Remarks on some determinant inequalities with a geometrical flavour

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Conference in Hono(u)r of Michael Christ
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16th May 2016

Outline

Basic questions – I

Let us work in euclidean n-space $\mathbb{R}^n$. Given indices $0 < p_j < \infty$ we consider inequalities of the form

$$\|f_0\|_{p_0} \dots \|f_n\|_{p_n} \leq C \sup_{x_0,\dots,x_n} f_0(x_0)\dots f_n(x_n) \mathrm{Vol}\ \mathrm{co}(x_0, x_1, \dots x_n)$$

where $C = C(p_j, n)$ is independent of the nonnegative measurable functions f_j and

$$\mathrm{Vol}\ \mathrm{co}(x_0, x_1, \dots x_n)$$

denotes the volume of the convex hull of the simplex in $\mathbb{R}^n$ spanned by $x_0, \dots, x_n$.

Of course

$$n!\mathrm{Vol}\ \mathrm{co}(x_0, x_1, \dots x_n) = |\det(x_1 - x_0 \ \dots x_n - x_0)|$$

$$= \left| \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \end{pmatrix} \right|$$

where $\det(y_1\ y_2\ \dots\ y_n)$ denotes the determinant of the $n \times n$ matrix whose columns are $y_1, \dots, y_n$.

Basic questions – II

$$\|f_0\|_{p_0}\dots\|f_n\|_{p_n} \le C \sup_{x_0,\dots,x_n} f_0(x_0)\dots f_n(x_n)\,\mathrm{Vol\ co}(x_0,x_1,\dots x_n)$$

Closely related are inequalities like:

$$\|f_1\|_{L^{p_1,\infty}}\dots\|f_n\|_{L^{p_n,\infty}} \le C \sup_{x_1,\dots,x_n} f_1(x_1)\dots f_n(x_n)|\det(x_1\ x_2\ \dots\ x_n)|$$

where $C = C(p_j, n)$ is independent of the nonnegative measurable functions f_j.
When $n = 1$ this reduces to

$$\|f\|_{L^{p,\infty}} \le C \sup_x f(x)|x|$$

– which is true for $p = 1$, with sharp constant $C = 2$.

Connection with Gressman's inequalities

Gressman considered (much more generally) multilinear determinant inequalities generalising 1-dimensional fractional integration:

$$\int_{(\mathbb{R}^n)^{n+1}} f_0(x_0)\ldots f_n(x_n)\mathrm{Vol}\,\mathrm{co}(x_0\ \ldots x_n)^{-\alpha}\mathrm{d}x_0\ldots\mathrm{d}x_n \leq C\|f_0\|_{p_0}\ldots\|f_n\|_{p_n}$$

where $f_j \geq 0$, $\alpha > 0$ and $p_j > 1$. As with classical fractional integration, these are equivalent to their reverse forms

$$\|f_0\|_{p_0}\ldots\|f_n\|_{p_n} \leq C\int_{(\mathbb{R}^n)^{n+1}} f_0(x_0)\ldots f_n(x_n)\mathrm{Vol}\,\mathrm{co}\,(x_0\ \ldots x_n)^{+\alpha}\mathrm{d}x_0\ldots\mathrm{d}x_n$$

where now $p_j < 1$. Changing notation, these are in turn equivalent to

$$\|f_0\|_{p_0}\ldots\|f_n\|_{p_n} \leq C\|f_0(x_0)\ldots f_n(x_n)\mathrm{Vol}\,\mathrm{co}(x_0\ \ldots\ x_n)^{+\alpha}\|_{L^r((\mathbb{R}^n)^{n+1})}$$

for $p_j > 0$ and $\max\{p_j\} < r < \infty$.

Sending $r \to \infty$ leads us to consider inequalities of the form

$$\|f_0\|_{p_0}\ldots\|f_n\|_{p_n} \leq C\sup_{x_0,\ldots,x_n} f_0(x_0)\ldots f_n(x_n)\mathrm{Vol}\,\mathrm{co}(x_0\ \ldots\ x_n).$$

Connection with old work of Macbeath

$$\|f_0\|_{p_0}\ldots\|f_n\|_{p_n} \leq C \sup_{x_0,\ldots,x_n} f_0(x_0)\ldots f_n(x_n)\mathrm{Vol}\,\mathrm{co}(x_0\ \ldots\ x_n).$$

Macbeath in the 1950's was concerned with various geometrical questions, including finding extremal sets $E \subseteq \mathbb{R}^n$ for the inequality

$$|E| \leq C_n \sup_{x_0,x_1,\ldots,x_n\in E} \mathrm{Vol}\,\mathrm{co}(x_0,x_1,\ldots x_n).$$

This is a special case $f_0 = f_1 = \cdots = f_n = \chi_E$ of our problem, and asks, amongst all sets E of given volume, which ones have simplices of *least* maximal volume with vertices in E?

Or, in other words, what is the sharp constant c_n so that given a set E of volume 1, one is guaranteed to find $n+1$ points in E which span a simplex of volume at least c_n, and which sets realise that constant?

Solution to Macbeath's problem

$$|E| \leq C_n \sup_{x_0,x_1,\dots,x_n \in E} \mathrm{Vol}\,\mathrm{co}(x_0, x_1, \dots x_n)$$

It is clear that E can be assumed to be convex and that $C_n \leq n^n$ since if E has centroid 0 and T is a simplex of maximal voulme contained in E then $T \subseteq nE$.

