Weighted inequalities for oscillatory integrals

Jonathan Bennett

U. Birmingham

19 May 2016

Conference in honour of Michael Christ, Madison 2016

Supported by ERC grant 307617.

Plan

Part 1: Inequalities with *general* weights.

(Recent work with David Beltran.)

Plan

Part 1: Inequalities with *general* weights.

(Recent work with David Beltran.)

Part 2: Some inequalities with *specific* weights.

(Recent work with Neal Bez, Susana Gutierrez, Taryn Flock and Marina Iliopoulou.)

General-weighted inequalities: the broad setting

General-weighted inequalities: the broad setting

Object: Given an operator T, identify a meaningful "geometrically-defined" operator $\mathcal{M}$ for which

$$\int |Tf|^2 w \lesssim \int |f|^2 \mathcal{M} w \qquad (\dagger)$$

holds for *all* weight functions w.

General-weighted inequalities: the broad setting

Object: Given an operator T, identify a meaningful "geometrically-defined" operator $\mathcal{M}$ for which

$$\int |Tf|^2 w \lesssim \int |f|^2 \mathcal{M} w \tag{$\dagger$}$$

holds for *all* weight functions w. (A classical question raised by many – see, for example, C. Fefferman and Stein 1971, or A. Córdoba and C. Fefferman 1976.)

General-weighted inequalities: the broad setting

Object: Given an operator T, identify a meaningful "geometrically-defined" operator $\mathcal{M}$ for which

$$\int |Tf|^2 w \lesssim \int |f|^2 \mathcal{M}w \qquad (\dagger)$$

holds for *all* weight functions w. (A classical question raised by many – see, for example, C. Fefferman and Stein 1971, or A. Córdoba and C. Fefferman 1976.)

Very much like a *pointwise* estimate

General-weighted inequalities: the broad setting

Object: Given an operator T, identify a meaningful "geometrically-defined" operator $\mathcal{M}$ for which

$$\int |Tf|^2 w \lesssim \int |f|^2 \mathcal{M} w \qquad (\dagger)$$

holds for *all* weight functions w. (A classical question raised by many – see, for example, C. Fefferman and Stein 1971, or A. Córdoba and C. Fefferman 1976.)

Very much like a *pointwise* estimate – *a local norm estimate at all scales*: for example, taking $w = \chi_{B(x,r)}$ we have

$$\int_{B(x,r)} |Tf|^2 \lesssim \int |f|^2 \mathcal{M}\chi_{B(x,r)} \cdots$$

General-weighted inequalities: the broad setting

Object: Given an operator T, identify a meaningful "geometrically-defined" operator $\mathcal{M}$ for which

$$\int |Tf|^2 w \lesssim \int |f|^2 \mathcal{M} w \qquad (\dagger)$$

holds for *all* weight functions w. (A classical question raised by many – see, for example, C. Fefferman and Stein 1971, or A. Córdoba and C. Fefferman 1976.)

Very much like a *pointwise* estimate – *a local norm estimate at all scales*: for example, taking $w = \chi_{B(x,r)}$ we have

$$\int_{B(x,r)} |Tf|^2 \lesssim \int |f|^2 \mathcal{M}\chi_{B(x,r)} \cdots$$

(Recalls pointwise estimates via g-functions, sharp maximal functions etc.)

General-weighted inequalities: the broad setting

Object: Given an operator T, identify a meaningful "geometrically-defined" operator $\mathcal{M}$ for which

$$\int |Tf|^2 w \lesssim \int |f|^2 \mathcal{M} w \qquad (\dagger)$$

holds for *all* weight functions w. (A classical question raised by many – see, for example, C. Fefferman and Stein 1971, or A. Córdoba and C. Fefferman 1976.)

Very much like a *pointwise* estimate – *a local norm estimate at all scales*: for example, taking $w = \chi_{B(x,r)}$ we have

$$\int_{B(x,r)} |Tf|^2 \lesssim \int |f|^2 \mathcal{M}\chi_{B(x,r)} \cdots$$

(Recalls pointwise estimates via g-functions, sharp maximal functions etc.)

Interested in finding $\mathcal{M}$ with

(1) *purely geometric character,*

General-weighted inequalities: the broad setting

Object: Given an operator T, identify a meaningful "geometrically-defined" operator $\mathcal{M}$ for which

$$\int |Tf|^2 w \lesssim \int |f|^2 \mathcal{M} w \qquad (\dagger)$$

holds for *all* weight functions w. (A classical question raised by many – see, for example, C. Fefferman and Stein 1971, or A. Córdoba and C. Fefferman 1976.)

Very much like a *pointwise* estimate – *a local norm estimate at all scales*: for example, taking $w = \chi_{B(x,r)}$ we have

$$\int_{B(x,r)} |Tf|^2 \lesssim \int |f|^2 \mathcal{M} \chi_{B(x,r)} \cdots$$

(Recalls pointwise estimates via g-functions, sharp maximal functions etc.)

Interested in finding $\mathcal{M}$ with

(1) *purely geometric character*, and

(2) *optimal bounds*, in the sense that the resulting inequality $\|T\|_{p-q} \lesssim \|\mathcal{M}\|_{(q/2)'-(p/2)'}^{1/2}$ is optimal in $p, q \geq 2$.

