Some Radon transforms and their discrete analogue

Xiaochun Li

Department of Mathematics
University of Illinois at Urbana-Champaign

Conference in Harmonic analysis in honor of Michael Christ
May 20, 2016

Bilinear Hilbert transforms

Let f_1, f_2 be Schwartz functions. The bilinear Hilbert transform is given by

$$H(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x+t) \frac{dt}{t}\,. \tag{1}$$

Theorem (Lacey-Thiele, '97)

$$\|H(f_1, f_2)\|_r \leq C \|f_1\|_{p_1} \|f_2\|_{p_2}\,, \tag{2}$$

provided that p_1, p_2, r *obey* $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$, $p_1, p_2 > 1$ *and* $r > 2/3$.

Bilinear Hilbert transforms

Let f_1, f_2 be Schwartz functions. The bilinear Hilbert transform is given by

$$H(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x+t) \frac{dt}{t} \,. \tag{1}$$

Theorem (Lacey-Thiele, '97)

$$\|H(f_1, f_2)\|_r \leq C \|f_1\|_{p_1} \|f_2\|_{p_2} \,, \tag{2}$$

provided that p_1, p_2, r *obey* $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$, $p_1, p_2 > 1$ *and* $r > 2/3$.

Let Γ denote curve $(t, \gamma(t))$, where γ is a continuous function on $\mathbb{R}$. The bilinear Hilbert transform along the curve Γ is defined by

$$H_\Gamma(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x-\gamma(t)) \frac{dt}{t}. \tag{3}$$

Theorem (L., '08)

If $\Gamma = (t, t^\alpha)$ for some real number $\alpha \neq 1$, then H_Γ can be extended to a bounded operator from $L^2 \times L^2$ to L^1.

Theorem (Lie, '11)

H_Γ is bounded from $L^2 \times L^2$ to L^1 if Γ is a "non-flat" smooth curve.

Let Γ denote curve $(t, \gamma(t))$, where γ is a continuous function on $\mathbb{R}$. The bilinear Hilbert transform along the curve Γ is defined by

$$H_\Gamma(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x-\gamma(t)) \frac{dt}{t} . \tag{3}$$

Theorem (L., '08)

If $\Gamma = (t, t^\alpha)$ for some real number $\alpha \neq 1$, then H_Γ can be extended to a bounded operator from $L^2 \times L^2$ to L^1.

Theorem (Lie, '11)

H_Γ is bounded from $L^2 \times L^2$ to L^1 if Γ is a "non-flat" smooth curve.

Bilinear Hilbert transform along curves

Let Γ denote curve $(t, \gamma(t))$, where γ is a continuous function on $\mathbb{R}$. The bilinear Hilbert transform along the curve Γ is defined by

$$H_\Gamma(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x-\gamma(t)) \frac{dt}{t} . \tag{3}$$

Theorem (L., '08)

If $\Gamma = (t, t^\alpha)$ for some real number $\alpha \neq 1$, then H_Γ can be extended to a bounded operator from $L^2 \times L^2$ to L^1.

Theorem (Lie, '11)

H_Γ is bounded from $L^2 \times L^2$ to L^1 if Γ is a "non-flat" smooth curve.

Bilinear Hilbert transform along curves

Let Γ denote curve $(t, \gamma(t))$, where γ is a continuous function on $\mathbb{R}$. The bilinear Hilbert transform along the curve Γ is defined by

$$H_\Gamma(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x-\gamma(t)) \frac{dt}{t}\,. \tag{3}$$

Theorem (L., '08)

If $\Gamma = (t, t^\alpha)$ for some real number $\alpha \neq 1$, then H_Γ can be extended to a bounded operator from $L^2 \times L^2$ to L^1.

Theorem (Lie, '11)

H_Γ is bounded from $L^2 \times L^2$ to L^1 if Γ is a "non-flat" smooth curve.

Uniform estimates for Bilinear Hilbert transform along polynomial curves

Let Γ be a polynomial curve given by $(t, P(t))$ for some real polynomial P. The bilinear Hilbert transform H_Γ along $(t, P(t))$ is

$$H_\Gamma(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x-P(t)) \frac{dt}{t} \tag{4}$$

Bilinear maximal function along Γ is given by

$$M_\Gamma(f_1, f_2)(x) = \sup_{\varepsilon>0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \left| f_1(x-t) f_2(x-P(t)) \right| dt \, . \tag{5}$$

Theorem (L.-Xiao, '13)

Suppose that Γ is a polynomial curve $(t, P(t))$ and P is a real polynomial without a linear term. Then both of H_Γ and M_Γ are bounded from $L^{p_1} \times L^{p_2}$ to L^r for $r > \frac{d-1}{d}$, $p_1, p_2 > 1$, and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$. Moreover, the bound is uniform in a sense that it depends on the degree of the polynomial P but independent of the coefficients of P.

Uniform estimates for Bilinear Hilbert transform along polynomial curves

Let Γ be a polynomial curve given by $(t, P(t))$ for some real polynomial P. The bilinear Hilbert transform H_Γ along $(t, P(t))$ is

$$H_\Gamma(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x - P(t)) \frac{dt}{t} \tag{4}$$

Bilinear maximal function along Γ is given by

$$M_\Gamma(f_1, f_2)(x) = \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \left| f_1(x-t) f_2(x - P(t)) \right| dt\,. \tag{5}$$

Theorem (L.-Xiao, '13)

Suppose that Γ is a polynomial curve $(t, P(t))$ and P is a real polynomial without a linear term. Then both of H_Γ and M_Γ are bounded from $L^{p_1} \times L^{p_2}$ to L^r for $r > \frac{d-1}{d}$, $p_1, p_2 > 1$, and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$. Moreover, the bound is uniform in a sense that it depends on the degree of the polynomial P but independent of the coefficients of P.

Uniform estimates for Bilinear Hilbert transform along polynomial curves

Let Γ be a polynomial curve given by $(t, P(t))$ for some real polynomial P. The bilinear Hilbert transform H_Γ along $(t, P(t))$ is

$$H_\Gamma(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x - P(t)) \frac{dt}{t} \tag{4}$$

Bilinear maximal function along Γ is given by

$$M_\Gamma(f_1, f_2)(x) = \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \left| f_1(x-t) f_2(x - P(t)) \right| dt \,. \tag{5}$$

Theorem (L.-Xiao, '13)

Suppose that Γ is a polynomial curve $(t, P(t))$ and P is a real polynomial without a linear term. Then both of H_Γ and M_Γ are bounded from $L^{p_1} \times L^{p_2}$ to L^r for $r > \frac{d-1}{d}$, $p_1, p_2 > 1$, and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$. Moreover, the bound is uniform in a sense that it depends on the degree of the polynomial P but independent of the coefficients of P.

Main Ingredients in the proof for the uniform estimates

- After removing finitely many intervals, the polynomial P is a monomial plus a tiny perturbation. The perturbation is handled via a quantitative version of inverse function theorem.
- Using σ-uniformity method, locally there is a decay estimate from $L^2 \times L^2$ to L^1, after removing finitely many paraproducts.
- Those paraproducts are uniformly bounded from $L^p \times L^q$ to L^r for all $p, q > 1$ and $r > 1/2$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$.
- Locally there is an appropriate upper bound that grows slowly enough, in contract to the decay estimate. This is one of the main difficulties in the uniform estimate. It can be achieved by a Whitney type decomposition.

Main Ingredients in the proof for the uniform estimates

- After removing finitely many intervals, the polynomial P is a monomial plus a tiny perturbation. The perturbation is handled via a quantitative version of inverse function theorem.
- Using σ-uniformity method, locally there is a decay estimate from $L^2 \times L^2$ to L^1, after removing finitely many paraproducts.
- Those paraproducts are uniformly bounded from $L^p \times L^q$ to L^r for all $p, q > 1$ and $r > 1/2$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$.
- Locally there is an appropriate upper bound that grows slowly enough, in contract to the decay estimate. This is one of the main difficulties in the uniform estimate. It can be achieved by a Whitney type decomposition.

Main Ingredients in the proof for the uniform estimates

- After removing finitely many intervals, the polynomial P is a monomial plus a tiny perturbation. The perturbation is handled via a quantitative version of inverse function theorem.
- Using σ-uniformity method, locally there is a decay estimate from $L^2 \times L^2$ to L^1, after removing finitely many paraproducts.
- Those paraproducts are uniformly bounded from $L^p \times L^q$ to L^r for all $p, q > 1$ and $r > 1/2$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$.
- Locally there is an appropriate upper bound that grows slowly enough, in contract to the decay estimate. This is one of the main difficulties in the uniform estimate. It can be achieved by a Whitney type decomposition.

Main Ingredients in the proof for the uniform estimates

- After removing finitely many intervals, the polynomial P is a monomial plus a tiny perturbation. The perturbation is handled via a quantitative version of inverse function theorem.
- Using σ-uniformity method, locally there is a decay estimate from $L^2 \times L^2$ to L^1, after removing finitely many paraproducts.
- Those paraproducts are uniformly bounded from $L^p \times L^q$ to L^r for all $p, q > 1$ and $r > 1/2$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$.
- Locally there is an appropriate upper bound that grows slowly enough, in contract to the decay estimate. This is one of the main difficulties in the uniform estimate. It can be achieved by a Whitney type decomposition.

Main Ingredients in the proof for the uniform estimates

- After removing finitely many intervals, the polynomial P is a monomial plus a tiny perturbation. The perturbation is handled via a quantitative version of inverse function theorem.
- Using σ-uniformity method, locally there is a decay estimate from $L^2 \times L^2$ to L^1, after removing finitely many paraproducts.
- Those paraproducts are uniformly bounded from $L^p \times L^q$ to L^r for all $p, q > 1$ and $r > 1/2$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$.
- Locally there is an appropriate upper bound that grows slowly enough, in contract to the decay estimate. This is one of the main difficulties in the uniform estimate. It can be achieved by a Whitney type decomposition.

The constraint of r

- In the uniform estimates, the range of $r > \frac{d-1}{d}$, $p_1, p_2 > 1$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$ is the best range.
- Let Γ be a polynomial curve $(t, P(t))$. It is natural to ask that, for a given polynomial P, what is the lower bound of r such that H_Γ and M_Γ are bounded from $L^{p_1} \times L^{p_2}$ to L^r.

Theorem (L.-Xiao, '13)

Let P be a polynomial without a linear term. The following are equivalent
i) All the roots of $P'(t) - 1 = 0$ have order at most $k - 1$.
ii) There is a constant C_P such that, for sufficiently small $\varepsilon > 0$, the following level set estimate holds:

$$\left|\{t : |P'(t) - 1| < \varepsilon\}\right| \leq C_P \varepsilon^{\frac{1}{k-1}}, \tag{6}$$

iii) H_Γ and M_Γ map from $L^{p_1} \times L^{p_2}$ to L^r for all $r > \frac{k-1}{k}$, $p_1, p_2 > 1$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$.

The constraint of r

- In the uniform estimates, the range of $r > \frac{d-1}{d}$, $p_1, p_2 > 1$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$ is the best range.
- Let Γ be a polynomial curve $(t, P(t))$. It is natural to ask that, for a given polynomial P, what is the lower bound of r such that H_Γ and M_Γ are bounded from $L^{p_1} \times L^{p_2}$ to L^r.