Moreover the problem is clearly affine invariant and the extremising sets turn out to be balls and ellipsoids. Thus

$$C_n = \frac{\mathrm{Vol}(\mathbb{B}_n)}{\mathrm{Vol}(\Sigma_n)}$$

where Σ_n is a maximal regular simplex inscribed in the unit ball $\mathbb{B}_n$ of $\mathbb{R}^n$.

Why *regular*?

A nice argument for why it has to be *regular* simplices maximimising the volume of simplices inscribed in the unit ball:

Consider an extremal simplex. Consider an arbitrary face. Then the remaining vertex must be on the unit sphere at maximal distance to the hyperplane in which this face lies.

So, when $n = 2$, the line through the remaining vertex and the origin meets the chosen face at right angles, forcing the triangle to be isoceles. By symmetry it must be equilateral.

When $n \geq 3$, proceed inductively starting in the same way. The "height" of the simplex above the face under consideration is determined, and the face itself is a simplex of one lower dimension inscribed in a sphere of one lower dimension, and so must be of a *regular* simplex in one lower dimension, by induction. By symmetry on the faces we conclude that the maximising simplex is indeed regular.

Finding structures in large sets

Given a set E, one can find $n+1$ points in E which span a simplex of volume at least $c_n \mathrm{Vol}(E)$.

Some work Mike, Jim Wright and I did on sublevel sets in the mid-1990's is in the same spirit.

A variant in the plane: if $E \subseteq Q = [0,1]^2$ then there will always exist an axis-parallel rectangle R with vertices in E whose area is "large".

Natural examples suggest that we should have

$$|R| \geq C|E|^2,$$

but to date the best known is still the slightly weaker

$$|R| \geq C_\epsilon \frac{|E|^2}{\log(1/|E|)^{2-}}.$$

Open problems: Decide whether the log term is really there. Figure out what is going on in higher dimensions.

Connection with recent work of Christ

In recent work concerning near-extremisers for the Hausdorff-Young inequality, Mike has needed to study some questions analogous to those he, Jim & I studied, but with euclidean n-space replaced by the space of $n \times n$ real matrices, and volume replaced by determinant.

So a primordial question in this context is whether for suitable sets $E \subseteq \mathcal{M}^{n\times n}(\mathbb{R})$ we can assert the existence of matrices $A \in E$ with large determinant.

A moment's thought reveals that the correct scaling is

$$\mathrm{Vol}(E)^{1/n} \leq C_n \sup_{A\in E} |\det A|.$$

However, the set $\{A \ : \ |\det A| \leq 1\}$ has infinite volume, so we cannot expect such inequalities for *general* subsets of $\mathcal{M}^{n\times n}(\mathbb{R})$: some sort of convexity hypotheses will be needed.

A simple lemma in one dimension

For a set $E \subseteq \mathbb{R}$, let E^* be the interval centred at 0 with the same length as E.

Lemma

For $a_j \in \mathbb{R}$ we have

$$\sup_{x_j^* \in E_j^*} \Big|\sum_{j=1}^{k} a_j x_j^*\Big| \leq \sup_{x_j \in E_j} \Big|\sum_{j=1}^{k} a_j x_j\Big|.$$

For $k = 1$ this says there is an $x \in E$ such that $|x| \geq |E|/2$.

Replacing E_j by $a_j E_j$ reduces us to the case $a_j = 1$ for all j.

So we want to see there are $x_j \in E_j$ such that $|\sum_j x_j| \geq \frac{1}{2} \sum_j |E_j|$.

If not, for all $x_j \in E_j$ we have $|\sum_j x_j| < \frac{1}{2} \sum_j |E_j|$.

Proof

For all $x_j \in E_j$ we have $|\sum_j x_j| < \frac{1}{2}\sum_j |E_j|$.

Pick $x_j^{\pm} \in E_j$ with $x_j^+ \sim \sup E_j$ and $x_j^- \sim \inf E_j$. Then

$$x_j^+ - x_j^- \geq |E_j|,$$

$$|\sum_j x_j^+| < \frac{1}{2}\sum_j |E_j| \quad \text{and} \quad |\sum_j x_j^-| < \frac{1}{2}\sum_j |E_j|.$$

Hence

$$\sum_j |E_j| \leq \sum_j (x_j^+ - x_j^-) = (\sum_j x_j^+) - (\sum_j x_j^-)$$
$$< \frac{1}{2}\sum_j |E_j| + \frac{1}{2}\sum_j |E_j| = \sum_j |E_j|,$$

contradiction.

A restatement

For $a_j \in \mathbb{R}$ we have

$$\sup_{x_j^* \in E_j^*} \left|\sum_{j=1}^{k} a_j x_j^*\right| \le \sup_{x_j \in E_j} \left|\sum_{j=1}^{k} a_j x_j\right|.$$

So for $E = E_1 \times E_2 \times \cdots \times E_k \subseteq \mathbb{R}^k$ and $\mathcal{S}E$ its rectangular rearrangement, we have for all $a \in \mathbb{R}^k$,

$$\sup_{x \in \mathcal{S}E} |a \cdot x| \le \sup_{x \in E} |a \cdot x|.$$

A consequence for determinants

For $\xi \in \mathbb{R}^n$ write $\xi = (\xi_1, \xi')$ where $\xi' \in \mathbb{R}^{n-1}$. Let $E_1, \ldots, E_n \subseteq \mathbb{R}^n$. We have, for $x_j \in E_j$,

$$\det(x_1 \ldots x_n) = \det \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ \vdots & \vdots & \ldots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{pmatrix} = x_{11}A_1 + x_{21}A_1 + \ldots x_{n1}A_n,$$

where the A_j depend only on $\{x_1', \ldots, x_n'\}$.

