr{U}_s} \big(\eta\big(2^{nd}(\xi - a/q)\big) - \eta\big(2^{nd+1}(\xi - a/q)\big)\big)\eta\big(2^{s(d-\chi)}(\xi - a/q)\big),$$

where η a smooth cut-off function and

$$\mathscr{U}_s = \{a/q \in \mathbb{T} : \ (a,q) = 1 \text{ and } q \in \mathbf{P}_s\},$$

where the denominators $q \in \mathbf{P}_s$ have appropriate limitation in terms of their prime power factorization.

- We were able to show that for each $p > 1$ there is a constant $C > 0$ such that

$$\Big\| \Big(\sum_{n \in \mathbb{Z}} \big| \mathcal{F}^{-1}\big(\Delta_{n,s}\hat{f}\big)\big|^2 \Big)^{1/2} \Big\|_{\ell^p} \leq C \log(s+2) \|f\|_{\ell^p}.$$

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Do we need number theory?

Assume that $\mathcal{P}(x) = x^d$ and $d \geq 2$ and observe that $M_N^{\mathcal{P}} f(x) = K_N * f(x)$, where

$$K_N(x) = \frac{1}{N} \sum_{k=1}^{N} \delta_{\mathcal{P}(k)}(x).$$

Then

$$m_N(\xi) = \widehat{K}_N(\xi) = \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i \xi k^d} \quad (\xi \in \mathbb{T}).$$

If $\xi = a/q$ and $(a, q) = 1$ then we see that $m_N(a/q)$ behaves like a complete Gaussian sum

$$G(a/q) = \frac{1}{q} \sum_{r=1}^{q} e^{2\pi i \frac{a}{q} r^d}.$$

This suggests that the asymptotics for m_N should be concentrated in some neighbourhoods of Diophantine approximations of ξ with small denominators. Indeed, if $|\xi - a/q| \leq$ 'small in terms of N' with small q in terms of N then

$$m_N(\xi) \simeq \left(\frac{1}{q} \sum_{r=1}^{q} e^{2\pi i \frac{a}{q} r^d} \right) \cdot \left(\int_0^1 e^{2\pi i (\xi - \frac{a}{q})(Nx)^d} dx \right).$$

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Thank You!