Behaviour of the Brascamp-Lieb Constant and Applications

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Brascamp-Lieb Inequalities

A very general family of multilinear inequalities.

Let m, n be positive integers. For each $j \in [1, m]$, let $n_j \in \{1, \ldots, n\}$ and take $L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ to be a surjective linear map. The associated Brascamp-Lieb inequality is

$$\int_{\mathbb{R}^n} \Big| \prod_{j=1}^{m} (f_j \circ L_j) \Big| \leq BL(\mathbf{L}, \mathbf{p}) \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^{n_j})}$$

Think Young's convolution inequality: $\sum p_j^{-1} = 2$

$$\int_{\mathbb{R}^{2n}} |f_1(x) f_2(x-y) f_3(y)| dxdy \leq C \prod_{j=1}^{3} \|f_j\|_{L^{p_j}(\mathbb{R}^n)}$$

(or Hölder's inequality, Loomis-Whitney, Geometric Brascamp-Lieb, ...)

Results

Theorem (Bennett, Bez, F., Lee, '15)

Suppose that $(\mathbf{L}^0, \mathbf{p})$ *is a Brascamp–Lieb datum for which* $BL(\mathbf{L}^0, \mathbf{p}) < \infty$*. Then there exists* $\delta > 0$ *and a constant* $C < \infty$ *such that*

$$BL(\mathbf{L}, \mathbf{p}) \leq C$$

whenever $\|\mathbf{L} - \mathbf{L}^0\| < \delta$*.*

This result already enough for our applications.

Theorem (Bennett, Bez, Cowling, F. '16)

$BL(\mathbf{L}, \mathbf{p})$ *is a continuous function of* $\mathbf{L}$*.*

Finiteness Characterization

Theorem (Bennett, Christ, Carbery, and Tao, '08)

Let $(\mathbf{L}, \mathbf{p})$ *be a Brascamp-Lieb datum. Then* $BL(\mathbf{L}, \mathbf{p})$ *is finite if and only if we have the scaling condition*

$$n = \sum_j p_j^{-1} n_j \tag{1}$$

and the dimension condition

$$\dim(V) \leq \sum_{j=1}^{m} p_j^{-1} \dim(L_j V) \text{ for all subspaces } V \subseteq \mathbb{R}^n. \tag{2}$$

A subspace V such that equality holds in (2) is called critical. If V is critical, then

$$BL(\mathbf{L}, \mathbf{p}) = BL(\mathbf{L}|_V, \mathbf{p}) BL(\mathbf{L}|_{V^\perp}, \mathbf{p})$$

Proof of continuity

Theorem (Lieb '90)

The Brascamp-Lieb constant is exhausted by Gaussians $\{exp(-\langle A_j x, x\rangle)\}$,

$$BL(\mathbf{L}, \mathbf{p})^2 = \sup_{\{A_j > 0\}} \left(\frac{\prod_{j=1}^m (\det A_j)^{p_j^{-1}}}{\det \left(\sum_{j=1}^m p_j^{-1} L_j^* A_j L_j \right)} \right),$$

Therefore, $\mathrm{BL}(\mathbf{L}, \mathbf{p})$ is lower semicontinuous.

But not necessarilly locally bounded,

$$F(x) = \sup_{y>0} \frac{x}{x^2 + y}$$

Proof of continuity in the rank-1 case

Suppose for each j, $L_j x = \langle v_j, x \rangle$.

In this case, Barthe has shown that

$$BL(\{v_i\}, \mathbf{p})^2 = \sup_{\{\lambda_i > 0\}} \frac{\prod_{i=1}^m \lambda_i^{1/p_i}}{\sum_{|I|=n} d_I p_I^{-1} \lambda_I} = F(\mathbf{d})$$

where $\lambda_I = \prod_{i \in I} \lambda_i$, $p_I = \prod_{i \in I} p_i$, and $d_I = \det(\{v_i\}_{i \in I})^2 \geq 0$.