For each j fix $x_j' := (x_{j2}, \ldots, x_{jn})$, let $E_j(x_j') = \{x_{j1} \in \mathbb{R} : (x_{j1}, x_j') \in E_j\}$, and apply the lemma with $k = n$ to get

$$\sup_{x_{j1}^* \in E_j(x_j')^*} |\sum_{j=1}^n A_j x_{j1}^*| \leq \sup_{x_{j1} \in E_j(x_j')} |\sum_{j=1}^n A_j x_{j1}|.$$

Hence, with $\widehat{E}$ the symmetrisation of E with respect to $e_1^\perp$,

$$\sup_{x_j \in \widehat{E}_j} |\det(x_1 \ldots x_n)| \leq \sup_{x_j \in E_j} |\det(x_1 \ldots x_n)|.$$

Similarly for convex hulls...

Similarly,

$$n!\mathrm{Vol}\,\mathrm{co}(x_0, x_1, \dots, x_n) = |\det(x_1 - x_0\ x_2 - x_0\ \dots\ x_n - x_0)|$$

$$= \left|\det\begin{pmatrix} x_{11} - x_{01} & x_{21} - x_{01} & \cdots & x_{n1} - x_{01} \\ \vdots & \vdots & \dots & \vdots \\ x_{1n} - x_{0n} & x_{2n} - x_{0n} & \cdots & x_{nn} - x_{0n} \end{pmatrix}\right|$$

$$= |x_{01}A_0 + x_{11}A_1 + x_{21}A_1 + \dots x_{n1}A_n|,$$

where the A_j depend only on $\{x_0', \dots, x_n'\}$. So, by the case $k = n+1$ of the lemma,

$$\sup_{x_{j1}^* \in E_j(x_j')^*} |\sum_{j=0}^{n} A_j x_{j1}^*| \leq \sup_{x_{j1} \in E_j(x_j')} |\sum_{j=0}^{n} A_j x_{j1}|,$$

and hence

$$\sup_{x_j \in \widehat{E}_j} |\mathrm{Vol}\,\mathrm{co}(x_0, \dots, x_n)| \leq \sup_{x_j \in E_j} |\mathrm{Vol}\,\mathrm{co}(x_0, \dots, x_n)|.$$

Rotation invariance

$$\sup_{x_j \in \widehat{E}_j} |\det(x_1 \dots x_n)| \le \sup_{x_j \in E_j} |\det(x_1 \dots x_n)|.$$

$$\sup_{x_j \in \widehat{E}_j} |\mathrm{Vol}\,\mathrm{co}(x_0, \dots, x_n)| \le \sup_{x_j \in E_j} |\mathrm{Vol}\,\mathrm{co}(x_0, \dots, x_n)|.$$

For $\omega \in \mathbb{S}^{n-1}$ let $\widehat{E}^\omega$ be symmetrisation of E with respect to $\omega^\perp$:

$$\sup_{x_j \in \widehat{E}_j^{\,\omega}} |\det(x_1 \dots x_n)| \le \sup_{x_j \in E_j} |\det(x_1 \dots x_n)|$$

$$\sup_{x_j \in \widehat{E}_j^{\,\omega}} |\mathrm{Vol}\,\mathrm{co}(x_0, \dots, x_n)| \le \sup_{x_j \in E_j} |\mathrm{Vol}\,\mathrm{co}(x_0, \dots, x_n)|.$$

Repeating gives, with E^* the radial rearrangement of E,

$$\sup_{x_j \in E_j^{*}} |\det(x_1 \dots x_n)| \le \sup_{x_j \in E_j} |\det(x_1 \dots x_n)|,$$

$$\sup_{x_j \in E_j^{*}} |\mathrm{Vol}\,\mathrm{co}(x_0, \dots, x_n)| \le \sup_{x_j \in E_j} |\mathrm{Vol}\,\mathrm{co}(x_0, \dots, x_n)|.$$

Now with functions...

Let f_j be nonnegative functions defined on $\mathbb{R}^n$ and let f_j^* be their radial nonincreasing rearrangements.

Proposition

$$\sup f_1^*(x_1)\dots f_n^*(x_n)|\det(x_1\dots x_n)|$$
$$\leq \sup f_1(x_1)\dots f_n(x_n)|\det(x_1\dots x_n)|.$$

Proposition

$$\sup f_0^*(x_0)\dots f_n^*(x_n)\mathrm{Vol}\,\mathrm{co}(x_0,\dots,x_n)$$
$$\leq \sup f_0(x_0)\dots f_n(x_n)\mathrm{Vol}\,\mathrm{co}(x_0,\dots,x_n).$$

The proofs are straightforward from the special case of characteristic functions of sets already established. (There is also a functional version of the simple lemma.)

An n-linear corollary

For arbitrary sets $E_j \subseteq \mathbb{R}^n$,

$$\sup_{x_j \in E_j{}^*} |\det(x_1 \dots x_n)| \leq \sup_{x_j \in E_j} |\det(x_1 \dots x_n)|,$$

or equivalently

$$|E_1|^{1/n} \dots |E_n|^{1/n} \leq \mathrm{Vol}(\mathbb{B}_n) \sup_{x_j \in E_j} |\det(x_1 \dots x_n)|.$$

And, for arbitrary f_j, by taking $E_j = \{x : |f_j(x)| > \lambda_j\}$, we get the (sharp)

$$\|f_1\|_{L^{n,\infty}} \dots \|f_n\|_{L^{n,\infty}} \leq \mathrm{Vol}(\mathbb{B}_n) \sup f_1(x_1) \dots f_n(x_n) |\det(x_1 \dots x_n)|.$$

An $(n+1)$-linear Macbeath theorem

Applying

$$\sup_{x_j \in E_j^*} |\mathrm{Vol}\,\mathrm{co}(x_0, \ldots, x_n)| \leq \sup_{x_j \in E_j} |\mathrm{Vol}\,\mathrm{co}(x_0, \ldots, x_n)|$$

with $E_j = E$ for all j immediately gives Macbeath's theorem.

