Existence and non-existence of extremizers for a k-plane transform inequality.

Alexis Drouot

May 18th 2016

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- Other L^p to L^q boundedness properties on Lebesgue spaces follow from interpolation theory between the L^1 estimate and the endpoint case.

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Theorem (Christ '11, Drouot '11, Flock '13)

In the **endpoint case** $p = \frac{d+1}{k+1}, q = d+1$ **the extremizers are all given by** $a\langle Lx\rangle^{-k-1}$**,** *where L is an affine map and a is a non-zero constant.*

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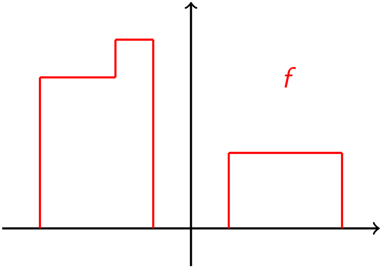

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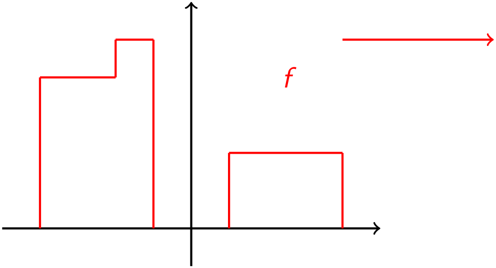

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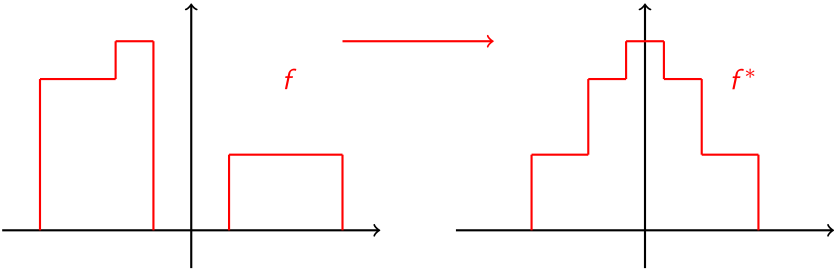

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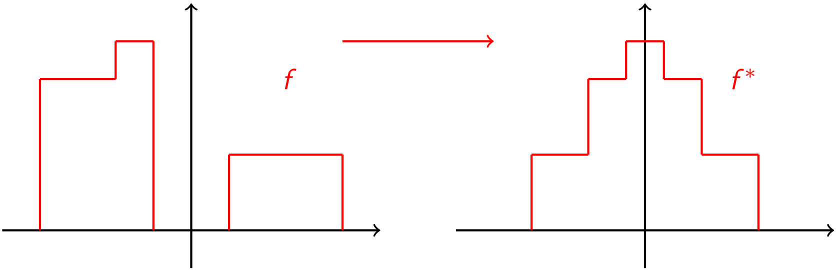

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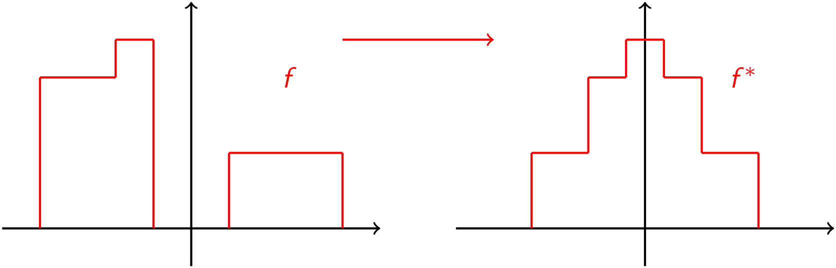

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Hence (if one puts aside the uniqueness question) we can restrict ourselves to **radial non-increasing extremizers.**

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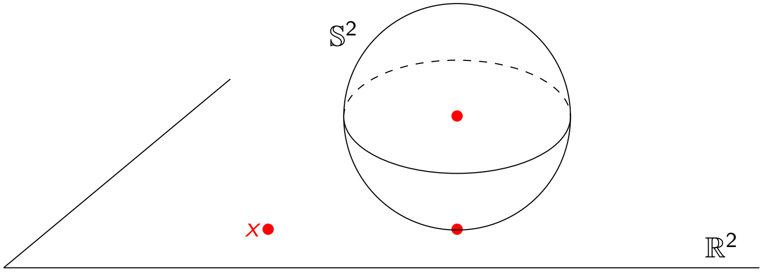