Theorem (L.-Xiao, '13)

Let P be a polynomial without a linear term. The following are equivalent
i) All the roots of $P'(t) - 1 = 0$ have order at most $k-1$.
ii) There is a constant C_P such that, for sufficiently small $\varepsilon > 0$, the following level set estimate holds:

$$\left|\{t : |P'(t) - 1| < \varepsilon\}\right| \leq C_P \varepsilon^{\frac{1}{k-1}}, \tag{6}$$

iii) H_Γ and M_Γ map from $L^{p_1} \times L^{p_2}$ to L^r for all $r > \frac{k-1}{k}$, $p_1, p_2 > 1$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$.

The constraint of r

- In the uniform estimates, the range of $r > \frac{d-1}{d}$, $p_1, p_2 > 1$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$ is the best range.
- Let Γ be a polynomial curve $(t, P(t))$. It is natural to ask that, for a given polynomial P, what is the lower bound of r such that H_Γ and M_Γ are bounded from $L^{p_1} \times L^{p_2}$ to L^r.

Theorem (L.-Xiao, '13)

Let P be a polynomial without a linear term. The following are equivalent
i) All the roots of $P'(t) - 1 = 0$ have order at most $k-1$.
ii) There is a constant C_P such that, for sufficiently small $\varepsilon > 0$, the following level set estimate holds:

$$\left|\{t : |P'(t) - 1| < \varepsilon\}\right| \leq C_P \varepsilon^{\frac{1}{k-1}}, \tag{6}$$

iii) H_Γ and M_Γ map from $L^{p_1} \times L^{p_2}$ to L^r for all $r > \frac{k-1}{k}$, $p_1, p_2 > 1$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$.

The constraint of r

- In the uniform estimates, the range of $r > \frac{d-1}{d}$, $p_1, p_2 > 1$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$ is the best range.
- Let Γ be a polynomial curve $(t, P(t))$. It is natural to ask that, for a given polynomial P, what is the lower bound of r such that H_Γ and M_Γ are bounded from $L^{p_1} \times L^{p_2}$ to L^r.

Theorem (L.-Xiao, '13)

Let P be a polynomial without a linear term. The following are equivalent

i) All the roots of $P'(t) - 1 = 0$ have order at most $k - 1$.

ii) There is a constant C_P such that, for sufficiently small $\varepsilon > 0$, the following level set estimate holds:

$$\left|\{t : |P'(t) - 1| < \varepsilon\}\right| \leq C_P \varepsilon^{\frac{1}{k-1}}, \tag{6}$$

iii) H_Γ and M_Γ map from $L^{p_1} \times L^{p_2}$ to L^r for all $r > \frac{k-1}{k}$, $p_1, p_2 > 1$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$.

The constraint of r

- In the uniform estimates, the range of $r > \frac{d-1}{d}$, $p_1, p_2 > 1$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$ is the best range.
- Let Γ be a polynomial curve $(t, P(t))$. It is natural to ask that, for a given polynomial P, what is the lower bound of r such that H_Γ and M_Γ are bounded from $L^{p_1} \times L^{p_2}$ to L^r.

Theorem (L.-Xiao, '13)

Let P be a polynomial without a linear term. The following are equivalent
i) All the roots of $P'(t) - 1 = 0$ have order at most $k-1$.
ii) There is a constant C_P such that, for sufficiently small $\varepsilon > 0$, the following level set estimate holds:

$$\left|\{t : |P'(t) - 1| < \varepsilon\}\right| \leq C_P \varepsilon^{\frac{1}{k-1}}, \tag{6}$$

iii) H_Γ and M_Γ map from $L^{p_1} \times L^{p_2}$ to L^r for all $r > \frac{k-1}{k}$, $p_1, p_2 > 1$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$.

The constraint of r

- In the uniform estimates, the range of $r > \frac{d-1}{d}$, $p_1, p_2 > 1$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$ is the best range.
- Let Γ be a polynomial curve $(t, P(t))$. It is natural to ask that, for a given polynomial P, what is the lower bound of r such that H_Γ and M_Γ are bounded from $L^{p_1} \times L^{p_2}$ to L^r.

Theorem (L.-Xiao, '13)

Let P be a polynomial without a linear term. The following are equivalent
i) All the roots of $P'(t) - 1 = 0$ have order at most $k-1$.
ii) There is a constant C_P such that, for sufficiently small $\varepsilon > 0$, the following level set estimate holds:

$$\left|\{t : |P'(t) - 1| < \varepsilon\}\right| \leq C_P \varepsilon^{\frac{1}{k-1}}, \tag{6}$$

iii) H_Γ and M_Γ map from $L^{p_1} \times L^{p_2}$ to L^r for all $r > \frac{k-1}{k}$, $p_1, p_2 > 1$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$.

Trilnear Hilbert transform along curves

Let $d \geq 2$. Consider the trilinear Hilbert transform along curve $\Gamma = (t, -t, t^d)$, given by

$$H_d(f_1, f_2, f_3)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x+t) f_3(x-t^d) \frac{dt}{t}\,. \tag{7}$$

Question 1. Is H_d bounded from $L^2 \times L^2 \times L^2$ to $L^{\frac{2}{3}}$?
This question can be reduced to the boundedness of the following trilinear operator T, defined by

$$T(f_1, f_2, f_3)(x) := \sum_{k \geq 0} H_k(f_1, f_2)(x) f_{3,k}(x) \tag{8}$$

where $f_{3,k}$ is a Fourier (smooth) restriction of f_3 to $[0, 2^{dk}]$, and H_k is given by

$$H_k(f_1, f_2)(x) = \iint \widehat{f_1}(\xi_1) \widehat{f_2}(\eta) e^{2\pi i(\xi+\eta)x} \phi\Big(\frac{\xi - \eta}{2^k}\Big)\, d\xi\, d\eta\,. \tag{9}$$

Here ϕ is a smooth cut-off away from the origin.

Trilnear Hilbert transform along curves

Let $d \geq 2$. Consider the trilinear Hilbert transform along curve $\Gamma = (t, -t, t^d)$, given by

$$H_d(f_1, f_2, f_3)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x+t) f_3(x-t^d) \frac{dt}{t}\,. \tag{7}$$

Question 1. Is H_d bounded from $L^2 \times L^2 \times L^2$ to $L^{\frac{2}{3}}$?

This question can be reduced to the boundedness of the following trilinear operator T, defined by

$$T(f_1, f_2, f_3)(x) := \sum_{k \geq 0} H_k(f_1, f_2)(x) f_{3,k}(x) \tag{8}$$

where $f_{3,k}$ is a Fourier (smooth) restriction of f_3 to $[0, 2^{dk}]$, and H_k is given by

$$H_k(f_1, f_2)(x) = \iint \widehat{f_1}(\xi_1) \widehat{f_2}(\eta) e^{2\pi i(\xi+\eta)x} \phi\left(\frac{\xi - \eta}{2^k}\right) d\xi d\eta\,. \tag{9}$$

Here ϕ is a smooth cut-off away from the origin.

Trilnear Hilbert transform along curves

Let $d \geq 2$. Consider the trilinear Hilbert transform along curve $\Gamma = (t, -t, t^d)$, given by

$$H_d(f_1, f_2, f_3)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x+t) f_3(x-t^d) \frac{dt}{t}\,. \tag{7}$$

Question 1. Is H_d bounded from $L^2 \times L^2 \times L^2$ to $L^{\frac{2}{3}}$?

This question can be reduced to the boundedness of the following trilinear operator T, defined by

$$T(f_1, f_2, f_3)(x) := \sum_{k \geq 0} H_k(f_1, f_2)(x) f_{3,k}(x) \tag{8}$$

where $f_{3,k}$ is a Fourier (smooth) restriction of f_3 to $[0, 2^{dk}]$, and H_k is given by

$$H_k(f_1, f_2)(x) = \iint \widehat{f_1}(\xi_1) \widehat{f_2}(\eta) e^{2\pi i(\xi+\eta)x} \phi\left(\frac{\xi - \eta}{2^k}\right) d\xi d\eta\,. \tag{9}$$

Here ϕ is a smooth cut-off away from the origin.

Let $d \geq 2$. Consider the trilinear Hilbert transform along curve $\Gamma = (t, -t, t^d)$, given by

$$H_d(f_1, f_2, f_3)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x+t) f_3(x-t^d) \frac{dt}{t}\,. \tag{7}$$

Question 1. Is H_d bounded from $L^2 \times L^2 \times L^2$ to $L^{\frac{2}{3}}$?
This question can be reduced to the boundedness of the following trilinear operator T, defined by

$$T(f_1, f_2, f_3)(x) := \sum_{k \geq 0} H_k(f_1, f_2)(x) f_{3,k}(x) \tag{8}$$

where $f_{3,k}$ is a Fourier (smooth) restriction of f_3 to $[0, 2^{dk}]$, and H_k is given by

$$H_k(f_1, f_2)(x) = \iint \widehat{f_1}(\xi_1) \widehat{f_2}(\eta) e^{2\pi i(\xi+\eta)x} \phi\left(\frac{\xi - \eta}{2^k}\right) d\xi d\eta\,. \tag{9}$$

Here ϕ is a smooth cut-off away from the origin.

Discrete bilinear operators

The discrete bilinear Hilbert transform along $(m, P(m))$ is given by

$$T(f_1, f_2)(n) = \sum_{m \neq 0} \frac{1}{m} f_1(n-m) f_2(n - P(m)) , \tag{10}$$

where $m \in \mathbb{Z}$.

The discrete bilinear maximal operator along $(m, P(m))$ is given by

$$T^*(f_1, f_2)(n) = \sup_{M \in \mathbb{N}} \frac{1}{M} \Big| \sum_{m \in \mathbb{Z}} f_1(n-m) f_2(n - P(m)) \Big| . \tag{11}$$

Question 2. Is T (or T^*) maps boundedly from $L^2(\mathbb{Z}) \times L^2(\mathbb{Z})$ to $L^1(\mathbb{Z})$? Here $L^p(\mathbb{Z})$ is the L^p space associated to the counting measure.

T and T^* are discrete analogue of H_Γ and M_Γ, respectively, for $\Gamma = (t, P(t))$. Because there is no transference principle available, it is not clear the boundedness of H_Γ (or M_Γ) implies any boundedness of T (or T^*).

Discrete bilinear operators

The discrete bilinear Hilbert transform along $(m, P(m))$ is given by

$$T(f_1, f_2)(n) = \sum_{m \neq 0} \frac{1}{m} f_1(n-m) f_2(n-P(m)), \tag{10}$$

where $m \in \mathbb{Z}$.

The discrete bilinear maximal operator along $(m, P(m))$ is given by

$$T^*(f_1, f_2)(n) = \sup_{M \in \mathbb{N}} \frac{1}{M} \Big| \sum_{m \in \mathbb{Z}} f_1(n-m) f_2(n-P(m)) \Big|. \tag{11}$$

Question 2. Is T (or T^*) maps boundedly from $L^2(\mathbb{Z}) \times L^2(\mathbb{Z})$ to $L^1(\mathbb{Z})$? Here $L^p(\mathbb{Z})$ is the L^p space associated to the counting measure.