One can also take different sets E_j. For example in $\mathbb{R}^2$ let E_j have measure πr_j^2. Then the maximising simplex for the left hand side must be such that its vertices are distant r_0, r_1 and r_2 from its orthocentre.

Let $A(r_0, r_1, r_2)$ be the area of such a triangle. It can be shown that

$$A(r_0, r_1, r_2) = \frac{1}{2}\left(r_0 r_1 \sqrt{1 - \lambda^2 r_2^2} + r_1 r_2 \sqrt{1 - \lambda^2 r_0^2} + r_2 r_0 \sqrt{1 - \lambda^2 r_1^2} \right)$$

where λ is the unique negative solution of the cubic equation

$$2 r_0 r_1 r_2 \lambda^3 - (r_0^2 + r_1^2 + r_2^2)\lambda^2 + 1 = 0.$$

(In higher dimensions not every simplex is orthocentric so the calculations are somewhat "in principle"...)

Sharp $(n+1)$-linear weak-type inequalities

Let $V(r_0,\ldots,r_n)$ be the volume of the maximal simplex with vertices in $E_0^*,\ldots,E_n^*$ where $\mathrm{Vol}(E_j)=\mathrm{Vol}(\mathbb{B}_n)r_j^n$. Then for $\sum_{j=0}^n \alpha_j=1$ and $0\leq\alpha_j\leq 1/n$ we have

$$r_0^{n\alpha_0}\ldots r_n^{n\alpha_n}\leq C_\alpha V(r_0,\ldots,r_n)$$

with a sharp C_α, leading to inequalities

$$|E_0|^{\alpha_0}\ldots|E_n|^{\alpha_n}\leq C_\alpha \sup_{x_j\in E_j}|\mathrm{Vol}\,\mathrm{co}(x_0,\ldots,x_n)|$$

which are sharp in the sense that for each α there are extremal sets E_j which are balls of certain radii depending on the α_j's.

These in turn lead to sharp inequalities

$$\|f_0\|_{p_0,\infty}\ldots\|f_n\|_{p_n,\infty}\leq C_p \sup_{x_0,\ldots,x_n} f_0(x_0)\ldots f_n(x_n)\mathrm{Vol}\,\mathrm{co}(x_0,\ldots,x_n)$$

for $\sum_{j=0}^n 1/p_j=1$ and $p_j\geq n$ for all j.

$(n+1)$-linear strong-type inequalities

$\|f_0\|_{p_0,\infty} \dots \|f_n\|_{p_n,\infty} \le C_p \sup_{x_0,\dots,x_n} f_0(x_0) \dots f_n(x_n) \mathrm{Vol}\,\mathrm{co}(x_0,\dots,x_n)$ for $\sum_{j=0}^n 1/p_j = 1$ and $p_j \ge n$ for all j.

By interpolation or otherwise we get (non-sharp) limiting Gressman inequalities

$$\|f_0\|_{p_0} \dots \|f_n\|_{p_n} \le C \sup_{x_0,\dots,x_n} f_0(x_0) \dots f_n(x_n) \mathrm{Vol}\,\mathrm{co}(x_0,\dots,x_n)$$

for $\sum_{j=0}^n 1/p_j = 1$ and $p_j > n$ for all j. (These conditions are necessary for the limiting Gressman inequalities to hold.)

Extremals on the diagonal

We have the endpoint limiting Gressman inequalities

$$\|f_0\|_{p_0} \dots \|f_n\|_{p_n} \leq C \sup_{x_0,\dots,x_n} f_0(x_0) \dots f_n(x_n) \mathrm{Vol}\,\mathrm{co}(x_0,\dots,x_n)$$

for $\sum_{j=0}^{n} 1/p_j = 1$ and $p_j > n$ for all j.

On the diagonal ($p_j = n+1$ for all j) Ting Chen has established that there are extremals of the form

$$f_j(x) = \left(\frac{1}{1+|x|^2}\right)^{1/2}.$$

The argument uses the interplay between euclidean and spherical symmetries via stereographic projection.

Probably one cannot hope for more at present.

Two results of Christ

In this section E will be a subset of $\mathcal{M}^{n\times n}(\mathbb{R})$ regarded as the euclidean space $\mathbb{R}^{n^2}$, together with Lebesgue measure.

Proposition (M. Christ)

There exists a constant A_n such that if $E \subseteq \mathcal{M}^{n\times n}(\mathbb{R})$ is convex (and symmetric), then

$$|E|^{1/n} \leq A_n \sup_{A\in E} |\det(A)|.$$

Theorem (M. Christ)

There exists a constant B_n such that for every $E \subseteq \mathcal{M}^{n\times n}(\mathbb{R})$

$$|E|^{1/n} \leq B_n \sum_{j=1}^{n} \sup_{A_1,\dots,A_j\in E} |\det(A_1 + \cdots + A_j)|.$$

Note the need for sums of fewer than n matrices on the right-hand side in the second result; result is false if we take only single summands.

Comments on Christ's results

Mike reduced the second result to the first using an auxiliary polynomial argument, and established the first using Loomis–Whitney.