General-weighted inequalities: the broad setting

Object: Given an operator T, identify a meaningful "geometrically-defined" operator $\mathcal{M}$ for which

$$\int |Tf|^2 w \lesssim \int |f|^2 \mathcal{M}w \qquad (\dagger)$$

holds for *all* weight functions w. (A classical question raised by many – see, for example, C. Fefferman and Stein 1971, or A. Córdoba and C. Fefferman 1976.)

Very much like a *pointwise* estimate – *a local norm estimate at all scales*: for example, taking $w = \chi_{B(x,r)}$ we have

$$\int_{B(x,r)} |Tf|^2 \lesssim \int |f|^2 \mathcal{M}\chi_{B(x,r)} \cdots$$

(Recalls pointwise estimates via g-functions, sharp maximal functions etc.)

Interested in finding $\mathcal{M}$ with

(1) *purely geometric character*, and

(2) *optimal bounds*, in the sense that the resulting inequality $\|T\|_{p-q} \lesssim \|\mathcal{M}\|^{1/2}_{(q/2)'-(p/2)'}$ is optimal in $p, q \geq 2$.

A variety of well-known results in the realm of *Calderón–Zygmund theory*, *involving the Hardy–Littlewood maximal operator* (Fefferman–Stein 1971, Córdoba–Fefferman 1976, Wilson 1985, Pérez 1995...)

Classical examples pertain to the classical Calderón–Zygmund theory

Classical examples pertain to the classical Calderón–Zygmund theory – the operators T are either

- *positive* – such as the Hardy–Littlewood maximal function or fractional integrals,

Classical examples pertain to the classical Calderón–Zygmund theory – the operators T are either

- *positive* – such as the Hardy–Littlewood maximal function or fractional integrals,
- or cancellative but with *critical homogeneity* (in terms of kernel integrability) – such as the Hilbert transform.

Classical examples pertain to the classical Calderón–Zygmund theory – the operators T are either

- *positive* – such as the Hardy–Littlewood maximal function or fractional integrals,
- or cancellative but with *critical homogeneity* (in terms of kernel integrability) – such as the Hilbert transform.

Can anything sensible be said about operators of the form

$$Tf(x) = \int e^{i\Phi(x,y)} a(x,y) f(y) dy \quad ?$$

– that is, with kernels that are *oscillatory*?

Classical examples pertain to the classical Calderón–Zygmund theory – the operators T are either

- *positive* – such as the Hardy–Littlewood maximal function or fractional integrals,
- or cancellative but with *critical homogeneity* (in terms of kernel integrability) – such as the Hilbert transform.

Can anything sensible be said about operators of the form

$$Tf(x) = \int e^{i\Phi(x,y)} a(x,y) f(y) dy \quad ?$$

– that is, with kernels that are *oscillatory*?

Important examples:

- The Fourier extension operator

$$Tg(x) = \widehat{g d\sigma}(x) := \int_{\mathbb{S}^{d-1}} e^{ix\cdot\xi} g(\xi) d\sigma(\xi) \qquad ?$$

Classical examples pertain to the classical Calderón–Zygmund theory – the operators T are either

- *positive* – such as the Hardy–Littlewood maximal function or fractional integrals,
- or cancellative but with *critical homogeneity* (in terms of kernel integrability) – such as the Hilbert transform.

Can anything sensible be said about operators of the form

$$Tf(x) = \int e^{i\Phi(x,y)} a(x,y) f(y) dy \quad ?$$

– that is, with kernels that are *oscillatory*?

Important examples:

- The Fourier extension operator

$$Tg(x) = \widehat{gd\sigma}(x) := \int_{\mathbb{S}^{d-1}} e^{ix\cdot\xi} g(\xi) d\sigma(\xi) \qquad ?$$

- The disc multiplier

$$\widehat{Tf}(\xi) = \chi_{B(0,1)}(\xi)\widehat{f}(\xi) \qquad ?$$

Classical examples pertain to the classical Calderón–Zygmund theory – the operators T are either

- *positive* – such as the Hardy–Littlewood maximal function or fractional integrals,
- or cancellative but with *critical homogeneity* (in terms of kernel integrability) – such as the Hilbert transform.

Can anything sensible be said about operators of the form

$$Tf(x) = \int e^{i\Phi(x,y)} a(x,y) f(y) dy \quad ?$$

– that is, with kernels that are *oscillatory*?

Important examples:

- The Fourier extension operator

$$Tg(x) = \widehat{gd\sigma}(x) := \int_{\mathbb{S}^{d-1}} e^{ix\cdot\xi} g(\xi) d\sigma(\xi) \qquad ?$$

- The disc multiplier

$$\widehat{Tf}(\xi) = \chi_{B(0,1)}(\xi)\widehat{f}(\xi) \qquad ?$$

Here $Tf = K * f$, where

$$K(x) := \mathcal{F}^{-1}(\chi_{B(0,1)})(x) = \frac{cJ_{d/2}(2\pi|x|)}{|x|^{\frac{d}{2}}} = c\frac{e^{2\pi i|x|} + e^{-2\pi i|x|} + o(1)}{|x|^{\frac{d+1}{2}}}$$

Classical examples pertain to the classical Calderón–Zygmund theory – the operators T are either

- *positive* – such as the Hardy–Littlewood maximal function or fractional integrals,
- or cancellative but with *critical homogeneity* (in terms of kernel integrability) – such as the Hilbert transform.