T and T^* are discrete analogue of H_Γ and M_Γ, respectively, for $\Gamma = (t, P(t))$. Because there is no transference principle available, it is not clear the boundedness of H_Γ (or M_Γ) implies any boundedness of T (or T^*).

Discrete bilinear operators

The discrete bilinear Hilbert transform along $(m, P(m))$ is given by

$$T(f_1, f_2)(n) = \sum_{m \neq 0} \frac{1}{m} f_1(n-m) f_2(n - P(m)), \tag{10}$$

where $m \in \mathbb{Z}$.

The discrete bilinear maximal operator along $(m, P(m))$ is given by

$$T^*(f_1, f_2)(n) = \sup_{M \in \mathbb{N}} \frac{1}{M} \Big| \sum_{m \in \mathbb{Z}} f_1(n-m) f_2(n - P(m)) \Big| . \tag{11}$$

Question 2. Is T (or T^*) maps boundedly from $L^2(\mathbb{Z}) \times L^2(\mathbb{Z})$ to $L^1(\mathbb{Z})$? Here $L^p(\mathbb{Z})$ is the L^p space associated to the counting measure.

T and T^* are discrete analogue of H_Γ and M_Γ, respectively, for $\Gamma = (t, P(t))$. Because there is no transference principle available, it is not clear the boundedness of H_Γ (or M_Γ) implies any boundedness of T (or T^*).

Discrete bilinear operators

The discrete bilinear Hilbert transform along $(m, P(m))$ is given by

$$T(f_1, f_2)(n) = \sum_{m \neq 0} \frac{1}{m} f_1(n-m) f_2(n-P(m))\,, \tag{10}$$

where $m \in \mathbb{Z}$.
The discrete bilinear maximal operator along $(m, P(m))$ is given by

$$T^*(f_1, f_2)(n) = \sup_{M \in \mathbb{N}} \frac{1}{M} \Big| \sum_{m \in \mathbb{Z}} f_1(n-m) f_2(n-P(m)) \Big|\,. \tag{11}$$

Question 2. Is T (or T^*) maps boundedly from $L^2(\mathbb{Z}) \times L^2(\mathbb{Z})$ to $L^1(\mathbb{Z})$? Here $L^p(\mathbb{Z})$ is the L^p space associated to the counting measure.
T and T^* are discrete analogue of H_Γ and M_Γ, respectively, for $\Gamma = (t, P(t))$. Because there is no transference principle available, it is not clear the boundedness of H_Γ (or M_Γ) implies any boundedness of T (or T^*).

Discrete bilinear operators

The discrete bilinear Hilbert transform along $(m, P(m))$ is given by

$$T(f_1, f_2)(n) = \sum_{m \neq 0} \frac{1}{m} f_1(n-m) f_2(n - P(m)), \tag{10}$$

where $m \in \mathbb{Z}$.
The discrete bilinear maximal operator along $(m, P(m))$ is given by

$$T^*(f_1, f_2)(n) = \sup_{M \in \mathbb{N}} \frac{1}{M} \Big| \sum_{m \in \mathbb{Z}} f_1(n-m) f_2(n - P(m)) \Big| . \tag{11}$$

Question 2. Is T (or T^*) maps boundedly from $L^2(\mathbb{Z}) \times L^2(\mathbb{Z})$ to $L^1(\mathbb{Z})$? Here $L^p(\mathbb{Z})$ is the L^p space associated to the counting measure.
T and T^* are discrete analogue of H_Γ and M_Γ, respectively, for $\Gamma = (t, P(t))$. Because there is no transference principle available, it is not clear the boundedness of H_Γ (or M_Γ) implies any boundedness of T (or T^*).

Ionescu and Wainger's Theorem

Define a discrete singular Radon transform operator

$$Tf(n_1, n_2) = \sum_{m \neq 0} \frac{1}{m} f(n_1 - m, n_2 - P(m)) \tag{12}$$

where $P(m)$ is a real polynomial.

Theorem (Ionescu-Wainger, '05)

The discrete singular Radon transform T can be extended to a bounded operator on $L^p(\mathbb{Z}^2)$ for any $1 < p < \infty$.

Define a discrete singular Radon transform operator

$$Tf(n_1, n_2) = \sum_{m \neq 0} \frac{1}{m} f(n_1 - m, n_2 - P(m)) \tag{12}$$

where $P(m)$ is a real polynomial.

Theorem (Ionescu-Wainger, '05)

The discrete singular Radon transform T can be extended to a bounded operator on $L^p(\mathbb{Z}^2)$ for any $1 < p < \infty$.

Define a discrete singular Radon transform operator

$$Tf(n_1, n_2) = \sum_{m \neq 0} \frac{1}{m} f(n_1 - m, n_2 - P(m)) \tag{12}$$

where $P(m)$ is a real polynomial.

Theorem (Ionescu-Wainger, '05)

The discrete singular Radon transform T can be extended to a bounded operator on $L^p(\mathbb{Z}^2)$ for any $1 < p < \infty$.

Lemma (Dirichlet Principle)

For any given $N \in \mathbb{N}$ and any $x \in (0,1)$, there exist $a, q \in \mathbb{N}$ such that

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{Nq} ,$$

$$1 \leq q \leq N, \quad a \in \mathcal{P}_q .$$

Here

$$\mathcal{P}_q = \{ y \in \mathbb{N} : 1 \leq y \leq q, (y,q) = 1 \} .$$

For $a \in \mathcal{P}_q$, let

$$J_{a/q} = \left(-\frac{1}{Nq} + \frac{a}{q}, \frac{1}{Nq} + \frac{a}{q} \right) .$$

If $q \leq N/100$, then $J_{a/q}$ is called a major arc. Otherwise a minor arc.

Major arcs and Minor arcs

Lemma (Dirichlet Principle)

For any given $N \in \mathbb{N}$ and any $x \in (0,1)$, there exist $a, q \in \mathbb{N}$ such that

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{Nq},$$

$$1 \leq q \leq N, \quad a \in \mathcal{P}_q.$$

Here

$$\mathcal{P}_q = \{y \in \mathbb{N} : 1 \leq y \leq q, (y,q) = 1\}.$$

For $a \in \mathcal{P}_q$, let

$$J_{a/q} = \left(-\frac{1}{Nq} + \frac{a}{q}, \frac{1}{Nq} + \frac{a}{q}\right).$$

If $q \leq N/100$, then $J_{a/q}$ is called a major arc. Otherwise a minor arc.

Major arcs and Minor arcs

Lemma (Dirichlet Principle)

For any given $N \in \mathbb{N}$ *and any* $x \in (0,1)$, *there exist* $a, q \in \mathbb{N}$ *such that*

$$\left| x - \frac{a}{q} \right| \le \frac{1}{Nq},$$

$$1 \le q \le N, \quad a \in \mathcal{P}_q.$$

Here

$$\mathcal{P}_q = \{y \in \mathbb{N} : 1 \le y \le q, (y,q) = 1\}.$$

For $a \in \mathcal{P}_q$, let

$$J_{a/q} = \left(-\frac{1}{Nq} + \frac{a}{q}, \frac{1}{Nq} + \frac{a}{q} \right).$$

If $q \le N/100$, then $J_{a/q}$ is called a major arc. Otherwise a minor arc.

Lemma (Dirichlet Principle)

For any given $N \in \mathbb{N}$ and any $x \in (0,1)$, there exist $a, q \in \mathbb{N}$ such that

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{Nq},$$

$$1 \leq q \leq N, \quad a \in \mathcal{P}_q .$$

Here

$$\mathcal{P}_q = \{y \in \mathbb{N} : 1 \leq y \leq q, (y,q) = 1\} .$$

For $a \in \mathcal{P}_q$, let

$$J_{a/q} = \left(-\frac{1}{Nq} + \frac{a}{q}, \frac{1}{Nq} + \frac{a}{q} \right) .$$

If $q \leq N/100$, then $J_{a/q}$ is called a major arc. Otherwise a minor arc.

Lemma (Dirichlet Principle)

For any given $N \in \mathbb{N}$ and any $x \in (0,1)$, there exist $a, q \in \mathbb{N}$ such that

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{Nq},$$

$$1 \leq q \leq N, \quad a \in \mathcal{P}_q.$$

Here

$$\mathcal{P}_q = \{ y \in \mathbb{N} : 1 \leq y \leq q, (y,q) = 1 \}.$$

For $a \in \mathcal{P}_q$, let

$$J_{a/q} = \left(-\frac{1}{Nq} + \frac{a}{q}, \frac{1}{Nq} + \frac{a}{q} \right).$$

If $q \leq N/100$, then $J_{a/q}$ is called a major arc. Otherwise a minor arc.

Major arcs and Minor arcs

Lemma (Dirichlet Principle)

For any given $N \in \mathbb{N}$ *and any* $x \in (0,1)$, *there exist* $a, q \in \mathbb{N}$ *such that*

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{Nq}\,,$$

$$1 \leq q \leq N, \quad a \in \mathcal{P}_q\,.$$

Here

$$\mathcal{P}_q = \{ y \in \mathbb{N} : 1 \leq y \leq q, (y,q) = 1 \}\,.$$

For $a \in \mathcal{P}_q$, let

$$J_{a/q} = \left(-\frac{1}{Nq} + \frac{a}{q}, \frac{1}{Nq} + \frac{a}{q} \right)\,.$$

If $q \leq N/100$, then $J_{a/q}$ is called a major arc. Otherwise a minor arc.

Major arcs and Minor arcs

Lemma (Dirichlet Principle)

For any given $N \in \mathbb{N}$ and any $x \in (0,1)$, there exist $a, q \in \mathbb{N}$ such that

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{Nq} \,,$$

$$1 \leq q \leq N, \quad a \in \mathcal{P}_q \,.$$

Here

$$\mathcal{P}_q = \{ y \in \mathbb{N} : 1 \leq y \leq q, (y,q) = 1 \} \,.$$

For $a \in \mathcal{P}_q$, let

$$J_{a/q} = \left(-\frac{1}{Nq} + \frac{a}{q}, \frac{1}{Nq} + \frac{a}{q} \right) \,.$$

If $q \leq N/100$, then $J_{a/q}$ is called a major arc. Otherwise a minor arc.

Major arcs and Minor arcs

Lemma (Dirichlet Principle)

For any given $N \in \mathbb{N}$ and any $x \in (0,1)$, there exist $a, q \in \mathbb{N}$ such that

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{Nq},$$

$$1 \leq q \leq N, \quad a \in \mathcal{P}_q .$$

Here

$$\mathcal{P}_q = \{ y \in \mathbb{N} : 1 \leq y \leq q, (y,q) = 1 \} .$$

For $a \in \mathcal{P}_q$, let

$$J_{a/q} = \left(-\frac{1}{Nq} + \frac{a}{q}, \frac{1}{Nq} + \frac{a}{q} \right) .$$

If $q \leq N/100$, then $J_{a/q}$ is called a major arc. Otherwise a minor arc.