The second result implies the first: if is E convex, we have $A_k \in E$ for $1 \leq k \leq j$ implies $\frac{1}{j}(A_1 + \cdots + A_j) \in E$ so

$$\sup_{A_1,\ldots,A_j \in E} |\det(A_1 + \cdots + A_j)| \leq j^n \sup_{A \in E} |\det(A)|.$$

Now applying the theorem we obtain

$$|E|^{1/n} \lesssim \sum_{j=1}^{n} \sup_{A_1,\ldots,A_j \in E} |\det(A_1 + \cdots + A_j)| \lesssim \sup_{A \in E} |\det(A)|.$$

In fact convexity is crucial in the study of these inequalities – $E + \cdots + E$ is "more convex" than E – for any E, "$\lim_{m\to\infty} \frac{E+\cdots+E}{m}$" is convex.

Geometrical considerations

Lack of a large group of invariants – no euclidean or affine invariance at the level of actions on $\mathcal{M}^{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$.

Nevertheless, there is an important action of $SL(n, \mathbb{R})$ on $\mathcal{M}^{n\times n}(\mathbb{R})$ by (pre-)multiplication.

That is, if $T \in GL(n, \mathbb{R})$, $A \in \mathcal{M}^{n\times n}(\mathbb{R})$ and $E \subseteq \mathcal{M}^{n\times n}(\mathbb{R})$ then

$$\det(TA) = \det T \det A$$

and

$$|TE| = |\det T|^n |E|.$$

So both of Mike's results are invariant under premultiplication by a matrix of unimodular determinant.

This observation already featured in Mike's analysis.

A theorem of Ting Chen

Theorem (Ting Chen, 2016)

There exists a constant C_n such that for every $E_1, \ldots, E_n \subseteq \mathcal{M}^{n\times n}(\mathbb{R})$,

$$\prod_{j=1}^{n} |E_j|^{1/n^2} \leq C_n \sup_{A_1\in E_1 \ldots, A_n \in E_n} |\det(A_1 + \cdots + A_n)|.$$

Note the “multilinear” nature of the result.

Taking $E_j = E$ for all j we obtain a slightly improved version of Christ’s theorem: we no longer need to consider j-fold Minkowski sums for $1 \leq j \leq n-1$.

Ting Chen's argument – Notation

Let us work in the three-dimensional case.

We have three sets of matrices $E, F, G \subseteq \mathcal{M}^{3\times 3}(\mathbb{R})$.

Members of E will be denoted by

$$X = (x_1 x_2 x_3)$$

where x_j are 3×1 column vectors.

Members of F will be denoted by

$$Y = (y_1 y_2 y_3)$$

and members of G will be denoted by

$$Z = (z_1 z_2 z_3)\,.$$

We supsose that $\sup_{X\in E, Y\in F, Z\in G} |\det(X+Y+Z)| = s$.

We want to show that

$$|E|^{1/9}|F|^{1/9}|G|^{1/9} \lesssim s.$$

Ting Chen's argument – Projections and slices

We define $\Pi_{12} : \mathcal{M}^{3\times 3}(\mathbb{R}) \to \mathcal{M}^{3\times 2}(\mathbb{R})$ by $\Pi_{12}(x_1 x_2 x_3) = (x_1 x_2)$ and we define $\Pi_1 : \mathcal{M}^{3\times 2}(\mathbb{R}) \to \mathcal{M}^{3\times 1}(\mathbb{R})$ by $\Pi_1(x_1 x_2) = x_1$.

We use standard superscript notation to denote slices: For $x_1 \in \mathcal{M}^{3\times 1}(\mathbb{R})$ we define $E^{x_1} := \{(x_2 x_3)\ (x_1 x_2 x_3) \in E\}$; for $(x_1 x_2) \in \mathcal{M}^{3\times 2}(\mathbb{R})$ we define $E^{x_1 x_2} := \{x_3 : (x_1 x_2 x_3) \in E\}$; and for $x_1 \in \mathcal{M}^{3\times 1}(\mathbb{R})$ and $H \subseteq \mathcal{M}^{3\times 2}(\mathbb{R})$ we define $H^{x_1} := \{x_2 : (x_1 x_2) \in H\}$.

Since $|E| = \int_{\Pi_{12}(E)} |E^{x_1 x_2}| \mathrm{d}x_1 \mathrm{d}x_2$ there is an $(\overline{x_1 x_2}) \in \Pi_{12}(E)$ such that

$$|E^{\overline{x_1 x_2}}||\Pi_{12}(E)| \geq |E|.$$

Similarly there is a $\overline{y_1} \in \Pi_1(\Pi_{12}(F))$ such that

$$|(\Pi_{12}F)^{\overline{y_1}}||\Pi_1(\Pi_{12}(F))| \geq |\Pi_{12}(F)|.$$

Ting Chen's argument – I

Then for all $x_3 \in E^{\overline{x_1 x_2}}$, for all $(y_2 y_3) \in F^{\overline{y_1}}$ and all $Z \in G$ we have

$$|\det(\overline{x_1} + \overline{y_1} + z_1 \;\; \overline{x_2} + y_2 + z_2 \;\; x_3 + y_3 + z_3)| \le s,$$

i.e. for all $y_2 \in \Pi_{12}(F)^{\overline{y_1}}$, all $y_3 \in F^{\overline{y_1} y_2}$, and all $Z \in G$

$$\sup_{x_3 \in E^{\overline{x_1 x_2}} + y_3 + z_3} |\det(\overline{x_1} + \overline{y_1} + z_1 \;\; \overline{x_2} + y_2 + z_2 \;\; x_3)| \le s.$$

We now pretend that $E^{\overline{x_1 x_2}}$ – and hence its translate by $y_3 + z_3$ – is a set of product form in $\mathbb{R}^3 = \mathcal{M}^{3\times 1}(\mathbb{R})$.

