Remarks on Extremization Problems Related To Young's Inequality

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Part 1: Introduction

Young's convolution inequality for $\mathbb{R}^d$ [1913]

- For functions on $\mathbb{R}^d$

$$\|f * g\|_r \leq \mathbf{A}_{pq}^d \|f\|_p \|g\|_q$$

where $\mathbf{A}_{pq}$ is a certain explicit constant
$\mathbf{A}_{pq} < 1$ when $p, q, r \in (1, \infty)$

- Beckner and Brascamp-Lieb (1974/75)

- Extremizers are (compatible) ordered pairs of Gaussians.
(Lieb 1990)

Young's inequality (continued)

- Monotonicity of $\langle f * g, h\rangle$ under a certain nonlinear heat flow establishes these results.
- The heat flow analysis was developed for arbitrary Hölder-Brascamp-Lieb multilinear inequalities by Bennett-Carbery-Tao circa 2008.
- Ordered pairs that satisfy

$$\|f * g\|_r \geq (1-\delta)\mathbf{A}_{pq}^d \|f\|_p \|g\|_q \qquad \text{for small } \delta > 0$$

are those close in $L^p \times L^q$ to the extremizing Gaussians. (C. 2011)

Riesz-Sobolev inequality (1930, 1938)

For any Lebesgue measurable sets $A, B, C \subset \mathbb{R}^d$ with finite Lebesgue measures,

$$\langle \mathbf{1_A} * \mathbf{1_B}, \mathbf{1_C} \rangle \le \langle \mathbf{1_{A^\star}} * \mathbf{1_{B^\star}}, \mathbf{1_{C^\star}} \rangle.$$

where $A^\star =$ ball centered at 0 with $|A^\star| = |A|$, et cetera.

Corresponding result for general nonnegative functions follows by expressing f as superposition of indicator functions of superlevel sets $\{x : f(x) > t\}$.

Equality in Riesz-Sobolev (Burchard [1998])

Let $d = 1$. If E_j satisfy

$$\langle \mathbf{1}_{E_1} * \mathbf{1}_{E_2}, \mathbf{1}_{E_3} \rangle = \langle \mathbf{1}_{E_1^\star} * \mathbf{1}_{E_2^\star}, \mathbf{1}_{E_3^\star} \rangle$$

then (up to null sets) $\boxed{E_n \text{ are } \textbf{intervals}}$ whose centers satisfy $\boxed{c_3 = c_1 + c_2}$

provided

$$|E_k| \leq |E_i| + |E_j| \qquad \boxed{\textbf{called admissibility}}$$

for all permutations (i, j, k) of $(1, 2, 3)$.

Equality in Riesz-Sobolev (continued)

For $d > 1$, Burchard showed that equality occurs

- Only for (homothetic, compatibly centered) **ellipsoids** in the strictly admissible case $|E_k|^{1/d} < |E_i|^{1/d} + |E_j|^{1/d}$,
- Only for (homothetic, compatibly centered) **convex sets** in the borderline admissible case.

There are two entirely different components in the analysis.

$\boxed{d = 1}$ A clever device, based on the ordering property of $\mathbb{R}$ (attributed by Burchard to Riesz). This device builds on the characterization of equality in the Brunn-Minkowski inequality.

$\boxed{d > 1}$ Combination of the $d = 1$ theorem with (repeated) *Steiner symmetrization*.

Brascamp-Lieb-Luttinger inequality (1974)

Let $m \geq 2$ and $n \geq m$. Consider

$$\boxed{\Phi(f_1, f_2, \ldots, f_n) = \int_{\mathbb{R}^m} \prod_{j=1}^{n} (f_j \circ L_j)\, dx}$$

where $L_j : \mathbb{R}^m \to \mathbb{R}^{\mathbf{1}}$ are (distinct) surjective linear mappings (with no common nullspace).

Theorem.

$$\Phi(f_1, \ldots, f_n) \leq \Phi(f_1^\star, \ldots, f_n^\star).$$

The case of nonnegative functions follows directly from the fundamental case of indicator functions of arbitrary sets.

Part 2

Which sets maximize BLL functionals with $L_j : \mathbb{R}^m \to \mathbb{R}^{\mathbf{1}}$?