Major arcs

The $L^p(\mathbb{Z}^2)$ boundedness of the discrete singular Radon transform T is equivalent to the $L^p(\mathbb{R}^2)$ boundedness of $\tilde{T}$, given by

$$\tilde{T}f(x_1, x_2) = \sum_{m \neq 0} \frac{1}{m} f(x_1 - m, x_2 - P(m)) . \tag{13}$$

$$\widehat{\tilde{T}f}(\xi, \eta) = \sum_{j=0}^{\infty} \sum_{m} \rho\left(\frac{m}{2^j}\right) e^{-im\xi} e^{-iP(m)\eta} \widehat{f}(\xi, \eta) . \tag{14}$$

Definition

For any given positive integers a_1, a_2, q with $(a_1, a_2, q) = 1$ and $q \in [1, 2^{j/100}]$, define the major arc by

$$J_j(a_1/q, a_2/q) = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_k - a_k/q| \leq 2^{-(k-1/2)j}, k = 1, 2\} .$$

Major arcs

The $L^p(\mathbb{Z}^2)$ boundedness of the discrete singular Radon transform T is equivalent to the $L^p(\mathbb{R}^2)$ boundedness of $\tilde{T}$, given by

$$\tilde{T}f(x_1, x_2) = \sum_{m \neq 0} \frac{1}{m} f(x_1 - m, x_2 - P(m)) \,. \tag{13}$$

$$\widehat{\tilde{T}f}(\xi, \eta) = \sum_{j=0}^{\infty} \sum_{m} \rho\Big(\frac{m}{2^j}\Big) e^{-im\xi} e^{-iP(m)\eta} \widehat{f}(\xi, \eta) \,. \tag{14}$$

Definition

For any given positive integers a_1, a_2, q with $(a_1, a_2, q) = 1$ and $q \in [1, 2^{j/100}]$, define the major arc by

$$J_j(a_1/q, a_2/q) = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_k - a_k/q| \leq 2^{-(k-1/2)j}, k = 1, 2\} \,.$$

Major arcs

The $L^p(\mathbb{Z}^2)$ boundedness of the discrete singular Radon transform T is equivalent to the $L^p(\mathbb{R}^2)$ boundedness of $\tilde{T}$, given by

$$\tilde{T}f(x_1, x_2) = \sum_{m \neq 0} \frac{1}{m} f(x_1 - m, x_2 - P(m)) . \tag{13}$$

$$\widehat{\tilde{T}f}(\xi, \eta) = \sum_{j=0}^{\infty} \sum_{m} \rho\left(\frac{m}{2^j}\right) e^{-im\xi} e^{-iP(m)\eta} \widehat{f}(\xi, \eta) . \tag{14}$$

Definition

For any given positive integers a_1, a_2, q with $(a_1, a_2, q) = 1$ and $q \in [1, 2^{j/100}]$, define the major arc by

$$J_j(a_1/q, a_2/q) = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_k - a_k/q| \leq 2^{-(k-1/2)j}, k = 1, 2\} .$$

Major arcs

The $L^p(\mathbb{Z}^2)$ boundedness of the discrete singular Radon transform T is equivalent to the $L^p(\mathbb{R}^2)$ boundedness of $\tilde{T}$, given by

$$\tilde{T}f(x_1, x_2) = \sum_{m \neq 0} \frac{1}{m} f(x_1 - m, x_2 - P(m)) . \tag{13}$$

$$\widehat{\tilde{T}f}(\xi, \eta) = \sum_{j=0}^{\infty} \sum_{m} \rho\left(\frac{m}{2^j}\right) e^{-im\xi} e^{-iP(m)\eta} \widehat{f}(\xi, \eta) . \tag{14}$$

Definition

For any given positive integers a_1, a_2, q with $(a_1, a_2, q) = 1$ and $q \in [1, 2^{j/100}]$, define the major arc by

$$J_j(a_1/q, a_2/q) = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_k - a_k/q| \leq 2^{-(k-1/2)j}, k = 1, 2\} .$$

Major arcs

The $L^p(\mathbb{Z}^2)$ boundedness of the discrete singular Radon transform T is equivalent to the $L^p(\mathbb{R}^2)$ boundedness of $\tilde{T}$, given by

$$\tilde{T}f(x_1, x_2) = \sum_{m \neq 0} \frac{1}{m} f(x_1 - m, x_2 - P(m)) . \tag{13}$$

$$\widehat{\tilde{T}f}(\xi, \eta) = \sum_{j=0}^{\infty} \sum_{m} \rho\left(\frac{m}{2^j}\right) e^{-im\xi} e^{-iP(m)\eta} \widehat{f}(\xi, \eta) . \tag{14}$$

Definition

For any given positive integers a_1, a_2, q with $(a_1, a_2, q) = 1$ and $q \in [1, 2^{j/100}]$, define the major arc by

$$J_j(a_1/q, a_2/q) = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_k - a_k/q| \leq 2^{-(k-1/2)j}, k = 1, 2\} .$$

Minor arcs make a small contribution in the linear operator

By the Dirichlet principle and the Weyl sum estimate, if (ξ, η) does not belong to any major arc, then

$$\sum_m \rho\left(\frac{m}{2^j}\right) e^{-im\xi} e^{-iP(m)\eta} = O(2^{-\delta j}) \tag{15}$$

for some positive δ. By Plancherel theorem, it follows that the contribution from minor arcs is negligible. Thus the main story in Ionescu-Wainger's proof is to establish the almost orthogonality for the major arcs. Each major arc here is a tiny neighborhood of $(a_1/q, a_2/q)$ for some $q \in [1, 2^{j/100}]$.

Minor arcs make a small contribution in the linear operator

By the Dirichlet principle and the Weyl sum estimate, if (ξ, η) does not belong to any major arc, then

$$\sum_m \rho\left(\frac{m}{2^j}\right) e^{-im\xi} e^{-iP(m)\eta} = O(2^{-\delta j}) \tag{15}$$

for some positive δ. By Plancherel theorem, it follows that the contribution from minor arcs is negligible. Thus the main story in Ionescu-Wainger's proof is to establish the almost orthogonality for the major arcs. Each major arc here is a tiny neighborhood of $(a_1/q, a_2/q)$ for some $q \in [1, 2^{j/100}]$.

Minor arcs make a small contribution in the linear operator

By the Dirichlet principle and the Weyl sum estimate, if (ξ, η) does not belong to any major arc, then

$$\sum_m \rho\Big(\frac{m}{2^j}\Big) e^{-im\xi} e^{-iP(m)\eta} = O(2^{-\delta j}) \tag{15}$$

for some positive δ. By Plancherel theorem, it follows that the contribution from minor arcs is negligible. Thus the main story in Ionescu-Wainger's proof is to establish the almost orthogonality for the major arcs. Each major arc here is a tiny neighborhood of $(a_1/q, a_2/q)$ for some $q \in [1, 2^{j/100}]$.

Minor arcs make a small contribution in the linear operator

By the Dirichlet principle and the Weyl sum estimate, if (ξ, η) does not belong to any major arc, then

$$\sum_m \rho\left(\frac{m}{2^j}\right) e^{-im\xi} e^{-iP(m)\eta} = O(2^{-\delta j}) \tag{15}$$

for some positive δ. By Plancherel theorem, it follows that the contribution from minor arcs is negligible. Thus the main story in Ionescu-Wainger's proof is to establish the almost orthogonality for the major arcs. Each major arc here is a tiny neighborhood of $(a_1/q, a_2/q)$ for some $q \in [1, 2^{j/100}]$.

Discrete bilinear Hilbert transform along (t, t^2)

The discrete bilinear Hilbert transform along (t, t^2) is

$$H_2(f,g)(x) = \sum_{m\neq 1} \frac{1}{m} f(x-m)g(x-m^2)\,. \tag{16}$$

It can be written as

$$H_2(f,g) = \iint \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\sigma(\xi,\eta)d\xi\eta\,, \tag{17}$$

where σ is given by

$$\sigma(\xi,\eta) = \sum_{j=0}^{\infty} \frac{1}{2^j} \sum_{m} \rho\Big(\frac{m}{2^j}\Big) e^{-2\pi i m\xi} e^{-2\pi i m^2\eta} \tag{18}$$

for a suitable smooth cut-off ρ in a small neighborhood of 1.

Discrete bilinear Hilbert transform along (t, t^2)

The discrete bilinear Hilbert transform along (t, t^2) is

$$H_2(f,g)(x) = \sum_{m\neq 1} \frac{1}{m} f(x-m)g(x-m^2)\,. \tag{16}$$

It can be written as

$$H_2(f,g) = \iint \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\sigma(\xi,\eta)d\xi\eta\,, \tag{17}$$

where σ is given by

$$\sigma(\xi,\eta) = \sum_{j=0}^{\infty} \frac{1}{2^j} \sum_m \rho\Big(\frac{m}{2^j}\Big) e^{-2\pi i m\xi} e^{-2\pi i m^2\eta} \tag{18}$$

for a suitable smooth cut-off ρ in a small neighborhood of 1.

Discrete bilinear Hilbert transform along (t, t^2)

The discrete bilinear Hilbert transform along (t, t^2) is

$$H_2(f,g)(x) = \sum_{m \neq 1} \frac{1}{m} f(x-m) g(x-m^2)\,. \tag{16}$$

It can be written as

$$H_2(f,g) = \iint \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)x} \sigma(\xi,\eta) d\xi\eta\,, \tag{17}$$

where σ is given by

$$\sigma(\xi,\eta) = \sum_{j=0}^{\infty} \frac{1}{2^j} \sum_m \rho\left(\frac{m}{2^j}\right) e^{-2\pi i m\xi} e^{-2\pi i m^2 \eta} \tag{18}$$

for a suitable smooth cut-off ρ in a small neighborhood of 1.

Discrete bilinear Hilbert transform along (t, t^2)

The discrete bilinear Hilbert transform along (t, t^2) is

$$H_2(f,g)(x) = \sum_{m \neq 1} \frac{1}{m} f(x-m) g(x-m^2) \,. \tag{16}$$

It can be written as

$$H_2(f,g) = \iint \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)x} \sigma(\xi,\eta) d\xi\eta \,, \tag{17}$$

where σ is given by

$$\sigma(\xi,\eta) = \sum_{j=0}^{\infty} \frac{1}{2^j} \sum_m \rho\Big(\frac{m}{2^j}\Big) e^{-2\pi i m\xi} e^{-2\pi i m^2\eta} \tag{18}$$

for a suitable smooth cut-off ρ in a small neighborhood of 1.

Discrete bilinear Hilbert transform along (t, t^2)

It seems that Hardy-Littlewood circle method can not be applied to the bilinear case, because the contribution from the minor arcs is significant. Let H_2^* denote the discrete bilinear maximal function defined by

$$H_2^*(f,g)(n) = \sup_{M \in \mathbb{N}} \frac{1}{M} \sum_{m=1}^{M} \left| f(n-m) g(n-m^2) \right| . \tag{19}$$

Using the exponential sum estimate, we have

Theorem (Hu-L., '13)

The discrete bilinear maximal function H_2^* is bounded from $L^2(\mathbb{Z}) \times L^2(\mathbb{Z})$ to $L^r(\mathbb{Z})$ if $r > 1$.

Discrete bilinear Hilbert transform along (t, t^2)

It seems that Hardy-Littlewood circle method can not be applied to the bilinear case, because the contribution from the minor arcs is significant.

Let H_2^* denote the discrete bilinear maximal function defined by

$$H_2^*(f,g)(n) = \sup_{M\in\mathbb{N}} \frac{1}{M} \sum_{m=1}^{M} \left| f(n-m)g(n-m^2) \right| . \tag{19}$$

Using the exponential sum estimate, we have

Theorem (Hu-L., '13)

The discrete bilinear maximal function H_2^* is bounded from $L^2(\mathbb{Z}) \times L^2(\mathbb{Z})$ to $L^r(\mathbb{Z})$ if $r > 1$.