Can anything sensible be said about operators of the form

$$Tf(x) = \int e^{i\Phi(x,y)} a(x,y) f(y) dy \quad ?$$

– that is, with kernels that are *oscillatory*?

Important examples:

- The Fourier extension operator

$$Tg(x) = \widehat{gd\sigma}(x) := \int_{\mathbb{S}^{d-1}} e^{ix\cdot\xi} g(\xi) d\sigma(\xi) \qquad ?$$

- The disc multiplier

$$\widehat{Tf}(\xi) = \chi_{B(0,1)}(\xi)\widehat{f}(\xi) \qquad ?$$

  Here $Tf = K * f$, where

$$K(x) := \mathcal{F}^{-1}(\chi_{B(0,1)})(x) = \frac{cJ_{d/2}(2\pi|x|)}{|x|^{\frac{d}{2}}} = c\frac{e^{2\pi i|x|} + e^{-2\pi i|x|} + o(1)}{|x|^{\frac{d+1}{2}}}$$

  – something **far from integrable** for $d \geq 2$.

Classical examples pertain to the classical Calderón–Zygmund theory – the operators T are either

- *positive* – such as the Hardy–Littlewood maximal function or fractional integrals,
- or cancellative but with *critical homogeneity* (in terms of kernel integrability) – such as the Hilbert transform.

Can anything sensible be said about operators of the form

$$Tf(x) = \int e^{i\Phi(x,y)} a(x,y) f(y) dy \quad ?$$

– that is, with kernels that are *oscillatory*?

Important examples:

- The Fourier extension operator

$$Tg(x) = \widehat{gd\sigma}(x) := \int_{\mathbb{S}^{d-1}} e^{ix\cdot\xi} g(\xi) d\sigma(\xi) \qquad ?$$

- The disc multiplier

$$\widehat{Tf}(\xi) = \chi_{B(0,1)}(\xi)\widehat{f}(\xi) \qquad ?$$

  Here $Tf = K * f$, where

$$K(x) := \mathcal{F}^{-1}(\chi_{B(0,1)})(x) = \frac{cJ_{d/2}(2\pi|x|)}{|x|^{\frac{d}{2}}} = c\frac{e^{2\pi i|x|} + e^{-2\pi i|x|} + o(1)}{|x|^{\frac{d+1}{2}}}$$

  – something **far from integrable** for $d \geq 2$.

What might $\mathcal{M}$ look like in general oscillatory contexts?

Given an oscillatory integral operator

$$Tf(x) = \int e^{i\Phi(x,y)} a(x,y) f(y) dy,$$

how might we go about identifying a controlling maximal operator $\mathcal{M}$?

What might $\mathcal{M}$ look like in general oscillatory contexts?

Given an oscillatory integral operator

$$Tf(x) = \int e^{i\Phi(x,y)} a(x,y) f(y) dy,$$

how might we go about identifying a controlling maximal operator $\mathcal{M}$?

Using *sublevel set operators* a la Carbery–Christ–Wright maybe?

What might $\mathcal{M}$ look like in general oscillatory contexts?

Given an oscillatory integral operator

$$Tf(x) = \int e^{i\Phi(x,y)} a(x,y) f(y) dy,$$

how might we go about identifying a controlling maximal operator $\mathcal{M}$?

Using *sublevel set operators* a la Carbery–Christ–Wright maybe?

Let

$$Sf(x) = \int_{\{y:|\Phi(x,y)|\leq 1\}} a(x,y) f(y) dy,$$

or, more generally,

$$S_{\psi,\phi} f(x) = \int_{\{y:|\Phi(x,y)-\psi(x)-\phi(y)|\leq 1\}} a(x,y) f(y) dy$$

for measurable functions ϕ, ψ, and look for its controlling maximal functions...

What might $\mathcal{M}$ look like in general oscillatory contexts?

Given an oscillatory integral operator

$$Tf(x) = \int e^{i\Phi(x,y)} a(x,y) f(y) dy,$$

how might we go about identifying a controlling maximal operator $\mathcal{M}$?

Using *sublevel set operators* a la Carbery–Christ–Wright maybe?

Let

$$Sf(x) = \int_{\{y : |\Phi(x,y)| \leq 1\}} a(x,y) f(y) dy,$$

or, more generally,

$$S_{\psi,\phi} f(x) = \int_{\{y : |\Phi(x,y) - \psi(x) - \phi(y)| \leq 1\}} a(x,y) f(y) dy$$

for measurable functions ϕ, ψ, and look for its controlling maximal functions...

Complicated

What might $\mathcal{M}$ look like in general oscillatory contexts?

Given an oscillatory integral operator

$$Tf(x) = \int e^{i\Phi(x,y)} a(x,y) f(y) dy,$$

how might we go about identifying a controlling maximal operator $\mathcal{M}$?

Using *sublevel set operators* a la Carbery–Christ–Wright maybe?