Hölder-Brascamp-Lieb/BLL functional symmetry group

• A BLL functional defined by integration over $\mathbb{R}^m$ has an *m*–**dimensional group of symmetries**:

$$\int_{\mathbb{R}^m} \prod_{j=1}^{n} \mathbf{1}_{E_j}(L_j(x))\,dx$$

is invariant under any translation $x \mapsto x + v$ of $\mathbb{R}^m$.

• This corresponds to translating each E_j by $L_j(v)$, so $|E_j|$ is unchanged.

• Maximizing tuples can be unique only up to the action of this group.

Example: Gowers forms and norms

- For $k \geq 2$ the Gowers **forms** are

$$\mathcal{T}_k(f_\alpha : \alpha \in \{0,1\}^k) = \int_{x\in\mathbb{R}} \int_{h\in\mathbb{R}^k} \prod_{\alpha\in\{0,1\}^k} f_\alpha(x + \alpha \cdot h)\, dh\, dx$$

where $f_\alpha : \mathbb{R} \to [0, \infty]$.

- There are 2^k functions f_α; integration is over $\mathbb{R}^{k+1}$.

- Gowers **norms** are

$$\|f\|_{U^k}^{2^k} = \mathcal{T}_k(f, f, \ldots, f).$$

Example: Gowers forms and norms (continued)

- $\mathcal{T}_k(f_\alpha : \alpha \in \{0,1\}^k) \le A_k \prod_\alpha \|f_\alpha\|_{L^{p_k}}$ where p_k is dictated by scaling.
- Extremizing tuples are tuples of Gaussians; Eisner-Tao found the optimal constant in the inequality.
- C. showed that sets whose indicator functions have nearly maximal Gowers <u>norms</u>, among sets of specified Lebesgue measure, are nearly equal to intervals.
- My student Anh Nguyen is working to prove the corresponding stability result: Functions of nearly maximal Gowers norm are nearly (in norm) equal to Gaussians.

Maximizers of the Brascamp-Lieb-Luttinger inequality?

- Burchard characterized (admissible) extremizers of BLL functionals for $n = m + 1$ (one more function than dimension). (A corollary of her theorem on the Riesz-Sobolev inequality ($m = 2,\ n = 3$).)

- Flock and C. treated

$$\int_{\mathbb{R}^2} \prod_j (f_j \circ L_j)$$

  for an arbitrarily large number of functions f_j

- but were unable to generalize further.

- C. used Burchard's theorem in characterizing extremizers for a natural inequality for the Radon transform. Flock and Drouot extended this to the k-plane transform for all k.

Theorem: Maximizers of the B-L-L inequality

Assumptions/notation.

- $L_j : \mathbb{R}^m \to \mathbb{R}^1$ surjective linear mappings. (Natural nondegeneracy hypotheses.)
- $n =$ number of sets E_j is $> m$.
- $(|E_j| : 1 \leq j \leq n)$ **strictly admissible**.

Conclusion. If $\Phi(E_1, \ldots, E_n) = \Phi(E_1^\star, \ldots, E_n^\star)$ then E_j are intervals with compatibly situated centers.

Admissibility is equivalent to this simple necessary condition: If E_j are intervals of the specified lengths centered at 0, and if any one of these intervals is translated, then the functional Φ decreases.

A flow on special sets (Brascamp-Lieb)

For any **finite union of intervals** $E \subset \mathbb{R}$, define a flow $t \mapsto E(t)$ for $t \in [0, 1]$, as follows:

- $E(0) = E$.
- Each constituent interval moves rigidly at a constant speed so that its center will arrive at the origin at $t = 1$
- **until the first time of collision.**
- Stop the clock, glue together any intervals that have collided. The total number of intervals decreases.
- Restart the clock, with each interval moving at a constant speed chosen so that its center will arrive at the origin at $t = 1$
- until the next collision . . .

If sets E_j each flow in this way then $\Phi(E_1(t), \ldots, E_n(t))$ **is a nondecreasing function of** t for any tuple of sets.