By the simple lemma we then have, for all $y_2 \in (\Pi_{12} F)^{\overline{y_1}}$ and all $(z_1 z_2) \in \Pi_{12} G$,

$$\sup_{x_3 \in \mathcal{S}(E^{\overline{x_1 x_2}})} |\det(\overline{x_1} + \overline{y_1} + z_1 \;\; \overline{x_2} + y_2 + z_2 \;\; x_3)| \le s.$$

Ting Chen's argument – II

For all $x_3 \in \mathcal{S}(E^{\overline{x_1 x_2}})$, all $y_2 \in (\Pi_{12} F)^{\overline{y_1}}$ and all $(z_1 z_2) \in \Pi_{12} G$,

$$|\det(\overline{x_1} + \overline{y_1} + z_1 \;\; \overline{x_2} + y_2 + z_2 \;\; x_3)| \leq s,$$

i.e. for all $x_3 \in \mathcal{S}(E^{\overline{x_1 x_2}})$, all $z_1 \in \Pi_1(\Pi_{12} G)$ and all $z_2 \in \Pi_{12}(G)^{z_1}$,

$$\sup_{y_2 \in (\Pi_{12} F)^{\overline{y_1}} + \overline{x_2} + z_2} |\det(\overline{x_1} + \overline{y_1} + z_1 \;\; y_2 \;\; x_3)| \leq s.$$

We now pretend that $(\Pi_{12} F)^{\overline{y_1}}$ – and hence its translate by $\overline{x_2} + z_2$ – is a set of product form in $\mathbb{R}^3 = \mathcal{M}^{3\times 1}(\mathbb{R})$.

By the simple lemma we therefore have, for all $x_3 \in \mathcal{S}(E^{\overline{x_1 x_2}})$, and all $z_1 \in \Pi_1(\Pi_{12} G)$,

$$\sup_{y_2 \in \mathcal{S}(\Pi_{12} F)^{\overline{y_1}}} |\det(\overline{x_1} + \overline{y_1} + z_1 \;\; y_2 \;\; x_3)| \leq s.$$

Ting Chen's argument – III

$$\sup_{z_1 \in \Pi_1(\Pi_{12} G)} \; \sup_{y_2 \in \mathcal{S}(\Pi_{12} F)^{\overline{y_1}}} \; \sup_{x_3 \in \mathcal{S}(E^{\overline{x_1 x_2}})} |\det(\overline{x_1} + \overline{y_1} + z_1 \;\; y_2 \;\; x_3)| \leq s.$$

By the trilinear measure estimate in $\mathbb{R}^3$, we therefore have

$$|\Pi_1(\Pi_{12} G)|^{1/3} |\mathcal{S}(\Pi_{12} F)^{\overline{y_1}}|^{1/3} |\mathcal{S}(E^{\overline{x_1 x_2}})|^{1/3} \lesssim s,$$

or

$$|\Pi_1(\Pi_{12} G)|^{1/3} |(\Pi_{12} F)^{\overline{y_1}}|^{1/3} |E^{\overline{x_1 x_2}}|^{1/3} \lesssim s.$$

(BOARD!)

This, together with the slicing inequalities

$$|E^{\overline{x_1 x_2}}| |\Pi_{12}(E)| \geq |E|$$

$$|(\Pi_{12} F)^{\overline{y_1}}| |\Pi_1(\Pi_{12}(F))| \geq |\Pi_{12}(F)|$$

is what we need to proceed.

Ting Chen's argument – IV

For certain $\overline{x_1}, \overline{x_2}$ and $\overline{y_1}$ we have

$$|\Pi_1(\Pi_{12}G)|^{1/3}|(\Pi_{12}F)^{\overline{y_1}}|^{1/3}|E^{\overline{x_1}\,\overline{x_2}}|^{1/3} \lesssim s$$

$$|E^{\overline{x_1}\,\overline{x_2}}||\Pi_{12}(E)| \geq |E| \text{ and } |(\Pi_{12}F)^{\overline{y_1}}||\Pi_1(\Pi_{12}(F))| \geq |\Pi_{12}(F)|$$

By repeating the same arguments starting with F and G respectively we get that for certain y_1^*, y_2^* and z_1^*

$$|\Pi_1(\Pi_{12}E)|^{1/3}|(\Pi_{12}G)^{z_1^*}|^{1/3}|F^{y_1^* y_2^*}|^{1/3} \lesssim s$$

$$|F^{y_1^* y_2^*}||\Pi_{12}(F)| \geq |F| \text{ and } |(\Pi_{12}G)^{z_1^*}||\Pi_1(\Pi_{12}(G))| \geq |\Pi_{12}(G)|$$

and for certain $\widehat{z_1}, \widehat{z_2}$ and $\widehat{x_1}$,

$$|\Pi_1(\Pi_{12}F)|^{1/3}|(\Pi_{12}E)^{\widehat{x_1}}|^{1/3}|G^{\widehat{z_1}\,\widehat{z_2}}|^{1/3} \lesssim s$$

$$|G^{\widehat{z_1}\,\widehat{z_2}}||\Pi_{12}(G)| \geq |G| \text{ and } |(\Pi_{12}E)^{\widehat{x_1}}||\Pi_1(\Pi_{12}(E))| \geq |\Pi_{12}(E)|.$$