Let

$$Sf(x) = \int_{\{y : |\Phi(x,y)| \leq 1\}} a(x,y) f(y) dy,$$

or, more generally,

$$S_{\psi,\phi} f(x) = \int_{\{y : |\Phi(x,y) - \psi(x) - \phi(y)| \leq 1\}} a(x,y) f(y) dy$$

for measurable functions ϕ, ψ, and look for its controlling maximal functions...

Complicated ... but in some specific contexts this reveals highly non-local maximal operators, sometimes involving *tubes* or *wide approach regions*.

Conjectural example 1: the Fourier extension operator

Recall Stein's restriction conjecture:

$$\|\widehat{gd\sigma}\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{S}^{d-1})}; \qquad \tfrac{1}{q} < \tfrac{d-1}{2d}, \tfrac{1}{q} \leq \tfrac{d-1}{d+1}\tfrac{1}{p'}.$$

Conjectural example 1: the Fourier extension operator

Recall Stein's restriction conjecture:

$$\|\widehat{gd\sigma}\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{S}^{d-1})}; \qquad \tfrac{1}{q} < \tfrac{d-1}{2d}, \tfrac{1}{q} \leq \tfrac{d-1}{d+1}\tfrac{1}{p'}.$$

A virtually equivalent formulation at the (missing) endpoint:

$$\|\widehat{gd\sigma}\|_{L^{\frac{2d}{d-1}}(B(0,R))} \lesssim_\epsilon R^\epsilon \|g\|_{L^{\frac{2d}{d-1}}(\mathbb{S}^{d-1})}; \qquad \epsilon > 0, R \gg 1.$$

Conjectural example 1: the Fourier extension operator

Recall Stein's restriction conjecture:

$$\|\widehat{gd\sigma}\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{S}^{d-1})}; \qquad \tfrac{1}{q} < \tfrac{d-1}{2d}, \tfrac{1}{q} \leq \tfrac{d-1}{d+1}\tfrac{1}{p'}.$$

A virtually equivalent formulation at the (missing) endpoint:

$$\|\widehat{gd\sigma}\|_{L^{\frac{2d}{d-1}}(B(0,R))} \lesssim_\epsilon R^\epsilon \|g\|_{L^{\frac{2d}{d-1}}(\mathbb{S}^{d-1})}; \qquad \epsilon > 0, R \gg 1.$$

One might hope for an inequality of the form

$$\int_{B(0,1)} |\widehat{gd\sigma}(R\xi)|^2 w(\xi) d\xi \lesssim \frac{1}{R^{d-1}} \int_{\mathbb{S}^{d-1}} |g|^2 \mathcal{M}_R w,$$

where $\mathcal{M}_R$ is some variant of the *Kakeya maximal operator*

$$\mathcal{K}_R w(\omega) := \sup_{T||\omega} \frac{1}{|T|} \int_T w;$$

here the supremum is taken over all R^{-1}-tubes T in $B(0,1) \subseteq \mathbb{R}^d$ with direction ω.

Conjectural example 1: the Fourier extension operator

Recall Stein's restriction conjecture:

$$\|\widehat{gd\sigma}\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{S}^{d-1})}; \qquad \tfrac{1}{q} < \tfrac{d-1}{2d}, \tfrac{1}{q} \leq \tfrac{d-1}{d+1}\tfrac{1}{p'}.$$

A virtually equivalent formulation at the (missing) endpoint:

$$\|\widehat{gd\sigma}\|_{L^{\frac{2d}{d-1}}(B(0,R))} \lesssim_\epsilon R^\epsilon \|g\|_{L^{\frac{2d}{d-1}}(\mathbb{S}^{d-1})}; \qquad \epsilon > 0, R \gg 1.$$

One might hope for an inequality of the form

$$\int_{B(0,1)} |\widehat{gd\sigma}(R\xi)|^2 w(\xi) d\xi \lesssim \frac{1}{R^{d-1}} \int_{\mathbb{S}^{d-1}} |g|^2 \mathcal{M}_R w,$$

where $\mathcal{M}_R$ is some variant of the *Kakeya maximal operator*

$$\mathcal{K}_R w(\omega) := \sup_{T || \omega} \frac{1}{|T|} \int_T w;$$

here the supremum is taken over all R^{-1}-tubes T in $B(0,1) \subseteq \mathbb{R}^d$ with direction ω.
If true then

$$\|\widehat{gd\sigma}\|_{L^{\frac{2d}{d-1}}(\mathbb{S}^{d-1}) \to L^{\frac{2d}{d-1}}(B(0,R))} \lesssim \|\mathcal{M}_R\|^{1/2}_{L^d(\mathbb{R}^d) \to L^d(\mathbb{S}^{d-1})};$$

Conjectural example 1: the Fourier extension operator

Recall Stein's restriction conjecture:

$$\|\widehat{gd\sigma}\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{S}^{d-1})}; \quad \tfrac{1}{q} < \tfrac{d-1}{2d}, \tfrac{1}{q} \leq \tfrac{d-1}{d+1}\tfrac{1}{p'}.$$

A virtually equivalent formulation at the (missing) endpoint:

$$\|\widehat{gd\sigma}\|_{L^{\frac{2d}{d-1}}(B(0,R))} \lesssim_\epsilon R^\epsilon \|g\|_{L^{\frac{2d}{d-1}}(\mathbb{S}^{d-1})}; \quad \epsilon > 0, R \gg 1.$$