Discrete bilinear Hilbert transform along (t, t^2)

It seems that Hardy-Littlewood circle method can not be applied to the bilinear case, because the contribution from the minor arcs is significant. Let H_2^* denote the discrete bilinear maximal function defined by

$$H_2^*(f, g)(n) = \sup_{M \in \mathbb{N}} \frac{1}{M} \sum_{m=1}^{M} \left| f(n-m) g(n-m^2) \right| . \tag{19}$$

Using the exponential sum estimate, we have

Theorem (Hu-L., '13)

The discrete bilinear maximal function H_2^* is bounded from $L^2(\mathbb{Z}) \times L^2(\mathbb{Z})$ to $L^r(\mathbb{Z})$ if $r > 1$.

Discrete bilinear Hilbert transform along (t, t^2)

It seems that Hardy-Littlewood circle method can not be applied to the bilinear case, because the contribution from the minor arcs is significant. Let H_2^* denote the discrete bilinear maximal function defined by

$$H_2^*(f,g)(n) = \sup_{M \in \mathbb{N}} \frac{1}{M} \sum_{m=1}^{M} \left| f(n-m)g(n-m^2) \right| . \tag{19}$$

Using the exponential sum estimate, we have

Theorem (Hu-L., '13)

The discrete bilinear maximal function H_2^* is bounded from $L^2(\mathbb{Z}) \times L^2(\mathbb{Z})$ to $L^r(\mathbb{Z})$ if $r > 1$.

Discrete bilinear Hilbert transform along (t, t^2)

It seems that Hardy-Littlewood circle method can not be applied to the bilinear case, because the contribution from the minor arcs is significant. Let H_2^* denote the discrete bilinear maximal function defined by

$$H_2^*(f,g)(n) = \sup_{M \in \mathbb{N}} \frac{1}{M} \sum_{m=1}^{M} \left| f(n-m)g(n-m^2) \right| . \tag{19}$$

Using the exponential sum estimate, we have

Theorem (Hu-L., '13)

The discrete bilinear maximal function H_2^* is bounded from $L^2(\mathbb{Z}) \times L^2(\mathbb{Z})$ to $L^r(\mathbb{Z})$ if $r > 1$.

Proposition (Dong-L., '16)

Let σ_j be defined by

$$\sigma_j(\xi,\eta) = \frac{1}{2^j}\sum_m \rho\Big(\frac{m}{2^j}\Big) e^{-2\pi i \xi m} e^{-2\pi i \eta m^2}\,. \tag{20}$$

T_0 is a bilinear operator given by

$$T_0(f,g)(x) = \iint \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)x} \sum_{j=0}^{\infty} \sigma_j(\xi,\eta)\psi_0\Big(\frac{\eta}{1/2^{j/2}}\Big)\, d\xi d\eta\,. \tag{21}$$

Here ψ_0 is a bump function near 0. Then T_0 is bounded from $L^2 \times L^2$ to L^1.

Question. Is the discrete bilinear Hilbert transform H_2 (or the discrete bilinear maximal operator H_2^*) bounded from $L^2 \times L^2$ to L^1?

Proposition (Dong-L., '16)

Let σ_j *be defined by*

$$\sigma_j(\xi,\eta) = \frac{1}{2^j}\sum_m \rho\Big(\frac{m}{2^j}\Big) e^{-2\pi i \xi m} e^{-2\pi i \eta m^2} . \tag{20}$$

T_0 *is a bilinear operator given by*

$$T_0(f,g)(x) = \iint \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)x} \sum_{j=0}^{\infty} \sigma_j(\xi,\eta)\psi_0\Big(\frac{\eta}{1/2^{j/2}}\Big) d\xi d\eta . \tag{21}$$

Here ψ_0 *is a bump function near* 0*. Then* T_0 *is bounded from* $L^2 \times L^2$ *to* L^1*.*

Question. Is the discrete bilinear Hilbert transform H_2 (or the discrete bilinear maximal operator H_2^*) bounded from $L^2 \times L^2$ to L^1?

Proposition (Dong-L., '16)

Let σ_j be defined by

$$\sigma_j(\xi,\eta) = \frac{1}{2^j}\sum_m \rho\Big(\frac{m}{2^j}\Big) e^{-2\pi i \xi m} e^{-2\pi i \eta m^2}\,. \tag{20}$$

T_0 is a bilinear operator given by

$$T_0(f,g)(x) = \iint \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)x} \sum_{j=0}^{\infty} \sigma_j(\xi,\eta)\psi_0\Big(\frac{\eta}{1/2^{j/2}}\Big)\, d\xi d\eta\,. \tag{21}$$

Here ψ_0 is a bump function near 0. Then T_0 is bounded from $L^2 \times L^2$ to L^1.

Question. Is the discrete bilinear Hilbert transform H_2 (or the discrete bilinear maximal operator H_2^*) bounded from $L^2 \times L^2$ to L^1?

Discrete Fourier restriction

Let $d \geq 3$ be a positive integer. Suppose $p > 2(d+1)$. Is it true that

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^d) \right|^2 \leq CN^{1-\frac{2(d+1)}{p}+\varepsilon} \|f\|_{p'}^2 \, ? \tag{22}$$

(22) is equivalent to, by duality,

$$\Big\| \sum_{n=1}^{N} a_n e^{2\pi i n x} e^{2\pi i t n^d} \Big\|_{L^p(\mathbb{T}^2)} \leq CN^{\frac{1}{2}-\frac{d+1}{p}+\varepsilon} \Big(\sum_n |a_n|^2 \Big)^{1/2} . \tag{23}$$

Discrete Fourier restriction

Let $d \geq 3$ be a positive integer. Suppose $p > 2(d+1)$. Is it true that

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^d) \right|^2 \leq C N^{1 - \frac{2(d+1)}{p} + \varepsilon} \| f \|_{p'}^2 \, ? \qquad (22)$$

(22) is equivalent to, by duality,

$$\Big\| \sum_{n=1}^{N} a_n e^{2\pi i n x} e^{2\pi i t n^d} \Big\|_{L^p(\mathbb{T}^2)} \leq C N^{\frac{1}{2} - \frac{d+1}{p} + \varepsilon} \Big(\sum_{n} |a_n|^2 \Big)^{1/2} . \qquad (23)$$

Discrete Fourier restriction

Let $d \geq 3$ be a positive integer. Suppose $p > 2(d+1)$. Is it true that

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^d) \right|^2 \leq C N^{1 - \frac{2(d+1)}{p} + \varepsilon} \|f\|_{p'}^2 \, ? \tag{22}$$

(22) is equivalent to, by duality,

$$\Big\| \sum_{n=1}^{N} a_n e^{2\pi i n x} e^{2\pi i t n^d} \Big\|_{L^p(\mathbb{T}^2)} \leq C N^{\frac{1}{2} - \frac{d+1}{p} + \varepsilon} \Big(\sum_{n} |a_n|^2 \Big)^{1/2} . \tag{23}$$

Discrete Fourier restriction

Let $d \geq 3$ be a positive integer. Suppose $p > 2(d+1)$. Is it true that

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^d) \right|^2 \leq C N^{1 - \frac{2(d+1)}{p} + \varepsilon} \|f\|_{p'}^2 \, ? \tag{22}$$

(22) is equivalent to, by duality,

$$\Big\| \sum_{n=1}^{N} a_n e^{2\pi i n x} e^{2\pi i t n^d} \Big\|_{L^p(\mathbb{T}^2)} \leq C N^{\frac{1}{2} - \frac{d+1}{p} + \varepsilon} \Big(\sum_{n} |a_n|^2 \Big)^{1/2} . \tag{23}$$

Let $d \geq 3$ be a positive integer. Suppose $p > 2(d+1)$. Is it true that

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^d) \right|^2 \leq CN^{1-\frac{2(d+1)}{p}+\varepsilon} \|f\|_{p'}^2 \, ? \tag{22}$$

(22) is equivalent to, by duality,

$$\Big\| \sum_{n=1}^{N} a_n e^{2\pi i n x} e^{2\pi i t n^d} \Big\|_{L^p(\mathbb{T}^2)} \leq CN^{\frac{1}{2}-\frac{d+1}{p}+\varepsilon} \Big(\sum_n |a_n|^2 \Big)^{1/2} . \tag{23}$$

Weighted restriction

For any integer $d \geq 3$, define the weighted restriction operator

$$\mathcal{R}_N f(x,t) = \int_0^1 f(\xi) e^{2\pi i (x\xi + t\xi^d)} \frac{1}{N^{d-1}} \sum_{k=1}^{N^{d-1}} e^{-2\pi i k N \xi} \, d\xi \,. \qquad (24)$$

We use $B(r)$ to denote a cube (or ball) in $\mathbb{R}^2$ with side length (or radius) $r > 0$.

$$\|\mathcal{R}_N f\|_{L^p(B(N^d))} \leq C N^{-(d-1)(\frac{1}{2}-\frac{1}{p})+\varepsilon} \|f\|_2 \,. \qquad (25)$$

(25) is equivalent to

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^d) \right|^2 \leq C N^{1-\frac{2(d+1)}{p}+\varepsilon} \|f\|_{p'}^2 \,.$$

Weighted restriction

For any integer $d \geq 3$, define the weighted restriction operator

$$\mathcal{R}_N f(x,t) = \int_0^1 f(\xi) e^{2\pi i (x\xi + t\xi^d)} \frac{1}{N^{d-1}} \sum_{k=1}^{N^{d-1}} e^{-2\pi i k N \xi} \, d\xi \,. \tag{24}$$

We use $B(r)$ to denote a cube (or ball) in $\mathbb{R}^2$ with side length (or radius) $r > 0$.

$$\|\mathcal{R}_N f\|_{L^p(B(N^d))} \leq C N^{-(d-1)(\frac{1}{2}-\frac{1}{p})+\varepsilon} \|f\|_2 \,. \tag{25}$$

(25) is equivalent to

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^d) \right|^2 \leq C N^{1-\frac{2(d+1)}{p}+\varepsilon} \|f\|_{p'}^2 \,.$$

Weighted restriction

For any integer $d \geq 3$, define the weighted restriction operator

$$\mathcal{R}_N f(x,t) = \int_0^1 f(\xi) e^{2\pi i (x\xi + t\xi^d)} \frac{1}{N^{d-1}} \sum_{k=1}^{N^{d-1}} e^{-2\pi i k N \xi} \, d\xi \, . \tag{24}$$

We use $B(r)$ to denote a cube (or ball) in $\mathbb{R}^2$ with side length (or radius) $r > 0$.

$$\|\mathcal{R}_N f\|_{L^p(B(N^d))} \leq C N^{-(d-1)(\frac{1}{2}-\frac{1}{p})+\varepsilon} \|f\|_2 \, . \tag{25}$$

(25) is equivalent to

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^d) \right|^2 \leq C N^{1-\frac{2(d+1)}{p}+\varepsilon} \|f\|_{p'}^2 \, .$$