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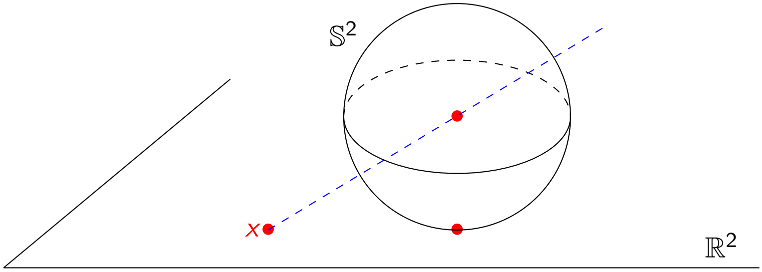

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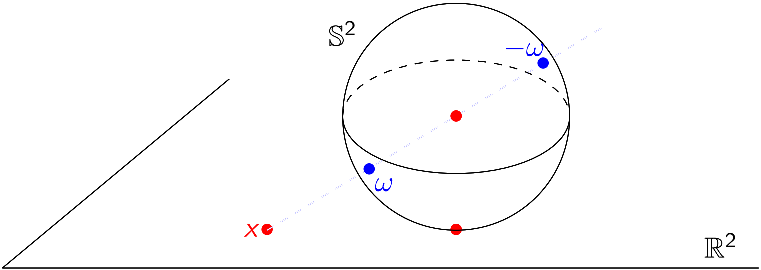

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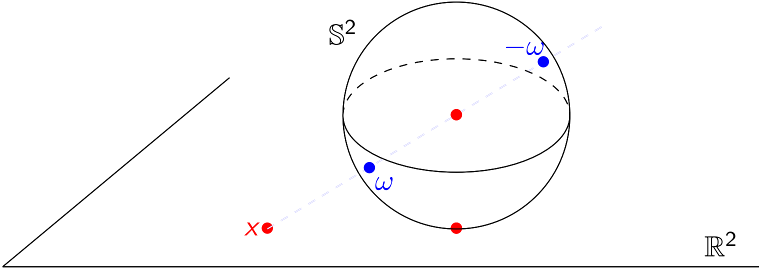

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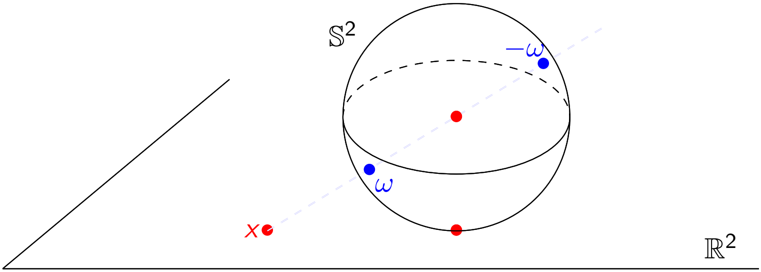

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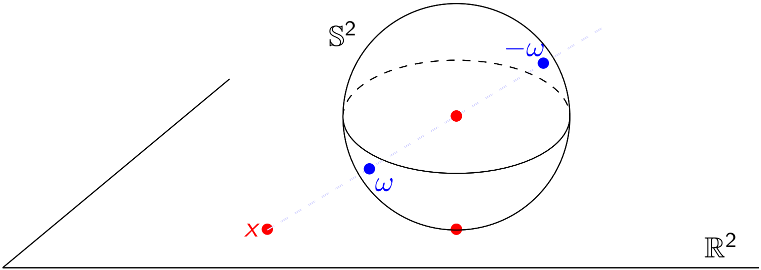

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The radial nonincreasing rearrangement on $\mathbb{R}^d$ **transfers to an asymmetric notion of rearrangement on** $\mathbb{S}^d$ through $f \mapsto F$.