"Theorem":[1] Flow on general (Lebesgue measurable) sets

There exists a flow $[0,1] \ni t \mapsto E(t)$ satisfying

- Flow is as already defined for finite unions of intervals.
- $E(0)$ arbitrary; $E(1) = E(0)^\star$.
- Continuous: $\lim_{s\to t} |E(s) \,\Delta\, E(t)| = 0$.
- Measure-preserving: $|E(t)| \equiv |E|$
- Inclusion monotonic: $A \subset B \Rightarrow A(t) \subset B(t)$
- Contractive: $|A(t) \,\Delta\, B(t)| \le |A \,\Delta\, B|$.
- Functional monotonic: For any BLL functional, $\Phi(E_1(t), \ldots, E_n(t))$ is a nondecreasing function of t.

[1] **Folklore; perhaps first noticed by Burchard?**

Aside: Smoothing property of the flow

For any set $E \subset \mathbb{R}$ (positive, finite Lebesgue measure), for any $t > 0$, $E(t)$ is (up to a null set) a countable **union of intervals**.

Corollary

Corollary of flow. In order to characterize maximizers of a BLL functional it suffices to prove that tuples of intervals centered at 0 are strict $\boxed{\textbf{LOCAL}}$ maximizers. □

This means that one need only examine tuples satisfying

$$|E_j \,\Delta\, E_j^\star| \le \delta_0 \qquad \forall j$$

where δ_0 may be chosen as small as desired (but fixed).

Part 3: Perturbative analysis

- I will next sketch how the proof of local maximality works for the Riesz-Sobolev inequality.
- This provides a **more complicated** alternative proof of Burchard's 1998 theorem for $\mathbb{R}^1$.
- The proof of the new theorem that I have stated — the analysis for general BLL functionals — requires additional ideas.
- *(The extension of the flow to general sets is not actually required in the argument below; I just like it and wanted to share it.)*

Expansion

- Express $\mathbf{1}_{E_j} = \mathbf{1}_{E_j^\star} + f_j$. Thus

$$f_j = \begin{cases} +1 \text{ on } E_j \setminus E_j^\star \\ -1 \text{ on } E_j^\star \setminus E_j \\ \quad 0 \text{ else.} \end{cases}$$

- By hypothesis, $\|f_j\|_1 \ll |E_j|$.
- Crucially, $\int f_j = 0 \quad \forall\, j$.
- Expand

$$\langle \mathbf{1}_{E_1} * \mathbf{1}_{E_2}, \mathbf{1}_{E_3} \rangle = \big\langle (\mathbf{1}_{E_1^\star} + f_1) * (\mathbf{1}_{E_2^\star} + f_2), (\mathbf{1}_{E_3^\star} + f_3) \big\rangle$$

into 8 terms.

Perturbative analysis (continued)

- The first variation term is $\boxed{\sum_{n=1}^{3} \int K_n f_n}$

where $\boxed{K_n = \mathbf{1}_{E_i^\star} * \mathbf{1}_{E_j^\star}}$ and $\{1,2,3\} = \{i,j,n\}$.

- The kernel K_n satisfies

$$\inf_{x \in E_n^\star} K_n(x) = \sup_{x \notin E_n^\star} K_n(x),$$

is even, is nonincreasing on $[0,\infty)$, and has strictly negative derivative at the right endpoint of $E_n^\star$.

- This is a constrained optimization problem. The first variation is not necessarily zero; it is typically *negative* and thus helps us.

- However, **the second variation typically has the wrong sign.**

• First variation term

$$\int K_n f_n \asymp -|E_n \,\Delta\, E_n^\star|^2$$

if nearly all points of $E_n \,\Delta\, E_n^\star$ are located within distance $C|E_n \,\Delta\, E_n^\star|$ of the boundary of $E_n^\star$, but is otherwise much more negative.

• The second and third order terms are $O(\max_j |E_j \,\Delta\, E_j^\star|^2)$.

• A simple analysis exploiting this and based on the above properties of K_n allows reduction to the case in which **every point of $E_j \,\Delta\, E_j^\star$ lies within distance $C|E_j \,\Delta\, E_j^\star|$ of an endpoint of the interval $E_j^\star$.**

Perturbations near edges of $E_j^\star$

Given $(E_j : j \in \{1,2,3\})$ define

$$\delta = \inf_{\mathbf{v}\in\mathbb{R}^{\mathbf{m}}} \max_j \left| \left(E_j + L_j(v) \right) \Delta\, E_j^\star \right|$$

where the infimum is taken over all $v \in \mathbb{R}^m$.