Weighted restriction

For any integer $d \geq 3$, define the weighted restriction operator

$$\mathcal{R}_N f(x,t) = \int_0^1 f(\xi) e^{2\pi i (x\xi + t\xi^d)} \frac{1}{N^{d-1}} \sum_{k=1}^{N^{d-1}} e^{-2\pi i k N \xi} \, d\xi \,. \qquad (24)$$

We use $B(r)$ to denote a cube (or ball) in $\mathbb{R}^2$ with side length (or radius) $r > 0$.

$$\|\mathcal{R}_N f\|_{L^p(B(N^d))} \leq C N^{-(d-1)(\frac{1}{2}-\frac{1}{p})+\varepsilon} \|f\|_2 \,. \qquad (25)$$

(25) is equivalent to

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^d) \right|^2 \leq C N^{1-\frac{2(d+1)}{p}+\varepsilon} \|f\|_{p'}^2 \,.$$

Periodic Strichartz estimate

The discrete Fourier restriction estimate (23) is periodic Strichartz estimate associated to dispersive equations. For instance, when $d = 3$, (23) is the Strichartz estimate for the periodic KDV-equation:

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0$$

with the initial periodic data

$$u(x, 0) = \phi(x) .$$

Periodic Strichartz estimate

The discrete Fourier restriction estimate (23) is periodic Strichartz estimate associated to dispersive equations. For instance, when $d = 3$, (23) is the Strichartz estimate for the periodic KDV-equation:

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0$$

with the initial periodic data

$$u(x, 0) = \phi(x) .$$

Periodic Strichartz estimate

The discrete Fourier restriction estimate (23) is periodic Strichartz estimate associated to dispersive equations. For instance, when $d = 3$, (23) is the Strichartz estimate for the periodic KDV-equation:

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0$$

with the initial periodic data

$$u(x, 0) = \phi(x)\,.$$

Application

The Strichartz estimates (23) with $p = 6$ can be used to obtain the local well-posedness for the dispersive equations. For instance,

Theorem (Hu-L., '13)

The Cauchy problem of periodic gKdV is the generalized KdV equation

$$u_t + u_{xxx} + F(u)u_x = 0 \tag{26}$$

with the initial condition $u(x, 0) = \phi(x)$ for $x \in \mathbb{T}$, $t \in \mathbb{R}$. The Cauchy problem (26) is locally well-posed provided that F is a C^5 function and the initial data $\phi \in H^s$ for $s > 1/2$.

Application

The Strichartz estimates (23) with $p = 6$ can be used to obtain the local well-posedness for the dispersive equations. For instance,

Theorem (Hu-L., '13)

The Cauchy problem of periodic gKdV is the generalized KdV equation

$$u_t + u_{xxx} + F(u)u_x = 0 \tag{26}$$

with the initial condition $u(x,0) = \phi(x)$ for $x \in \mathbb{T}$, $t \in \mathbb{R}$. The Cauchy problem (26) is locally well-posed provided that F is a C^5 function and the initial data $\phi \in H^s$ for $s > 1/2$.

Application

The Strichartz estimates (23) with $p = 6$ can be used to obtain the local well-posedness for the dispersive equations. For instance,

Theorem (Hu-L., '13)

The Cauchy problem of periodic gKdV is the generalized KdV equation

$$u_t + u_{xxx} + F(u)u_x = 0 \tag{26}$$

with the initial condition $u(x, 0) = \phi(x)$ for $x \in \mathbb{T}$, $t \in \mathbb{R}$. The Cauchy problem (26) is locally well-posed provided that F is a C^5 function and the initial data $\phi \in H^s$ for $s > 1/2$.

Application

The Strichartz estimates (23) with $p = 6$ can be used to obtain the local well-posedness for the dispersive equations. For instance,

Theorem (Hu-L., '13)

The Cauchy problem of periodic gKdV is the generalized KdV equation

$$u_t + u_{xxx} + F(u)u_x = 0 \tag{26}$$

with the initial condition $u(x,0) = \phi(x)$ *for* $x \in \mathbb{T}$, $t \in \mathbb{R}$. *The Cauchy problem (26) is locally well-posed provided that* F *is a* C^5 *function and the initial data* $\phi \in H^s$ *for* $s > 1/2$.

Waring's problem

Besides the relation to the dispersive equations, the L^p estimates for the exponential sum (23) are also motivated by Waring's problem.
For positive integers N and r, let $W_r(N)$ be the number of solutions of the Diophantine equation

$$x_1^d + \cdots + x_r^d = N\,, \tag{27}$$

with positive $x_1, \cdots, x_r$. The size of $W_r(N)$ is the main concern in Waring's problem.
Let $A_{p,N}$ denote the best constant obeying

$$\Big\| \sum_{n=1}^{N} a_n e^{2\pi i n x} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T}^2)} \leq A_{p,N} \Big(\sum_{n=1}^{N} |a_n|^2 \Big)^{1/2} . \tag{28}$$

Waring's problem

Besides the relation to the dispersive equations, the L^p estimates for the exponential sum (23) are also motivated by Waring's problem.

For positive integers N and r, let $W_r(N)$ be the number of solutions of the Diophantine equation

$$x_1^d + \cdots + x_r^d = N\,, \tag{27}$$

with positive $x_1, \cdots, x_r$. The size of $W_r(N)$ is the main concern in Waring's problem.

Let $A_{p,N}$ denote the best constant obeying

$$\Big\|\sum_{n=1}^{N} a_n e^{2\pi i n x} e^{2\pi i n^d t}\Big\|_{L^p(\mathbb{T}^2)} \leq A_{p,N}\Big(\sum_{n=1}^{N}|a_n|^2\Big)^{1/2}. \tag{28}$$

Waring's problem

Besides the relation to the dispersive equations, the L^p estimates for the exponential sum (23) are also motivated by Waring's problem.
For positive integers N and r, let $W_r(N)$ be the number of solutions of the Diophantine equation

$$x_1^d + \cdots + x_r^d = N\,, \tag{27}$$

with positive $x_1, \cdots, x_r$. The size of $W_r(N)$ is the main concern in Waring's problem.
Let $A_{p,N}$ denote the best constant obeying

$$\Big\|\sum_{n=1}^{N} a_n e^{2\pi i n x} e^{2\pi i n^d t}\Big\|_{L^p(\mathbb{T}^2)} \leq A_{p,N}\Big(\sum_{n=1}^{N} |a_n|^2\Big)^{1/2}. \tag{28}$$

Besides the relation to the dispersive equations, the L^p estimates for the exponential sum (23) are also motivated by Waring's problem.
For positive integers N and r, let $W_r(N)$ be the number of solutions of the Diophantine equation

$$x_1^d + \cdots + x_r^d = N\,, \tag{27}$$

with positive $x_1, \cdots, x_r$. The size of $W_r(N)$ is the main concern in Waring's problem.
Let $A_{p,N}$ denote the best constant obeying

$$\Big\| \sum_{n=1}^{N} a_n e^{2\pi i n x} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T}^2)} \leq A_{p,N} \Big(\sum_{n=1}^{N} |a_n|^2 \Big)^{1/2}. \tag{28}$$

Waring's problem

The estimate $A_{2(d+1),N} \leq N^{\varepsilon}$ implies, for $p \geq 2(d+1)$,

$$\Big\| \sum_{n=0}^{N} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon}. \tag{29}$$

(29) provides a new result for the asymptotic formula of $W_r(N)$. Hardy and Littlewood established an asymptotic formula for r very large by the circle method. Vinogradov refined Hardy-Littlewood's circle method and provided an asymptotic formula for $r \geq Cd^2 \log d$.

Waring's problem

The estimate $A_{2(d+1),N} \leq N^{\varepsilon}$ implies, for $p \geq 2(d+1)$,

$$\Big\| \sum_{n=0}^{N} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon}. \tag{29}$$

(29) provides a new result for the asymptotic formula of $W_r(N)$. Hardy and Littlewood established an asymptotic formula for r very large by the circle method. Vinogradov refined Hardy-Littlewood's circle method and provided an asymptotic formula for $r \geq Cd^2 \log d$.

Waring's problem

The estimate $A_{2(d+1),N} \leq N^{\varepsilon}$ implies, for $p \geq 2(d+1)$,

$$\Big\| \sum_{n=0}^{N} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon} . \tag{29}$$

(29) provides a new result for the asymptotic formula of $W_r(N)$. Hardy and Littlewood established an asymptotic formula for r very large by the circle method. Vinogradov refined Hardy-Littlewood's circle method and provided an asymptotic formula for $r \geq Cd^2 \log d$.

Waring's problem

The estimate $A_{2(d+1),N} \leq N^{\varepsilon}$ implies, for $p \geq 2(d+1)$,

$$\Big\| \sum_{n=0}^{N} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon} . \qquad (29)$$

(29) provides a new result for the asymptotic formula of $W_r(N)$. Hardy and Littlewood established an asymptotic formula for r very large by the circle method. Vinogradov refined Hardy-Littlewood's circle method and provided an asymptotic formula for $r \geq Cd^2 \log d$.

Waring's problem

The estimate $A_{2(d+1),N} \leq N^{\varepsilon}$ implies, for $p \geq 2(d+1)$,

$$\Big\| \sum_{n=0}^{N} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon} . \tag{29}$$

(29) provides a new result for the asymptotic formula of $W_r(N)$. Hardy and Littlewood established an asymptotic formula for r very large by the circle method. Vinogradov refined Hardy-Littlewood's circle method and provided an asymptotic formula for $r \geq Cd^2 \log d$.

Suppose the following estimate is true

$$\Big\|\sum_{n=0}^{N} e^{2\pi i n^d t}\Big\|_{L^p(\mathbb{T})} \le C N^{1-\frac{d}{p}+\varepsilon}.$$

Then, for $r \ge 2d+3$, the number $W_r(N)$ of representations of N as the sum of r d-th powers of positive integers satisfies

$$W_r(N) = \frac{\left(\Gamma(1+\frac{1}{d})\right)^r}{\Gamma(r/d)} N^{\frac{r}{d}-1}\mathfrak{S}(N,r) + O(N^{\frac{r}{d}-1-\frac{1}{d^2}}) + O(N^{\frac{r}{d}-1-\frac{\rho(r-2(d+1))}{d}+\varepsilon}), \tag{30}$$

where $\rho = \left(3d(d-1)\log(12d^2)\right)^{-1}$ and $\mathfrak{S}(N,r)$ is the singular series defined by

$$\mathfrak{S}(N,r) = \sum_{q=1}^{\infty} \frac{1}{q^r} \sum_{a\in\mathcal{P}_q} \left(S(a/q)\right)^r e^{-2\pi i \frac{a}{q} N}. \tag{31}$$

Here $S(a/q) = \sum_{n=0}^{q-1} e^{2\pi i \frac{a}{q} n^d}$ and $\mathcal{P}_q$ is the collection of all integers in $\{1, \cdots, q\}$ that are relatively prime to q.