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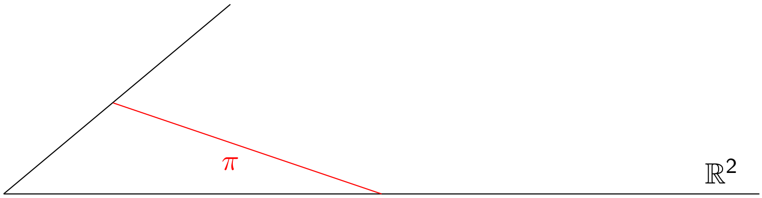

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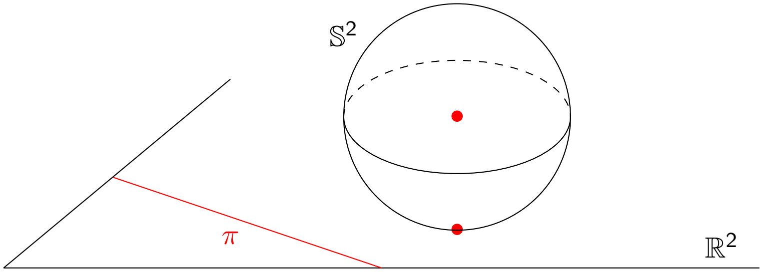

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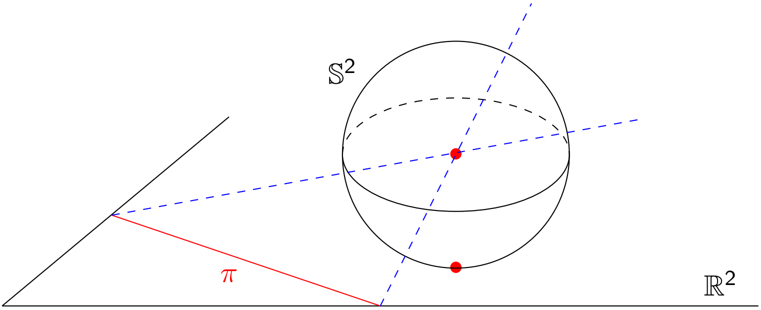

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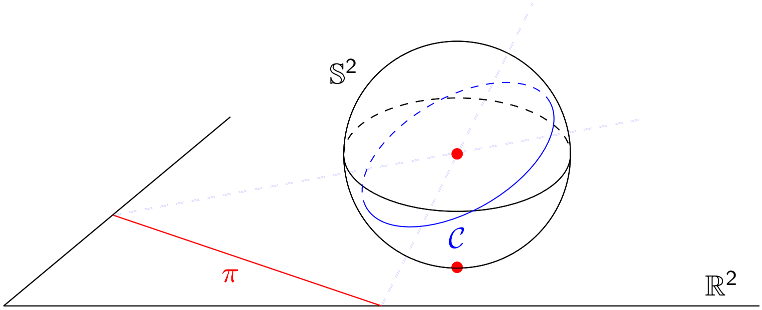

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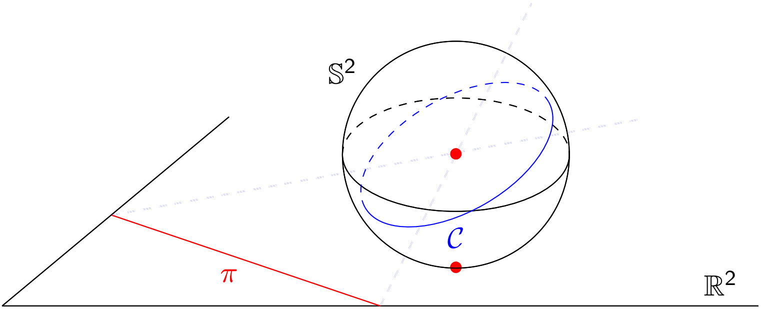

Correspondance $\pi \longleftrightarrow \mathcal{C}$, $g(\pi) \longleftrightarrow G(\mathcal{C})$. Let $\mathcal{R}_+$ be the k-circle transform (on $\mathbb{S}^d$) and recall the correspondance $f(x) \longleftrightarrow F(\omega)$.