Thus consider orbit of (E_1, E_2, E_3) under the symmetry group introduced above, and choose v that provides the optimal approximation by a system of compatibly centered intervals.

There are two cases

Case (1) **Not quite intervals.**
There exists n such that E_n has

$$\textbf{diameter}(E_n) \geq |E_n| + c\delta$$

where "diameter" means that any interval of diameter $< |E_n| + c\delta$ misses a subset of E_n having measure $\geq c'\delta$.

Case (2) **Not quite optimally situated intervals.**
Each set E_n is very nearly an interval, but their centers satisfy

$$|c_3 - c_1 - c_2| \geq c''\delta.$$

Cancellation

- Consider case in which E_1 is not quite an interval.
- Exploit symmetry group to make $\int_{\mathbb{R}^+} f_1 = \int_{\mathbb{R}^-} f_1 = 0$.
- Thus $f_j = f_j^+ + f_j^-$ with both terms supported on short intervals, and both having vanishing integrals for $j = 1$.
- $\langle f_1, f_2 * \mathbf{1}_{E_3^\star} \rangle = 0!$ Indeed,

$$\langle f_1, f_2 * \mathbf{1}_{E_3^\star} \rangle = \int_{E_2} \int_{\mathbb{R}} f_1(y) \mathbf{1}_{E_3^\star + x}(y)\, dy\, dx.$$

  If x belongs to the support of f_2 then for one choice of $\pm$ sign, $E_3^\star + x$ contains the support of $f_1^\pm$; for the other choice, $E_3^\star + x$ is disjoint from the support of $f_1^\pm$.
- Same for $\langle f_1, f_3 * \mathbf{1}_{E_2^\star} \rangle$.

Cancellation

- Thus

$$\langle \mathbf{1}_{E_1} * \mathbf{1}_{E_2},\, \mathbf{1}_{E_3} \rangle = \langle \mathbf{1}_{E_1^\star} * \mathbf{1}_{E_2},\, \mathbf{1}_{E_3} \rangle + \langle K_1,\, f_1 \rangle.$$

  No second or third variation terms.

- First term $\leq \langle \mathbf{1}_{E_1^\star} * \mathbf{1}_{E_2^\star},\, \mathbf{1}_{E_3^\star} \rangle$ by Riesz-Sobolev inequality, while $\langle K_1,\, f_1 \rangle < 0$.

I reiterate: Analysis for general BLL functionals requires supplementary ideas.

Part 4: The functional $\frac{\|\widehat{\mathbf{1}_E}\|_q}{|E|^{1/p}}$

The affine group is a group of symmetries.

Maximizers

Theorem. Let $d \geq 1$. Let $2m$ be an even integer ≥ 4.
Let $q > 2$ be sufficiently close to $2m$.

$$E \text{ maximizes } \frac{\|\widehat{\mathbf{1}_E}\|_q}{|E|^{1/p}} \text{ if and only if } E \text{ is an ellipsoid.}$$

Previous work:

- The case $q = 2m$ is a corollary of Burchard's theorem.
- True for
  1. $d = 1$
  2. $d = 2$ and q close to 4
  3. q sufficiently large (and close to $2m$) for all d
- Maximizers exist for all $2 < q < \infty$ and all $d \geq 1$.

Reduction

- Let $\mathbb{B}$ = unit ball.

- **Reduction:** It suffices to prove for $q \in \{4, 6, 8, \dots\}$:
If $|E| = |\mathbb{B}|$ then

$$\|\widehat{\mathbf{1}_E}\|_q \leq \|\widehat{\mathbf{1}_{\mathbb{B}}}\|_q - c_{d,q} \inf_{\mathcal{E} \text{ ellipsoid}} |\mathcal{E} \,\Delta\, E|^2.$$

This reduction relies on a compactness theorem, whose proof relies on additive combinatorics (Freiman's theorem).