One might hope for an inequality of the form

$$\int_{B(0,1)} |\widehat{gd\sigma}(R\xi)|^2 w(\xi)d\xi \lesssim \frac{1}{R^{d-1}} \int_{\mathbb{S}^{d-1}} |g|^2 \mathcal{M}_R w,$$

where $\mathcal{M}_R$ is some variant of the *Kakeya maximal operator*

$$\mathcal{K}_R w(\omega) := \sup_{T||\omega} \frac{1}{|T|} \int_T w;$$

here the supremum is taken over all R^{-1}-tubes T in $B(0,1) \subseteq \mathbb{R}^d$ with direction ω.
If true then

$$\|\widehat{gd\sigma}\|_{L^{\frac{2d}{d-1}}(\mathbb{S}^{d-1}) \to L^{\frac{2d}{d-1}}(B(0,R))} \lesssim \|\mathcal{M}_R\|^{1/2}_{L^d(\mathbb{R}^d)\to L^d(\mathbb{S}^{d-1})};$$

i.e. "Kakeya" $\implies$ Restriction!

Some partial results for the extension operator

A simple example:

Theorem (Barceló–B–Carbery 2008; $d = 2$, sacrificing optimality)

$$\int_{B(0,1)} |\widehat{gd\sigma}(R\xi)|^2 w(\xi)d\xi \lesssim \frac{\log R}{R} \int_{\mathbb{S}^1} |g(\omega)|^2 (\mathcal{K}_{R^{-1/2}}(\mathcal{N}_{R^{-1/2}} w)^2)^{1/2}. \tag{1}$$

A simple example:

Theorem (Barceló–B–Carbery 2008; $d = 2$, sacrificing optimality)

$$\int_{B(0,1)} |\widehat{gd\sigma}(R\xi)|^2 w(\xi)d\xi \lesssim \frac{\log R}{R} \int_{\mathbb{S}^1} |g(\omega)|^2 (\mathcal{K}_{R^{-1/2}}(\mathcal{N}_{R^{-1/2}} w)^2)^{1/2}. \tag{1}$$

Remark: We have $\|(\mathcal{K}_{R^{-1/2}}(\mathcal{N}_{R^{-1/2}} w)^2)^{1/2}\|_4 \lesssim (\log R)^{1/2} \|w\|_4$ by the Kakeya/Nikodym maximal theorem in the plane.

A simple example:

Theorem (Barceló–B–Carbery 2008; $d = 2$, sacrificing optimality)

$$\int_{B(0,1)} |\widehat{gd\sigma}(R\xi)|^2 w(\xi)d\xi \lesssim \frac{\log R}{R} \int_{\mathbb{S}^1} |g(\omega)|^2 (\mathcal{K}_{R^{-1/2}}(\mathcal{N}_{R^{-1/2}} w)^2)^{1/2}. \tag{1}$$

Remark: We have $\|(\mathcal{K}_{R^{-1/2}}(\mathcal{N}_{R^{-1/2}} w)^2)^{1/2}\|_4 \lesssim (\log R)^{1/2}\|w\|_4$ by the Kakeya/Nikodym maximal theorem in the plane. As a result (1) implies that

$$\|\widehat{gd\sigma}\|_{L^{8/3}(B(0,R))} \lesssim_\epsilon R^{1/4+\epsilon}\|g\|_{L^{8/3}(\mathbb{S}^1)},$$

which is sharp in the power of R;

A simple example:

Theorem (Barceló–B–Carbery 2008; $d = 2$, sacrificing optimality)

$$\int_{B(0,1)} |\widehat{gd\sigma}(R\xi)|^2 w(\xi) d\xi \lesssim \frac{\log R}{R} \int_{\mathbb{S}^1} |g(\omega)|^2 (\mathcal{K}_{R^{-1/2}}(\mathcal{N}_{R^{-1/2}} w)^2)^{1/2}. \tag{1}$$

Remark: We have $\|(\mathcal{K}_{R^{-1/2}}(\mathcal{N}_{R^{-1/2}} w)^2)^{1/2}\|_4 \lesssim (\log R)^{1/2} \|w\|_4$ by the Kakeya/Nikodym maximal theorem in the plane. As a result (1) implies that

$$\|\widehat{gd\sigma}\|_{L^{8/3}(B(0,R))} \lesssim_\epsilon R^{1/4+\epsilon} \|g\|_{L^{8/3}(\mathbb{S}^1)},$$

which is sharp in the power of R; we'd of course like to be on L^4!

A simple example:

Theorem (Barceló–B–Carbery 2008; $d = 2$, sacrificing optimality)

$$\int_{B(0,1)} |\widehat{gd\sigma}(R\xi)|^2 w(\xi)d\xi \lesssim \frac{\log R}{R} \int_{\mathbb{S}^1} |g(\omega)|^2 (\mathcal{K}_{R^{-1/2}}(\mathcal{N}_{R^{-1/2}}w)^2)^{1/2}. \tag{1}$$

Remark: We have $\|(\mathcal{K}_{R^{-1/2}}(\mathcal{N}_{R^{-1/2}}w)^2)^{1/2}\|_4 \lesssim (\log R)^{1/2}\|w\|_4$ by the Kakeya/Nikodym maximal theorem in the plane. As a result (1) implies that

$$\|\widehat{gd\sigma}\|_{L^{8/3}(B(0,R))} \lesssim_\epsilon R^{1/4+\epsilon}\|g\|_{L^{8/3}(\mathbb{S}^1)},$$

which is sharp in the power of R; we'd of course like to be on L^4!