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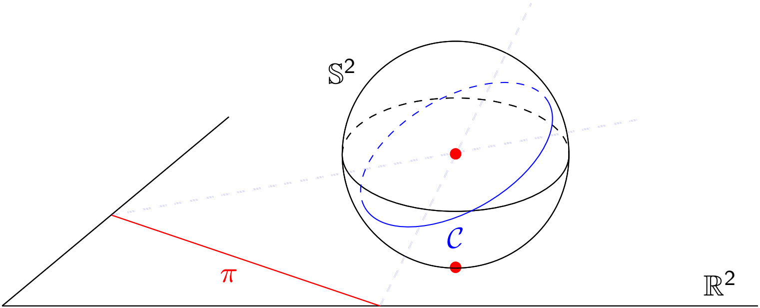

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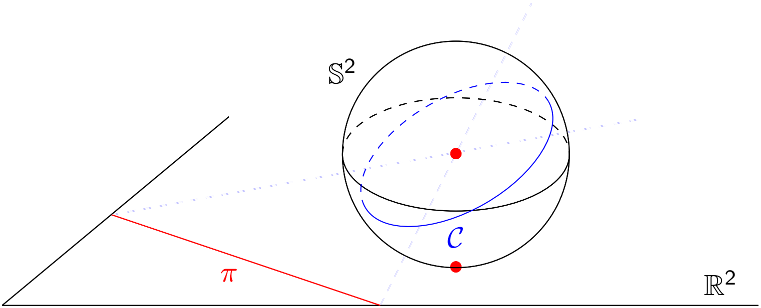

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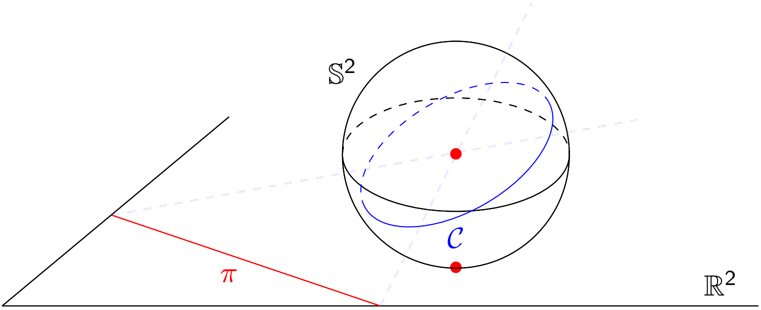

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We discover **new symmetries:** rotations about the e_x, e_y axis.

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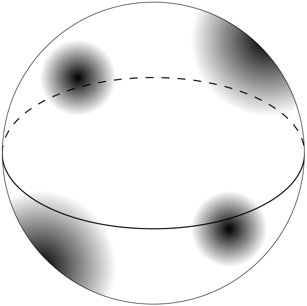

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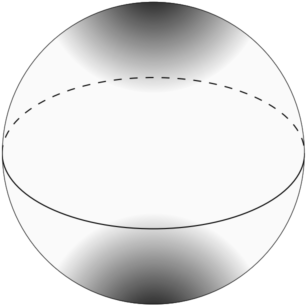

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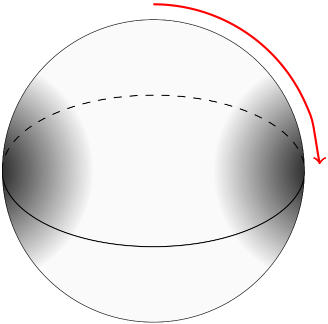

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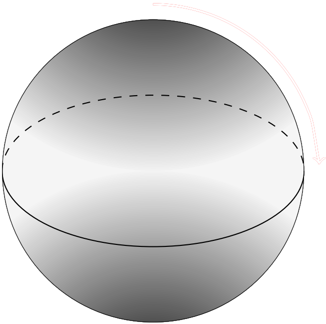

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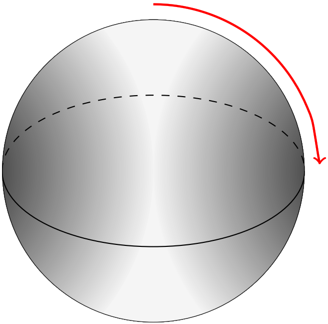

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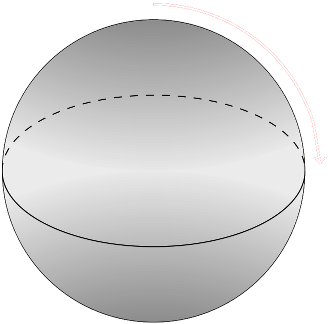