- Compactness/stability theorem plus Burchard's theorem reduce matters to small perturbations: $|E \,\Delta\, \mathbb{B}| \ll 1$.

Expansion

- Write $\mathbf{1}_E = \mathbf{1}_{\mathbb{B}} + f$ where $\int f = 0$
- Expansion

$$\|\widehat{\mathbf{1}_E}\|_q^q = \|\widehat{\mathbf{1}_{\mathbb{B}}}\|_q^q + q\langle \mathbf{K_q}, \mathbf{f}\rangle + \frac{1}{2}q(q-1)\langle \mathbf{f} * \mathbf{L_q}, \mathbf{f}\rangle + O(|E \,\Delta\, \mathbb{B}|^{2+\eta})$$

where

- $K_q \;=\;$ convolution of $q-1$ factors of $\mathbf{1}_{\mathbb{B}}$
- $L_q \;=\;$ convolution of $q-2$ factors of $\mathbf{1}_{\mathbb{B}}$

These are too complicated to calculate in closed form.

- Leading term $\langle K_q, f\rangle = \langle K_q, \mathbf{1}_E - \mathbf{1}_{\mathbb{B}}\rangle$ is nonpositive and behaves like $-|E \setminus \mathbb{B}|$ on the part of E that is not near boundary of $\mathbb{B}$
- This allows reduction to the case in which $E \,\Delta\, \mathbb{B}$ is contained in $C|E \,\Delta\, \mathbb{B}|$–neighborhood of S^{d-1}.
- Defining

$$F(\theta) = \int_{\mathbb{R}^+} f(r\theta) r^{d-1}\, dr = \int_{\mathbb{R}^+} \Big(\mathbf{1}_E(r\theta) - \mathbf{1}_{\mathbb{B}}(r\theta)\Big)\, r^{d-1}\, dr.$$

- one has

$$\begin{aligned}\|\widehat{\mathbf{1}_E}\|_q^q = \|\widehat{\mathbf{1}_{\mathbb{B}}}\|_q^q - \gamma_{d,q}\|F\|^2_{L^2(S^{d-1})} \\ + \iint_{S^{d-1}\times S^{d-1}} F(x)\, F(y)\, L_q(x-y)\, d\sigma(x)\, d\sigma(y) \\ + O(|E \,\Delta\, \mathbb{B}|^{2+\eta}).\end{aligned}$$

Bad news / good news

• ☹ I am not able to calculate the eigenvalues for the quadratic form

$$\iint_{S^{d-1}\times S^{d-1}} F(x)\, F(y)\, L_q(x-y)\, d\sigma(x)\, d\sigma(y).$$

• I am not able to calculate $\gamma_{d,q}$. ☺

• This inability is potentially good news! If one wants to determine whether $a - b < 0$, if a, b have the same order of magnitude and if one knows a exactly but one cannot calculate b, then one is stuck. But if one cannot calculate either quantity exactly, then there still may be hope.

- The quadratic form is diagonalized by spherical harmonics. Obviously its eigenvalues tend to 0.
- Spherical harmonics of degrees $1, 2$ can be eliminated by exploiting the affine symmetry group.

- Let Y be a spherical harmonic of degree ≥ 3. Define

$$E_t = \{x = r\theta : 0 \leq r \leq 1 + tY(\theta)\}.$$

Matters are reduced to showing that for small t,

$$\|\mathbf{1}_{E_t} * \mathbf{1}_{E_t} * \cdots * \mathbf{1}_{E_t}\|_2^2 \leq \|\mathbf{1}_{E_t^\star} * \mathbf{1}_{E_t^\star} * \cdots * \mathbf{1}_{E_t^\star}\|_2^2 - \boxed{c}\,t^2 \quad (*)$$

where $c > 0$ is allowed to depend on Y.

I have oversimplified a bit . . .

- Analyze this using Steiner symmetrization.
- Equality in 1D Riesz-Sobolev inequality must be achieved up to $o(t)$ for almost all interactions of vertical slices $E \cap \{x : x_d = s\}$.
- Vertical slices are intervals; this forces their centers to lie on a common hyperplane.
- This must hold for all rotates of E.
- This strong constraint is satisfied only by spherical harmonics of degrees $1, 2$.