Ting Chen's argument – Conclusion

Combining the blue inequalities $|G^{\widehat{z_1}\widehat{z_2}}||\Pi_{12}(G)| \geq |G|$ and $|(\Pi_{12}G)^{z_1^*}||\Pi_1(\Pi_{12}(G))| \geq |\Pi_{12}(G)|$ we get

$$|G| \leq |G^{\widehat{z_1}\widehat{z_2}}||(\Pi_{12}G)^{z_1^*}||\Pi_1(\Pi_{12}(G))|,$$

and similarly

$$|E| \leq |E^{\overline{x_1 x_2}}||(\Pi_{12}E)^{\widehat{x_1}}||\Pi_1(\Pi_{12}E)|$$

$$|F| \leq |F^{y_1^* y_2^*}||(\Pi_{12}F)^{\overline{y_1}}||\Pi_1(\Pi_{12}F)|.$$

Taking the geometric mean of the inequalities

$$|\Pi_1(\Pi_{12}G)|^{1/3}|(\Pi_{12}F)^{\overline{y_1}}|^{1/3}|E^{\overline{x_1 x_2}}|^{1/3} \lesssim s$$

$$|\Pi_1(\Pi_{12}E)|^{1/3}|(\Pi_{12}G)^{z_1^*}|^{1/3}|F^{y_1^* y_2^*}|^{1/3} \lesssim s$$

$$|\Pi_1(\Pi_{12}F)|^{1/3}|(\Pi_{12}E)^{\widehat{x_1}}|^{1/3}|G^{\widehat{z_1}\widehat{z_2}}|^{1/3} \lesssim s$$

and applying the previous displayed inequalities yields the result.

Ting Chen's argument – Succesive renormalisation

We still have to deal with the blatant lies.

And so far we have not used the advertised invariance of the problem.

We now exploit this invariance to renormalise multiple times ($3 \times 2 = 6$ times in the $n = 3$ case, and $n(n-1)$ in the general case) to deal with the lies. The invariance enters as a *catalyst* yielding measure theoretic consequences, and its presence vanishes without trace.

We do not use the invariance of the entire problem under the action of left-multiplication by members of $SL(n, \mathbb{R})$, but instead the facts which underly this invariance, i.e. that the action preserves determinants of individual matrices, and preserves volumes of sets.

Renormalisation details – I

"For all $y_2 \in \Pi_{12}(F)^{\overline{y_1}}$, all $y_3 \in F^{\overline{y_1 y_2}}$, and all $Z \in G$

$$\sup_{x_3 \in E^{\overline{x_1 x_2}} + y_3 + z_3} |\det(\overline{x_1} + \overline{y_1} + z_1 \;\; \overline{x_2} + y_2 + z_2 \;\; x_3)| \leq s.$$

We now pretend that $E^{\overline{x_1 x_2}}$ – and hence its translate by $y_3 + z_3$ – is a set of product form in $\mathbb{R}^3 = \mathcal{M}^{3\times 1}(\mathbb{R})$."

Of course we can do no such thing – $E^{\overline{x_1 x_2}}$ is an essentially arbitrary subset of $\mathbb{R}^3 = \mathcal{M}^{3\times 1}(\mathbb{R})$.

But, due to linearity of the determinant in each column, and hence convexity of $|\det|$ in each column, we do have:
For all $y_2 \in \Pi_{12}(F)^{\overline{y_1}}$, all $y_3 \in F^{\overline{y_1 y_2}}$, and all $Z \in G$

$$\sup_{x_3 \in \mathrm{co}\, E^{\overline{x_1 x_2}} + y_3 + z_3} |\det(\overline{x_1} + \overline{y_1} + z_1 \;\; \overline{x_2} + y_2 + z_2 \;\; x_3)| \leq s.$$

We now introduce a $T \in SL(3, \mathbb{R})$ such that $T \mathrm{co}\, E^{\overline{x_1 x_2}}$ contains an axis-parallel rectangle and is contained in a similar axis-parallel rectangle all with comparable volumes (cf. John ellipsoid).

Renormalistion details – II

For all $y_2 \in \Pi_{12}(F)^{\overline{y_1}}$, all $y_3 \in F^{\overline{y_1} y_2}$, and all $Z \in G$

$$\sup_{x_3 \in \mathrm{co}\, E^{\overline{x_1 x_2}} + y_3 + z_3} |\det(\overline{x_1} + \overline{y_1} + z_1 \;\; \overline{x_2} + y_2 + z_2 \;\; x_3)| \leq s$$

is the same as: for all $y_2 \in \Pi_{12}(F)^{\overline{y_1}}$, all $y_3 \in F^{\overline{y_1} y_2}$, and all $Z \in G$

$$\sup_{x_3 \in \mathrm{co}\, E^{\overline{x_1 x_2}} + y_3 + z_3} |\det(T\overline{x_1} + T\overline{y_1} + Tz_1 \;\; T\overline{x_2} + Ty_2 + Tz_2 \;\; Tx_3)| \leq s,$$

i.e.

$$\sup_{x_3 \in T\mathrm{co}\, E^{\overline{x_1 x_2}} + Ty_3 + Tz_3} |\det(T\overline{x_1} + T\overline{y_1} + Tz_1 \;\; T\overline{x_2} + Ty_2 + Tz_2 \;\; x_3)| \leq s.$$

By the simple lemma (since $T\mathrm{co}\, E^{\overline{x_1 x_2}}$ is essentially an axis parallel rectangle) we therefore have, for all $y_2 \in (\Pi_{12} TF)^{\overline{Ty_1}}$ and all $(z_1 z_2) \in \Pi_{12} TG$,

$$\sup_{x_3 \in \mathcal{S}(T\mathrm{co}\, E^{\overline{x_1 x_2}})} |\det(T\overline{x_1} + T\overline{y_1} + z_1 \;\; T\overline{x_2} + y_2 + z_2 \;\; x_3)| \leq s.$$