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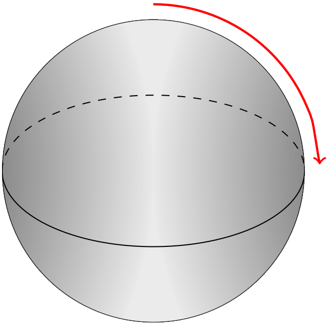

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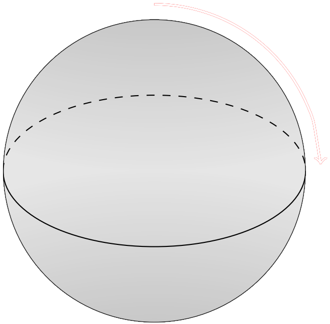

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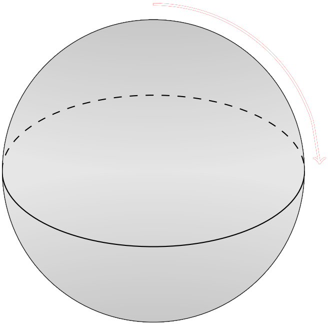

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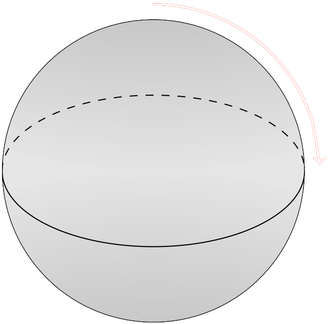

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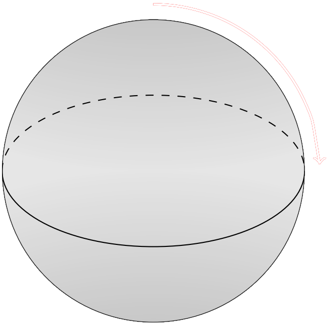

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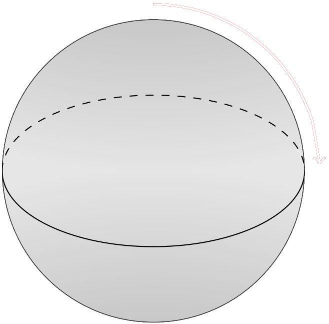

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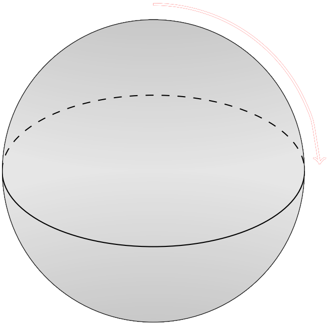
Hence 1 is an extremizer. Back to $\mathbb{R}^d$: $\langle x \rangle^{-k-1}$ is extremizer.

Competing symmetries (after Carlen and Loss).



Function F_0 localized near the black spots.
Radialize; get F_0^*. Note $|\mathcal{R}_+ F_0^*|_q \geq |\mathcal{R}_+ F_0|_q$.
Rotate; get F_1. Note $|\mathcal{R}_+ F_1|_q = |\mathcal{R}_+ F_0^*|_q$.
Radialize; get F_1^*. Note $|\mathcal{R}_+ F_1^*|_q \geq |\mathcal{R}_+ F_1|_q$.
Rotate; get F_2. Note $|\mathcal{R}_+ F_2|_q = |\mathcal{R}_+ F_1^*|_q$.
...

We get $F_0, F_1, \ldots$ with $|F_0|_p = |F_1|_p = \ldots$ and $|\mathcal{R}_+ F_0|_q \leq |\mathcal{R}_+ F_1|_q \leq \ldots$
It seems that $F_n \to 1$. **True by general result of Carlen, Loss.**

Since F_0 was arbitrary we have $\forall F_0$, $|\mathcal{R}_+ F_0|_q \leq |\mathcal{R}_+ 1|_q$.
Hence 1 is an extremizer. Back to $\mathbb{R}^d$: $\langle x \rangle^{-k-1}$ is extremizer.