Suppose the following estimate is true

$$\Big\| \sum_{n=0}^{N} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon}.$$

Then, for $r \geq 2d+3$, the number $W_r(N)$ of representations of N as the sum of r d-th powers of positive integers satisfies

$$W_r(N) = \frac{\left(\Gamma(1+\frac{1}{d})\right)^r}{\Gamma(r/d)} N^{\frac{r}{d}-1}\mathfrak{S}(N,r) + O(N^{\frac{r}{d}-1-\frac{1}{d^2}}) + O(N^{\frac{r}{d}-1-\frac{\rho(r-2(d+1))}{d}+\varepsilon}), \tag{30}$$

where $\rho = \left(3d(d-1)\log(12d^2)\right)^{-1}$ and $\mathfrak{S}(N,r)$ is the singular series defined by

$$\mathfrak{S}(N,r) = \sum_{q=1}^{\infty} \frac{1}{q^r} \sum_{a\in\mathcal{P}_q} \left(S(a/q)\right)^r e^{-2\pi i \frac{a}{q} N}. \tag{31}$$

Here $S(a/q) = \sum_{n=0}^{q-1} e^{2\pi i \frac{a}{q} n^d}$ and $\mathcal{P}_q$ is the collection of all integers in $\{1, \cdots, q\}$ that are relatively prime to q.

Suppose the following estimate is true

$$\Big\| \sum_{n=0}^{N} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon} .$$

Then, for $r \geq 2d+3$, the number $W_r(N)$ of representations of N as the sum of r d-th powers of positive integers satisfies

$$W_r(N) = \frac{\left(\Gamma(1+\frac{1}{d})\right)^r}{\Gamma(r/d)} N^{\frac{r}{d}-1} \mathfrak{S}(N,r) + O(N^{\frac{r}{d}-1-\frac{1}{d^2}}) + O(N^{\frac{r}{d}-1-\frac{\rho(r-2(d+1))}{d}+\varepsilon}) , \tag{30}$$

where $\rho = \left(3d(d-1)\log(12d^2)\right)^{-1}$ and $\mathfrak{S}(N,r)$ is the singular series defined by

$$\mathfrak{S}(N,r) = \sum_{q=1}^{\infty} \frac{1}{q^r} \sum_{a \in \mathcal{P}_q} \left(S(a/q)\right)^r e^{-2\pi i \frac{a}{q} N} . \tag{31}$$

Here $S(a/q) = \sum_{n=0}^{q-1} e^{2\pi i \frac{a}{q} n^d}$ and $\mathcal{P}_q$ is the collection of all integers in $\{1, \cdots, q\}$ that are relatively prime to q.

Suppose the following estimate is true

$$\|\sum_{n=0}^{N} e^{2\pi i n^d t}\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon}.$$

Then, for $r \geq 2d+3$, the number $W_r(N)$ of representations of N as the sum of r d-th powers of positive integers satisfies

$$W_r(N) = \frac{\left(\Gamma(1+\frac{1}{d})\right)^r}{\Gamma(r/d)} N^{\frac{r}{d}-1}\mathfrak{S}(N,r) + O(N^{\frac{r}{d}-1-\frac{1}{d^2}}) + O(N^{\frac{r}{d}-1-\frac{\rho(r-2(d+1))}{d}+\varepsilon}), \tag{30}$$

where $\rho = \left(3d(d-1)\log(12d^2)\right)^{-1}$ and $\mathfrak{S}(N,r)$ is the singular series defined by

$$\mathfrak{S}(N,r) = \sum_{q=1}^{\infty} \frac{1}{q^r} \sum_{a\in\mathcal{P}_q} \left(S(a/q)\right)^r e^{-2\pi i \frac{a}{q} N}. \tag{31}$$

Here $S(a/q) = \sum_{n=0}^{q-1} e^{2\pi i \frac{a}{q} n^d}$ and $\mathcal{P}_q$ is the collection of all integers in $\{1, \cdots, q\}$ that are relatively prime to q.

Suppose the following estimate is true

$$\Big\| \sum_{n=0}^{N} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon}.$$

Then, for $r \geq 2d+3$, the number $W_r(N)$ of representations of N as the sum of r d-th powers of positive integers satisfies

$$W_r(N) = \frac{\left(\Gamma(1+\frac{1}{d})\right)^r}{\Gamma(r/d)} N^{\frac{r}{d}-1}\mathfrak{S}(N,r) + O(N^{\frac{r}{d}-1-\frac{1}{d^2}}) + O(N^{\frac{r}{d}-1-\frac{\rho(r-2(d+1))}{d}+\varepsilon}), \tag{30}$$

where $\rho = \left(3d(d-1)\log(12d^2)\right)^{-1}$ and $\mathfrak{S}(N,r)$ is the singular series defined by

$$\mathfrak{S}(N,r) = \sum_{q=1}^{\infty} \frac{1}{q^r} \sum_{a\in\mathcal{P}_q} \left(S(a/q)\right)^r e^{-2\pi i \frac{a}{q} N}. \tag{31}$$

Here $S(a/q) = \sum_{n=0}^{q-1} e^{2\pi i \frac{a}{q} n^d}$ and $\mathcal{P}_q$ is the collection of all integers in $\{1, \cdots, q\}$ that are relatively prime to q.

Suppose the following estimate is true

$$\Big\|\sum_{n=0}^{N} e^{2\pi i n^d t}\Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon}\,.$$

Then, for $r \geq 2d+3$, the number $W_r(N)$ of representations of N as the sum of r d-th powers of positive integers satisfies

$$W_r(N) = \frac{\left(\Gamma(1+\frac{1}{d})\right)^r}{\Gamma(r/d)} N^{\frac{r}{d}-1}\mathfrak{S}(N,r) + O(N^{\frac{r}{d}-1-\frac{1}{d^2}}) + O(N^{\frac{r}{d}-1-\frac{\rho(r-2(d+1))}{d}+\varepsilon})\,, \tag{30}$$

where $\rho = \big(3d(d-1)\log(12d^2)\big)^{-1}$ and $\mathfrak{S}(N,r)$ is the singular series defined by

$$\mathfrak{S}(N,r) = \sum_{q=1}^{\infty}\frac{1}{q^r}\sum_{a\in\mathcal{P}_q}\left(S(a/q)\right)^r e^{-2\pi i \frac{a}{q} N}\,. \tag{31}$$

Here $S(a/q) = \sum_{n=0}^{q-1} e^{2\pi i \frac{a}{q} n^d}$ and $\mathcal{P}_q$ is the collection of all integers in $\{1,\cdots,q\}$ that are relatively prime to q.

Suppose the following estimate is true

$$\Big\| \sum_{n=0}^{N} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon} .$$

Then, for $r \geq 2d+3$, the number $W_r(N)$ of representations of N as the sum of r d-th powers of positive integers satisfies

$$W_r(N) = \frac{\left(\Gamma(1+\frac{1}{d})\right)^r}{\Gamma(r/d)} N^{\frac{r}{d}-1} \mathfrak{S}(N,r) + O(N^{\frac{r}{d}-1-\frac{1}{d^2}}) + O(N^{\frac{r}{d}-1-\frac{\rho(r-2(d+1))}{d}+\varepsilon}) , \tag{30}$$

where $\rho = \left(3d(d-1)\log(12d^2)\right)^{-1}$ and $\mathfrak{S}(N,r)$ is the singular series defined by

$$\mathfrak{S}(N,r) = \sum_{q=1}^{\infty} \frac{1}{q^r} \sum_{a \in \mathcal{P}_q} \left(S(a/q)\right)^r e^{-2\pi i \frac{a}{q} N} . \tag{31}$$

Here $S(a/q) = \sum_{n=0}^{q-1} e^{2\pi i \frac{a}{q} n^d}$ and $\mathcal{P}_q$ is the collection of all integers in $\{1, \cdots, q\}$ that are relatively prime to q.

Asymptotic formula

It is well-known that for $r \geq 4d$, the singular series $\mathfrak{S}(N, r)$ satisfies $\mathfrak{S}(N, r) \geq C_1$ The exponential sum

$$\Big\|\sum_{n=0}^{N} e^{2\pi i n^d t}\Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon}$$

yields

$$W_r(N) \sim C_{r,d} N^{\frac{r}{d}-1} \tag{32}$$

for some positive number $C_{r,d}$ depending on r, d only. Consequently, $A_{2(d+1),N} \leq N^{\varepsilon}$ implies that, for $r \geq 4d$ and N sufficiently large, the number of solutions of (27) is at least $C_{d,r} N^{r/d-1}$ for some positive constant $C_{r,d}$ depending on d and r only.

It is well-known that for $r \geq 4d$, the singular series $\mathfrak{S}(N, r)$ satisfies $\mathfrak{S}(N, r) \geq C_1$ The exponential sum

$$\Big\| \sum_{n=0}^{N} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon}$$

yields

$$W_r(N) \sim C_{r,d} N^{\frac{r}{d}-1} \tag{32}$$

for some positive number $C_{r,d}$ depending on r, d only. Consequently, $A_{2(d+1),N} \leq N^{\varepsilon}$ implies that, for $r \geq 4d$ and N sufficiently large, the number of solutions of (27) is at least $C_{d,r} N^{r/d-1}$ for some positive constant $C_{r,d}$ depending on d and r only.

Asymptotic formula

It is well-known that for $r \geq 4d$, the singular series $\mathfrak{S}(N, r)$ satisfies $\mathfrak{S}(N, r) \geq C_1$ The exponential sum

$$\Big\| \sum_{n=0}^{N} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon}$$

yields

$$W_r(N) \sim C_{r,d} N^{\frac{r}{d}-1} \tag{32}$$

for some positive number $C_{r,d}$ depending on r, d only. Consequently, $A_{2(d+1),N} \leq N^{\varepsilon}$ implies that, for $r \geq 4d$ and N sufficiently large, the number of solutions of (27) is at least $C_{d,r} N^{r/d-1}$ for some positive constant $C_{r,d}$ depending on d and r only.

Asymptotic formula

It is well-known that for $r \geq 4d$, the singular series $\mathfrak{S}(N,r)$ satisfies $\mathfrak{S}(N,r) \geq C_1$ The exponential sum

$$\Big\| \sum_{n=0}^{N} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon}$$

yields

$$W_r(N) \sim C_{r,d} N^{\frac{r}{d}-1} \tag{32}$$

for some positive number $C_{r,d}$ depending on r, d only. Consequently, $A_{2(d+1),N} \leq N^{\varepsilon}$ implies that, for $r \geq 4d$ and N sufficiently large, the number of solutions of (27) is at least $C_{d,r} N^{r/d-1}$ for some positive constant $C_{r,d}$ depending on d and r only.

Asymptotic formula

It is well-known that for $r \geq 4d$, the singular series $\mathfrak{S}(N, r)$ satisfies $\mathfrak{S}(N, r) \geq C_1$ The exponential sum

$$\Big\| \sum_{n=0}^{N} e^{2\pi i n^d t} \Big\|_{L^p(\mathbb{T})} \leq C N^{1-\frac{d}{p}+\varepsilon}$$

yields

$$W_r(N) \sim C_{r,d} N^{\frac{r}{d}-1} \tag{32}$$

for some positive number $C_{r,d}$ depending on r, d only. Consequently, $A_{2(d+1),N} \leq N^{\varepsilon}$ implies that, for $r \geq 4d$ and N sufficiently large, the number of solutions of (27) is at least $C_{d,r} N^{r/d-1}$ for some positive constant $C_{r,d}$ depending on d and r only.