Uses a Whitney decomposition and the bilinear extension estimate in the plane (as in Erdogan 2005) – an argument that *fails to exploit Córdoba's L^4 orthogonality*, but does at least extend to higher dimensions.

A simple example:

Theorem (Barceló–B–Carbery 2008; $d = 2$, sacrificing optimality)

$$\int_{B(0,1)} |\widehat{gd\sigma}(R\xi)|^2 w(\xi) d\xi \lesssim \frac{\log R}{R} \int_{\mathbb{S}^1} |g(\omega)|^2 (\mathcal{K}_{R^{-1/2}}(\mathcal{N}_{R^{-1/2}} w)^2)^{1/2}. \tag{1}$$

Remark: We have $\|(\mathcal{K}_{R^{-1/2}}(\mathcal{N}_{R^{-1/2}} w)^2)^{1/2}\|_4 \lesssim (\log R)^{1/2}\|w\|_4$ by the Kakeya/Nikodym maximal theorem in the plane. As a result (1) implies that

$$\|\widehat{gd\sigma}\|_{L^{8/3}(B(0,R))} \lesssim_\epsilon R^{1/4+\epsilon}\|g\|_{L^{8/3}(\mathbb{S}^1)},$$

which is sharp in the power of R; we'd of course like to be on L^4!

Uses a Whitney decomposition and the bilinear extension estimate in the plane (as in Erdogan 2005) – an argument that *fails to exploit Córdoba's L^4 orthogonality*, but does at least extend to higher dimensions.

Let's see another conjectural example...

Conjectural example 2: the disc multiplier

Conjectural example 2: the disc multiplier

In the Proceedings of the 1978 Williamstown Conference on Harmonic Analysis, Stein asked whether (†), i.e.

$$\int_{\mathbb{R}^d} |Tf|^2 w \lesssim \int_{\mathbb{R}^d} |f|^2 \mathcal{M} w,$$

might hold when T is the disc multiplier

$$\widehat{Tf}(\xi) = \chi_{B(0,1)}(\xi)\widehat{f}(\xi),$$

and $\mathcal{M}$ is some variant of the *universal maximal operator*

$$\mathcal{N} w(x) := \sup_{T \ni x} \frac{1}{|T|} \int_T w;$$

here the supremum is taken over arbitrary rectangles/tubes in $\mathbb{R}^d$ containing x.

Conjectural example 2: the disc multiplier

In the Proceedings of the 1978 Williamstown Conference on Harmonic Analysis, Stein asked whether (†), i.e.

$$\int_{\mathbb{R}^d} |Tf|^2 w \lesssim \int_{\mathbb{R}^d} |f|^2 \mathcal{M} w,$$

might hold when T is the disc multiplier

$$\widehat{Tf}(\xi) = \chi_{B(0,1)}(\xi)\widehat{f}(\xi),$$

and $\mathcal{M}$ is some variant of the *universal maximal operator*

$$\mathcal{N} w(x) := \sup_{T \ni x} \frac{1}{|T|} \int_T w;$$

here the supremum is taken over arbitrary rectangles/tubes in $\mathbb{R}^d$ containing x.

Some progress: Carbery 1984, Christ 1985, Carbery–Romera–Soria 1991, Carbery–Seeger 2000, Lee–Rogers–Seeger 2012, plus many related works...

Conjectural example 2: the disc multiplier

In the Proceedings of the 1978 Williamstown Conference on Harmonic Analysis, Stein asked whether (†), i.e.

$$\int_{\mathbb{R}^d} |Tf|^2 w \lesssim \int_{\mathbb{R}^d} |f|^2 \mathcal{M} w,$$

might hold when T is the disc multiplier

$$\widehat{Tf}(\xi) = \chi_{B(0,1)}(\xi)\widehat{f}(\xi),$$

and $\mathcal{M}$ is some variant of the *universal maximal operator*

$$\mathcal{N} w(x) := \sup_{T \ni x} \frac{1}{|T|} \int_T w;$$

here the supremum is taken over arbitrary rectangles/tubes in $\mathbb{R}^d$ containing x.

Some progress: Carbery 1984, Christ 1985, Carbery–Romera–Soria 1991, Carbery–Seeger 2000, Lee–Rogers–Seeger 2012, plus many related works...