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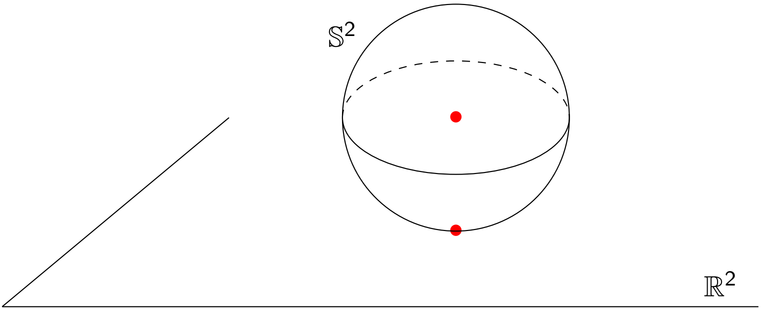

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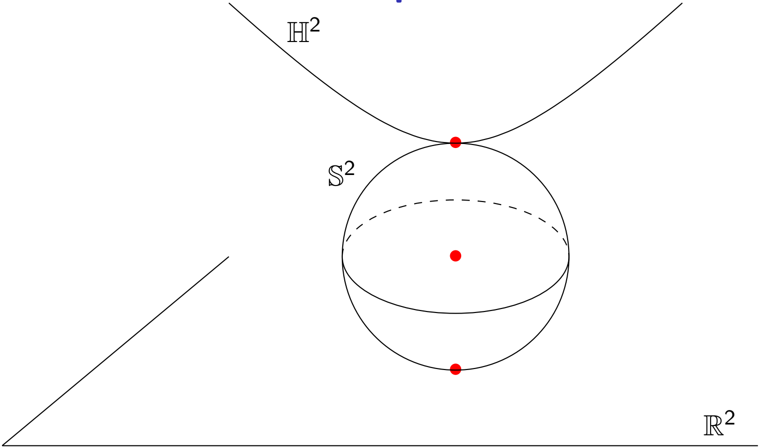

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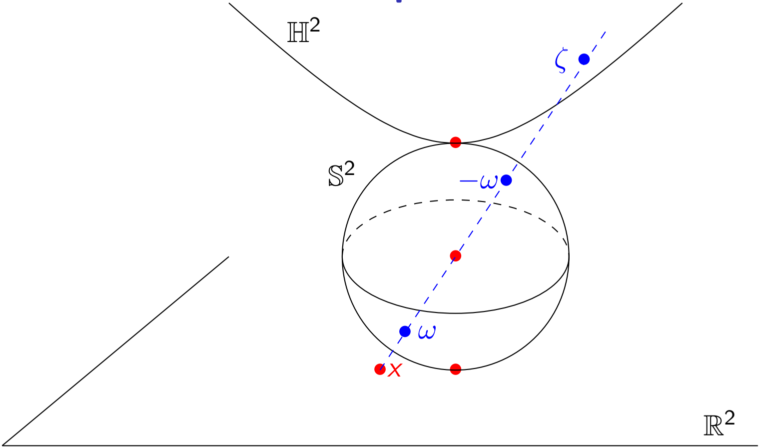

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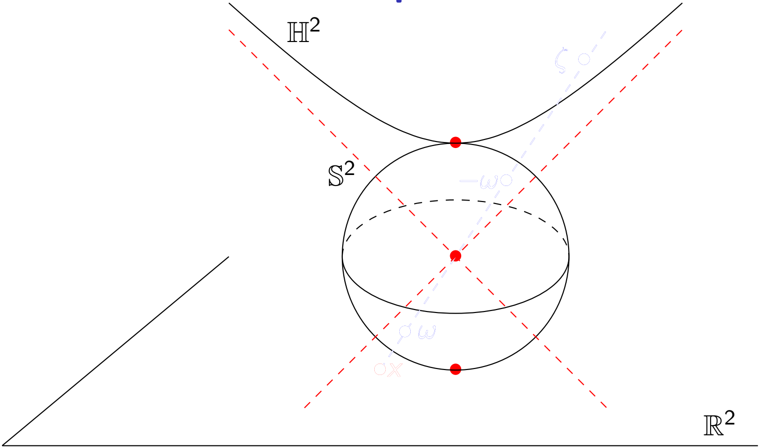