Renormalisation details – III

Now we have: for all $x_3 \in \mathcal{S}(T\mathrm{co}\, E^{\overline{x_1 x_2}})$, for all $y_2 \in (\Pi_{12} TF)^{T\overline{y_1}}$ and all $(z_1 z_2) \in \Pi_{12} TG$,

$$|\det(T\overline{x_1} + T\overline{y_1} + z_1 \;\; T\overline{x_2} + y_2 + z_2 \;\; x_3)| \leq s.$$

(Reality check: previously, in the cheating version, we had: For all $x_3 \in \mathcal{S}(E^{\overline{x_1 x_2}})$, all $y_2 \in (\Pi_{12} F)^{\overline{y_1}}$ and all $(z_1 z_2) \in \Pi_{12} G$,

$$|\det(\overline{x_1} + \overline{y_1} + z_1 \;\; \overline{x_2} + y_2 + z_2 \;\; x_3)| \leq s.)$$

Proceed as we did before: for all $x_3 \in \mathcal{S}(T\mathrm{co}\, E^{\overline{x_1 x_2}})$, all $z_1 \in \Pi_1(\Pi_{12} TG)$ and all $z_2 \in \Pi_{12}(TG)^{z_1}$,

$$\sup_{y_2 \in (\Pi_{12} TF)^{T\overline{y_1}} + T\overline{x_2} + z_2} |\det(T\overline{x_1} + T\overline{y_1} + z_1 \;\; y_2 \;\; x_3)| \leq s.$$

Renormalisation details – IV

For all $x_3 \in \mathcal{S}(T \operatorname{co} E^{\overline{x_1 x_2}})$, all $z_1 \in \Pi_1(\Pi_{12} TG)$ and all $z_2 \in \Pi_{12}(TG)^{z_1}$,

$$\sup_{y_2 \in (\Pi_{12} TF)^{T\overline{y_1}} + T\overline{x_2} + z_2} |\det(T\overline{x_1} + T\overline{y_1} + z_1 \;\; y_2 \;\; x_3)| \leq s.$$

As before, the same inequality persists if we pass to the convex hull of $(\Pi_{12} TF)^{T\overline{y_1}} + T\overline{x_2} + z_2$, and as before we choose an $S \in SL(3, \mathbb{R})$ such that $S\operatorname{co}(\Pi_{12} TF)^{T\overline{y_1}}$ is essentially an axis-parallel rectangle in $\mathbb{R}^3$ with comparable volume. So, under the same conditions on x_3, z_1 and z_2,

$$\sup_{y_2 \in \operatorname{co}(\Pi_{12} TF)^{T\overline{y_1}} + T\overline{x_2} + z_2} |\det(ST\overline{x_1} + ST\overline{y_1} + Sz_1 \;\; Sy_2 \;\; Sx_3)| \leq s,$$

i.e. for $x_3 \in S\mathcal{S}(T \operatorname{co} E^{\overline{x_1 x_2}})$, $z_1 \in \Pi_1(\Pi_{12} STG)$, $z_2 \in \Pi_{12}(STG)^{Sz_1}$ and $y_2 \in \operatorname{co} S(\Pi_{12} TF)^{T\overline{y_1}} + ST\overline{x_2} + Sz_2$,

$$|\det(ST\overline{x_1} + ST\overline{y_1} + z_1 \;\; y_2 \;\; x_3)| \leq s.$$

Renormalisation details – V

For $x_3 \in S\mathcal{S}(T\mathrm{co}\, E^{\overline{x_1 x_2}})$, $z_1 \in \Pi_1(\Pi_{12}STG)$, $z_2 \in \Pi_{12}(STG)^{Sz_1}$ and $y_2 \in \mathrm{co}S(\Pi_{12}TF)^{T\overline{y_1}} + ST\overline{x_2} + Sz_2$,

$$|\det(ST\overline{x_1} + ST\overline{y_1} + z_1 \;\; y_2 \;\; x_3)| \leq s.$$

By the simple lemma once more we therefore have

$$|\det(ST\overline{x_1} + ST\overline{y_1} + z_1 \;\; y_2 \;\; x_3)| \leq s$$

for $x_3 \in S\mathcal{S}(T\mathrm{co}\, E^{\overline{x_1 x_2}})$, $y_2 \in \mathcal{S}\mathrm{co}S(\Pi_{12}TF)^{T\overline{y_1}}$, $z_1 \in \Pi_1(\Pi_{12}STG)$.
By the measure estimate we therefore have

$$|\Pi_1(\Pi_{12}STG)|^{1/3}|\mathcal{S}\mathrm{co}S(\Pi_{12}TF)^{T\overline{y_1}}|^{1/3}|S\mathcal{S}(T\mathrm{co}\, E^{\overline{x_1 x_2}})|^{1/3} \leq s.$$

Since S, T and $\mathcal{S}$ preserve volumes this is the same as

$$|\Pi_1(\Pi_{12}G)|^{1/3}|\mathrm{co}(\Pi_{12}F)^{\overline{y_1}}|^{1/3}|\mathrm{co}\,(E^{\overline{x_1 x_2}})|^{1/3} \leq s,$$

which clearly implies

$$|\Pi_1(\Pi_{12}G)|^{1/3}|(\Pi_{12}F)^{\overline{y_1}}|^{1/3}|E^{\overline{x_1 x_2}}|^{1/3} \leq s,$$

thus closing the circle.

For full details see the upcoming PhD thesis of Chen Ting, University of Edinburgh, 2016, and look out for her article on the ArXiv.

Finally...

HAPPY BIRTHDAY MIKE!!