Recall that $Tf = K * f$ where

$$K(x) := \mathcal{F}^{-1}(\chi_{B(0,1)})(x) = \frac{cJ_{d/2}(2\pi|x|)}{|x|^{\frac{d}{2}}} = c\,\frac{e^{2\pi i|x|} + e^{-2\pi i|x|} + o(1)}{|x|^{\frac{d+1}{2}}}.$$

Conjectural example 2: the disc multiplier

In the Proceedings of the 1978 Williamstown Conference on Harmonic Analysis, Stein asked whether (†), i.e.

$$\int_{\mathbb{R}^d} |Tf|^2 w \lesssim \int_{\mathbb{R}^d} |f|^2 \mathcal{M}w,$$

might hold when T is the disc multiplier

$$\widehat{Tf}(\xi) = \chi_{B(0,1)}(\xi)\widehat{f}(\xi),$$

and $\mathcal{M}$ is some variant of the *universal maximal operator*

$$\mathcal{N}w(x) := \sup_{T \ni x} \frac{1}{|T|} \int_T w;$$

here the supremum is taken over arbitrary rectangles/tubes in $\mathbb{R}^d$ containing x.

Some progress: Carbery 1984, Christ 1985, Carbery–Romera–Soria 1991, Carbery–Seeger 2000, Lee–Rogers–Seeger 2012, plus many related works...

Recall that $Tf = K * f$ where

$$K(x) := \mathcal{F}^{-1}(\chi_{B(0,1)})(x) = \frac{cJ_{d/2}(2\pi|x|)}{|x|^{\frac{d}{2}}} = c\,\frac{e^{2\pi i|x|} + e^{-2\pi i|x|} + o(1)}{|x|^{\frac{d+1}{2}}}.$$

Here the kernel is oscillatory, but the multiplier is *not* - this is a rather special situation for oscillatory convolution kernels...

Oscillatory kernels and oscillatory multipliers: duality of phases

Oscillatory kernels and oscillatory multipliers: duality of phases

Stein (1993), Page 358:

> *The Fourier transform of* $e^{i\Phi(x)}a(x)$ *is essentially of the form* $e^{-i\Psi(\xi)}a^*(\xi)$*, where the pair* (Φ, Ψ) *are "dual" to each other*

Stein (1993), Page 358:

> *The Fourier transform of* $e^{i\Phi(x)}a(x)$ *is essentially of the form* $e^{-i\Psi(\xi)}a^*(\xi)$*, where the pair* (Φ, Ψ) *are "dual" to each other*

For example, $-|x|^p/p$ and $|\xi|^{p'}/p'$ are dual phases, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $p \in (1, \infty)$.

Stein (1993), Page 358:

> *The Fourier transform of $e^{i\Phi(x)}a(x)$ is essentially of the form $e^{-i\Psi(\xi)}a^*(\xi)$, where the pair (Φ, Ψ) are "dual" to each other*

For example, $-|x|^p/p$ and $|\xi|^{p'}/p'$ are dual phases, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $p \in (1, \infty)$.

Of course this is highly unstable as $p \to 1$; recall that

$$\widehat{\chi}_{B(0,1)}(\xi) = \frac{cJ_{d/2}(2\pi|\xi|)}{|\xi|^{\frac{d}{2}}} = c\frac{e^{2\pi i|\xi|} + e^{-2\pi i|\xi|} + o(1)}{|\xi|^{\frac{d+1}{2}}}.$$

Stein (1993), Page 358:

> *The Fourier transform of $e^{i\Phi(x)}a(x)$ is essentially of the form $e^{-i\Psi(\xi)}a^*(\xi)$, where the pair (Φ,Ψ) are "dual" to each other*

For example, $-|x|^p/p$ and $|\xi|^{p'}/p'$ are dual phases, where $\frac{1}{p}+\frac{1}{p'}=1$ and $p\in(1,\infty)$.

Of course this is highly unstable as $p\to 1$; recall that

$$\widehat{\chi}_{B(0,1)}(\xi)=\frac{cJ_{d/2}(2\pi|\xi|)}{|\xi|^{\frac{d}{2}}}=c\frac{e^{2\pi i|\xi|}+e^{-2\pi i|\xi|}+o(1)}{|\xi|^{\frac{d+1}{2}}}.$$

Point: oscillatory convolution kernels and oscillatory multipliers are, to an extent, the same thing.

Oscillatory kernels and oscillatory multipliers: duality of phases

Stein (1993), Page 358:

> *The Fourier transform of $e^{i\Phi(x)}a(x)$ is essentially of the form $e^{-i\Psi(\xi)}a^*(\xi)$, where the pair (Φ, Ψ) are "dual" to each other*

For example, $-|x|^p/p$ and $|\xi|^{p'}/p'$ are dual phases, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $p \in (1, \infty)$.

Of course this is highly unstable as $p \to 1$; recall that

$$\widehat{\chi}_{B(0,1)}(\xi) = \frac{cJ_{d/2}(2\pi|\xi|)}{|\xi|^{\frac{d}{2}}} = c\frac{e^{2\pi i|\xi|} + e^{-2\pi i|\xi|} + o(1)}{|\xi|^{\frac{d+1}{2}}}.$$

Point: oscillatory convolution kernels and oscillatory multipliers are, to an extent, the same thing.

Let us shift perspective to *oscillatory Fourier multipliers* (sacrificing the disc multiplier of course)...

The Fourier multiplier angle

Notation: For a multiplier m we define the operator T_m by $\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$.

The Fourier multiplier angle

Notation: For a multiplier m we define the operator T_m by $\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$.