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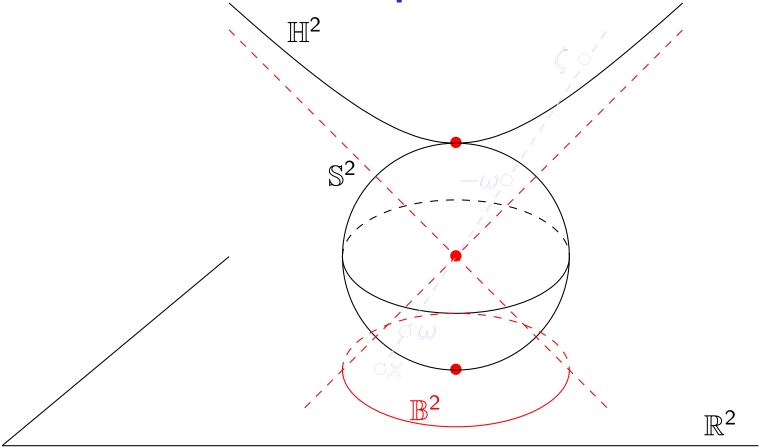

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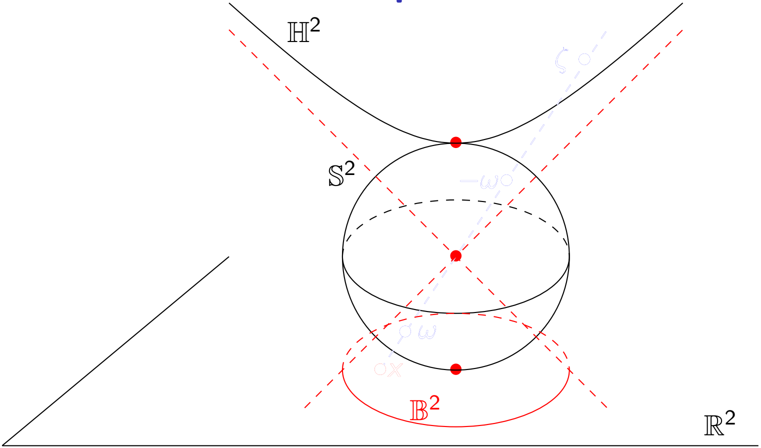

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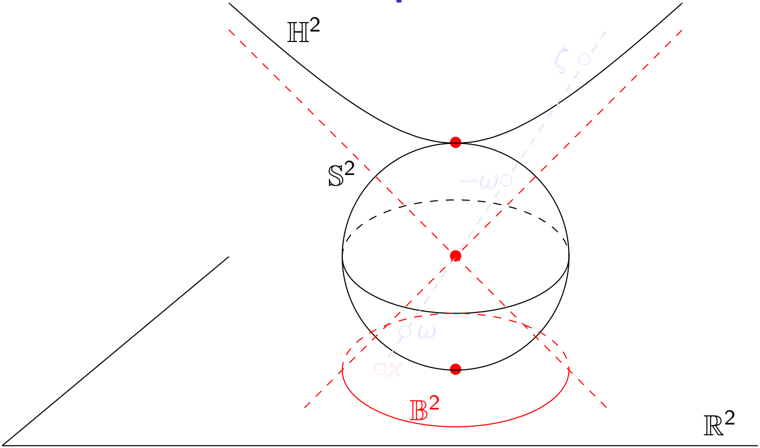

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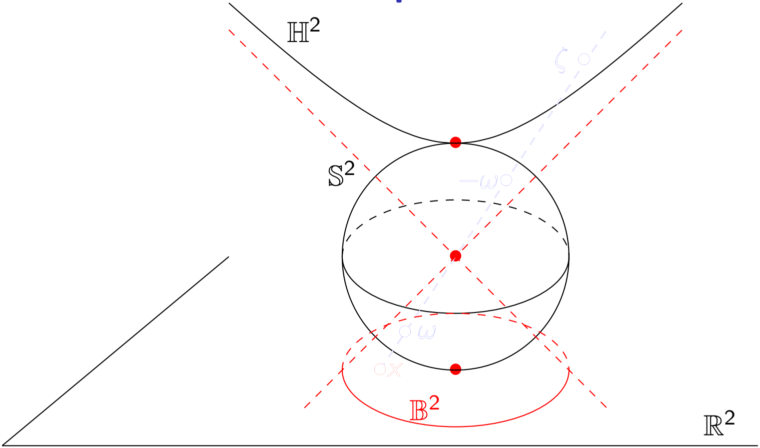

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Thanks for your attention!