G(d)

Let $G(d)$ denote the least value of r with the property that for every sufficiently large positive integer N, N can be represented as

$$N = x_1^d + \cdots + x_r^d \tag{33}$$

for positive integers $x_1, \cdots, x_r$. Then

$$A_{2(d+1),N} \leq N^{\varepsilon} \Rightarrow G(d) \leq 4d\,. \tag{34}$$

G(d)

Let $G(d)$ denote the least value of r with the property that for every sufficiently large positive integer N, N can be represented as

$$N = x_1^d + \cdots + x_r^d \tag{33}$$

for positive integers $x_1, \cdots, x_r$. Then

$$A_{2(d+1),N} \leq N^{\varepsilon} \Rightarrow G(d) \leq 4d\,. \tag{34}$$

Discrete restriction associated with (n, n^3)

Theorem (Hu-L., '13)

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^3) \right|^2 \leq C N^{1-\frac{8}{p}+\varepsilon} \|f\|_{p'}^2 \tag{35}$$

for $p \geq 14$.

Here the following Weyl sum estimate should be utilized

Lemma (Weyl)

Suppose that $|t - a/q| \leq 1/q^2$, *and that* $(a, q) = 1$. *Then*

$$\left| \sum_{n=1}^{N} e^{2\pi i (tn^3 + \alpha n^2 + \beta n)} \right| \leq C_\varepsilon N^{\frac{1}{4}+\varepsilon} q^{\frac{1}{4}}$$

if $q \geq N^2$.

Discrete restriction associated with (n, n^3)

Theorem (Hu-L., '13)

$$\sum_{n=1}^{N} \left|\widehat{f}(n, n^3)\right|^2 \leq CN^{1-\frac{8}{p}+\varepsilon}\|f\|_{p'}^2, \tag{35}$$

for $p \geq 14$.

Here the following Weyl sum estimate should be utilized

Lemma (Weyl)

Suppose that $|t - a/q| \leq 1/q^2$, *and that* $(a, q) = 1$. *Then*

$$\left|\sum_{n=1}^{N} e^{2\pi i(tn^3+\alpha n^2+\beta n)}\right| \leq C_\varepsilon N^{\frac{1}{4}+\varepsilon} q^{\frac{1}{4}}$$

if $q \geq N^2$.

Discrete restriction associated with (n, n^3)

Theorem (Hu-L., '13)

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^3) \right|^2 \leq C N^{1-\frac{8}{p}+\varepsilon} \|f\|_{p'}^2, \tag{35}$$

for $p \geq 14$.

Here the following Weyl sum estimate should be utilized

Lemma (Weyl)

Suppose that $|t - a/q| \leq 1/q^2$, *and that* $(a, q) = 1$. *Then*

$$\left| \sum_{n=1}^{N} e^{2\pi i (tn^3 + \alpha n^2 + \beta n)} \right| \leq C_\varepsilon N^{\frac{1}{4}+\varepsilon} q^{\frac{1}{4}}$$

if $q \geq N^2$.

Discrete restriction associated with (n, n^3)

Theorem (Hu-L., '13)

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^3) \right|^2 \leq C N^{1-\frac{8}{p}+\varepsilon} \|f\|_{p'}^2, \tag{35}$$

for $p \geq 14$.

Here the following Weyl sum estimate should be utilized

Lemma (Weyl)

Suppose that $|t - a/q| \leq 1/q^2$, *and that* $(a, q) = 1$. *Then*

$$\left| \sum_{n=1}^{N} e^{2\pi i (tn^3 + \alpha n^2 + \beta n)} \right| \leq C_\varepsilon N^{\frac{1}{4}+\varepsilon} q^{\frac{1}{4}}$$

if $q \geq N^2$.

Discrete restriction associated with (n, n^3)

Theorem (Hu-L., '13)

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^3) \right|^2 \leq C N^{1-\frac{8}{p}+\varepsilon} \|f\|_{p'}^2, \tag{35}$$

for $p \geq 14$.

Here the following Weyl sum estimate should be utilized

Lemma (Weyl)

Suppose that $|t - a/q| \leq 1/q^2$, *and that* $(a, q) = 1$. *Then*

$$\left| \sum_{n=1}^{N} e^{2\pi i (tn^3 + \alpha n^2 + \beta n)} \right| \leq C_\varepsilon N^{\frac{1}{4}+\varepsilon} q^{\frac{1}{4}}$$

if $q \geq N^2$.

Discrete restriction associated with (n, n^3)

Theorem (Hu-L., '13)

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^3) \right|^2 \leq C N^{1-\frac{8}{p}+\varepsilon} \|f\|_{p'}^2, \qquad (35)$$

for $p \geq 14$.

Here the following Weyl sum estimate should be utilized

Lemma (Weyl)

Suppose that $|t - a/q| \leq 1/q^2$, *and that* $(a, q) = 1$. *Then*

$$\left| \sum_{n=1}^{N} e^{2\pi i (tn^3 + \alpha n^2 + \beta n)} \right| \leq C_\varepsilon N^{\frac{1}{4}+\varepsilon} q^{\frac{1}{4}}$$

if $q \geq N^2$.

Discrete restriction associated with (n, n^3)

Theorem (Hu-L., '13)

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^3) \right|^2 \leq CN^{1-\frac{8}{p}+\varepsilon} \|f\|_{p'}^2, \tag{35}$$

for $p \geq 14$.

Here the following Weyl sum estimate should be utilized

Lemma (Weyl)

Suppose that $|t - a/q| \leq 1/q^2$, *and that* $(a, q) = 1$. *Then*

$$\left| \sum_{n=1}^{N} e^{2\pi i (tn^3 + \alpha n^2 + \beta n)} \right| \leq C_\varepsilon N^{\frac{1}{4}+\varepsilon} q^{\frac{1}{4}}$$

if $q \geq N^2$.

Discrete restriction associated with (n, n^3)

Theorem (Hu-L., '13)

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^3) \right|^2 \leq C N^{1-\frac{8}{p}+\varepsilon} \|f\|_{p'}^2, \tag{35}$$

for $p \geq 14$.

Here the following Weyl sum estimate should be utilized

Lemma (Weyl)

Suppose that $|t - a/q| \leq 1/q^2$, *and that* $(a, q) = 1$. *Then*

$$\left| \sum_{n=1}^{N} e^{2\pi i (tn^3 + \alpha n^2 + \beta n)} \right| \leq C_\varepsilon N^{\frac{1}{4}+\varepsilon} q^{\frac{1}{4}}$$

if $q \geq N^2$.

Discrete restriction associated with (n, n^3)

Theorem (Hu-L., '13)

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^3) \right|^2 \leq C N^{1-\frac{8}{p}+\varepsilon} \|f\|_{p'}^2, \tag{35}$$

for $p \geq 14$.

Here the following Weyl sum estimate should be utilized

Lemma (Weyl)

Suppose that $|t - a/q| \leq 1/q^2$, *and that* $(a, q) = 1$. *Then*

$$\left| \sum_{n=1}^{N} e^{2\pi i (tn^3 + \alpha n^2 + \beta n)} \right| \leq C_\varepsilon N^{\frac{1}{4}+\varepsilon} q^{\frac{1}{4}}$$

if $q \geq N^2$.

Discrete restriction associated with (n, n^3)

Theorem (Hu-L., '13)

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^3) \right|^2 \leq C N^{1-\frac{8}{p}+\varepsilon} \|f\|_{p'}^2, \tag{35}$$

for $p \geq 14$.

Here the following Weyl sum estimate should be utilized

Lemma (Weyl)

Suppose that $|t - a/q| \leq 1/q^2$, *and that* $(a, q) = 1$. *Then*

$$\left| \sum_{n=1}^{N} e^{2\pi i (tn^3 + \alpha n^2 + \beta n)} \right| \leq C_\varepsilon N^{\frac{1}{4}+\varepsilon} q^{\frac{1}{4}}$$

if $q \geq N^2$.

Discrete restriction associated with (n, n^4)

Theorem (Hua)

$$\int_{\mathbb{T}\times\mathbb{T}} \left| \sum_{n=1}^{N} e^{2\pi i(tn^3+xn)} \right|^{10} dxdt \le CN^{6+\varepsilon}. \tag{36}$$

Theorem (Hu-L., 14)

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^4) \right|^2 \le CN^{1-\frac{10}{p}+\varepsilon} \|f\|_{p'}^2, \tag{37}$$

for $p \ge 48$.

Discrete restriction associated with (n, n^4)

Theorem (Hua)

$$\int_{\mathbb{T}\times\mathbb{T}} \left| \sum_{n=1}^{N} e^{2\pi i(tn^3+xn)} \right|^{10} dxdt \leq CN^{6+\varepsilon}. \quad (36)$$

Theorem (Hu-L., 14)

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^4) \right|^2 \leq CN^{1-\frac{10}{p}+\varepsilon} \|f\|_{p'}^2 \quad (37)$$

for $p \geq 48$.

Discrete restriction associated with (n, n^4)

Theorem (Hua)

$$\int_{\mathbb{T}\times\mathbb{T}} \left| \sum_{n=1}^{N} e^{2\pi i(tn^3+xn)} \right|^{10} dxdt \leq CN^{6+\varepsilon}\,. \tag{36}$$

Theorem (Hu-L., 14)

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^4) \right|^2 \leq CN^{1-\frac{10}{p}+\varepsilon} \|f\|_{p'}^2 \tag{37}$$

for $p \geq 48$.

Discrete restriction associated with (n, n^4)

Theorem (Hua)

$$\int_{\mathbb{T}\times\mathbb{T}} \left| \sum_{n=1}^{N} e^{2\pi i(tn^3+xn)} \right|^{10} dxdt \leq CN^{6+\varepsilon}. \tag{36}$$

Theorem (Hu-L., 14)

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^4) \right|^2 \leq CN^{1-\frac{10}{p}+\varepsilon} \|f\|_{p'}^2 \tag{37}$$

for $p \geq 48$.

Discrete restriction associated with (n, n^4)

Theorem (Hua)

$$\int_{\mathbb{T}\times\mathbb{T}} \left| \sum_{n=1}^{N} e^{2\pi i (tn^3 + xn)} \right|^{10} dxdt \le CN^{6+\varepsilon}\,. \tag{36}$$

Theorem (Hu-L., 14)

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^4) \right|^2 \le CN^{1-\frac{10}{p}+\varepsilon} \|f\|_{p'}^2 \tag{37}$$

for $p \ge 48$.

Discrete restriction associated with (n, n^d)

Theorem

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^d) \right|^2 \leq C N^{1-\frac{2(d+1)}{p}+\varepsilon} \|f\|_{p'}^2, \tag{38}$$

for $p \geq O(d^2)$.

Discrete restriction associated with (n, n^d)

Theorem

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^d) \right|^2 \leq CN^{1-\frac{2(d+1)}{p}+\varepsilon} \|f\|_{p'}^2 \tag{38}$$

for $p \geq O(d^2)$.

Discrete restriction associated with (n, n^d)

Theorem

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^d) \right|^2 \leq CN^{1-\frac{2(d+1)}{p}+\varepsilon} \|f\|_{p'}^2 \tag{38}$$

for $p \geq O(d^2)$.

Discrete restriction associated with (n, n^d)

Theorem

$$\sum_{n=1}^{N} \left| \widehat{f}(n, n^d) \right|^2 \leq CN^{1-\frac{2(d+1)}{p}+\varepsilon} \|f\|_{p'}^2 \tag{38}$$

for $p \geq O(d^2)$.

THANK YOU !!!