Enough to show upper semicontinuity,

$$\limsup_{\mathbf{d} \to \mathbf{d}^0} F(\mathbf{d}^0) \leq F(\mathbf{d}).$$

Proof of continuity cont'd

Fix $\mathbf{d}^0$ and assume $|\mathbf{d} - \mathbf{d}^0| \leq \delta$,

$$
\begin{aligned}
F(\mathbf{d}) &\leq \sup_{\{\lambda_i > 0\}} \frac{\prod_{i=1}^m \lambda_i^{1/p_i}}{\sum_{I: d_I^0 \neq 0} d_I p_I^{-1} \lambda_I} \\
&\leq \sup_{\{\lambda_i > 0\}} \frac{\prod_{i=1}^m \lambda_i^{1/p_i}}{\sum_{I: d_I^0 \neq 0} (d_I^0 - \delta) p_I^{-1} \lambda_I} \\
&\leq \sup_{\{\lambda_i > 0\}} \frac{\prod_{i=1}^m \lambda_i^{1/p_i}}{\sum_{\mathcal{I}: d_I^0 \neq 0} d_I^0 p_I^{-1} \lambda_I} \left(1 - \delta \frac{\sum_{\mathcal{I}: d_I^0 \neq 0} \lambda_I p_I^{-1}}{\sum_{\mathcal{I}: d_I^0 \neq 0} d_I^0 \lambda_I p_I^{-1}}\right)^{-1} \\
&\leq F(\mathbf{d^0}) \left(1 - \frac{\delta}{D}\right)^{-1}
\end{aligned}
$$

where $D = min_{d_I^0 \neq 0} d_I^0$.

Extending to the general case

The essential ingrediant is an analog of Barthe's formula.

Lemma (Bennett, Bez, Cowling, F. '16)

Given a Brascamp-Lieb datum $(\mathbf{L}, \mathbf{p})$, *the Brascamp–Lieb constant* $BL(\mathbf{L}, \mathbf{p})$ *is given by*

$$BL(\mathbf{L}, \mathbf{p})^2 = \sup_{\mathbf{R},\lambda} \frac{\prod_{k=1}^{K} \lambda_k^{q_k}}{\sum_{\mathcal{I}} d_I q_I \lambda_I}$$

where λ *ranges over sequences* $\{\lambda_k\}_{k=1}^{K}$ *such that* $\lambda_k > 0$ *and* $\mathbf{R}$ *ranges over m-tuples* $\{R_i\}_{i=1}^{m}$ *where* $R_i \in \mathrm{SO(n_i)}$, *and* $d_I = d_I(\mathbf{L}, \mathbf{R})$ *is a continuous function such that* $d_I(\mathbf{L}, \mathbf{R}) \geq 0$.

Is $BL(\mathbf{L}, \mathbf{p})$ differentiable?

Known to be TRUE in special cases:
$BL(\mathbf{L}, \mathbf{p})$ is smooth when $\bigoplus Ker\ L_j = \mathbb{R}^n$ or, as shown by Valdimarson, when $(\mathbf{L}, \mathbf{p})$ is simple.

In general, FALSE. Consider,

$$\int_{\mathbb{R}^2} \prod_{j=1}^{4} f_j(\langle v_j, x\rangle) dxdy \leq \mathrm{BL}(\{v_i\}_{i=1}^4, \{2\}_{i=1}^4) \prod_{j=1}^{4} \|f_j\|_{L^2(\mathbb{R})}$$

Then $\mathrm{BL}(\{v_i\}_{i=1}^4, \{2\}_{i=1}^4)$ is given by

$$\left(\frac{2}{|\det(v_1 v_2)\det(v_3 v_4)| + |\det(v_1 v_3)\det(v_2 v_4)| + |\det(v_1 v_4)\det(v_2 v_3)|}\right)^{1/2}.$$

Applications

Very general related multilinear inequalities with an epsilon loss

- Nonlinear Brascamp-Lieb

$$\int_U \Big| \prod_{j=1}^m f_j \circ B_j \Big| \le C_\varepsilon \prod_{j=1}^m \|f_j\|_{L^{p_j}_\varepsilon(\mathbb{R}^{n_j})}.$$

 such inequalities have be applied to dispersive PDE e.g.

 I. Bejenaru, S. Herr, J. Holmer, D. Tataru, *On the 2d Zakharov system with L2 Schrodinger data*

- Multilinear Kayeya: there exists $\nu > 0$ (a tolerance on the angles of tubes) such that for every $\varepsilon > 0$,

$$\int_{[-1,1]^n} \prod_{j=1}^m \Big(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j} \Big)^{p_j} \le C_\varepsilon \delta^{n-\varepsilon} \prod_{j=1}^m (\#\mathbb{T}_j)^{p_j}$$

 our stability result is applied such a setting by Bourgain, Demeter, and Guth *... Vinogradov's mean value theorem for degrees higher than 3*

Basic idea of proofs: Induction on scales.

Thank you for your attention!