Goal: Identify classes of oscillatory Fourier multipliers m and maximal averaging operators $\mathcal{M}$ for which

$$\int |T_m f|^2 w \leq C \int |f|^2 \mathcal{M} w. \qquad (\dagger)$$

Notation: For a multiplier m we define the operator T_m by $\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$.

Goal: Identify classes of oscillatory Fourier multipliers m and maximal averaging operators $\mathcal{M}$ for which

$$\int |T_m f|^2 w \leq C \int |f|^2 \mathcal{M} w. \qquad (\dagger)$$

A classical (non-oscillatory) result of this type:

Theorem (∼Wilson 1980s)

If $m : \mathbb{R}^n \to \mathbb{C}$ is a Mikhlin multiplier, i.e.

$$|\partial^\gamma m(\xi)| \lesssim |\xi|^{-\gamma} \quad \text{for all} \quad |\gamma| \leq d/2 + 1,$$

or more generally, a Hörmander–Mikhlin multiplier, i.e.

$$\sup_j \|m\Psi(2^j \cdot)\|_{H^s} < \infty \quad \text{for some} \quad s > d/2,$$

then

$$\int |T_m f|^2 w \int |f|^2 M^{power} w.$$

(Here M is the Hardy–Littlewood maximal operator and $M^2 = M \circ M$.)

A proof using Stein's g-function method

For a Hörmander–Mikhlin multiplier m, we have

$$g(T_m f)(x) \lesssim g_\lambda^*(f)(x)$$

A proof using Stein's g-function method

For a Hörmander–Mikhlin multiplier m, we have

$$g(T_m f)(x) \lesssim g_\lambda^*(f)(x)$$

where

$$g(f)(x) := \left(\int_{|x-y| \leq t} |f * \phi_t(y)|^2 \frac{dy}{t^d} \frac{dt}{t} \right)^{1/2},$$

and

$$g_\lambda^*(f)(x) = \left(\int_0^\infty \int_{\mathbb{R}^d} |f * \phi_t(y)|^2 \left(1 + \frac{|x-y|}{t}\right)^{-d\lambda} \frac{dy}{t^d} \frac{dt}{t} \right)^{1/2}$$

with $\lambda > 1$. Here $\phi_t(x) = t^{-d}\phi(x/t)$ is a suitable approximate identity with $\int \phi = 0$.

A proof using Stein's g-function method

For a Hörmander–Mikhlin multiplier m, we have

$$g(T_m f)(x) \lesssim g_\lambda^*(f)(x)$$

where

$$g(f)(x) := \Big(\int_{|x-y|\leq t} |f * \phi_t(y)|^2 \frac{dy}{t^d} \frac{dt}{t} \Big)^{1/2},$$

and

$$g_\lambda^*(f)(x) = \Big(\int_0^\infty \int_{\mathbb{R}^d} |f * \phi_t(y)|^2 \Big(1 + \frac{|x-y|}{t}\Big)^{-d\lambda} \frac{dy}{t^d} \frac{dt}{t} \Big)^{1/2}$$

with $\lambda > 1$. Here $\phi_t(x) = t^{-d}\phi(x/t)$ is a suitable approximate identity with $\int \phi = 0$.

Theorem now follows from classical weighted L^2 inequalities for g and g_λ^*:

$$\int |T_m f|^2 w \lesssim \int g(T_m f)^2 M^{power} w \lesssim \int g_\lambda^*(f)^2 M^{power} w \int |f|^2 M^{power} w.$$

A proof using Stein's g-function method

For a Hörmander–Mikhlin multiplier m, we have

$$g(T_m f)(x) \lesssim g_\lambda^*(f)(x)$$

where

$$g(f)(x) := \Big(\int_{|x-y|\leq t} |f * \phi_t(y)|^2 \frac{dy}{t^d}\frac{dt}{t}\Big)^{1/2},$$

and

$$g_\lambda^*(f)(x) = \Big(\int_0^\infty \int_{\mathbb{R}^d} |f * \phi_t(y)|^2 \Big(1 + \frac{|x-y|}{t}\Big)^{-d\lambda} \frac{dy}{t^d}\frac{dt}{t}\Big)^{1/2}$$

with $\lambda > 1$. Here $\phi_t(x) = t^{-d}\phi(x/t)$ is a suitable approximate identity with $\int \phi = 0$.

Theorem now follows from classical weighted L^2 inequalities for g and g_λ^*:

$$\int |T_m f|^2 w \lesssim \int g(T_m f)^2 M^{power} w \lesssim \int g_\lambda^*(f)^2 M^{power} w \int |f|^2 M^{power} w.$$

As we shall see, Stein's g-function approach continues to be effective for certain classes of *highly oscillatory multipliers* (and thus kernels)...

A natural weakening of the Mikhlin condition $|\partial^\gamma m(\xi)| \lesssim |\xi|^{-|\gamma|}$ allowing more singular multipliers was considered by Miyachi in the 1980s.

Some well-known oscillatory multipliers

A natural weakening of the Mikhlin condition $|\partial^\gamma m(\xi)| \lesssim |\xi|^{-|\gamma|}$ allowing more singular multipliers was considered by Miyachi in the 1980s.

Let $\alpha, \beta \in \mathbb{R}$ be given and suppose m is supported on $\{|\xi|^\alpha \geq 1\}$ and satisfies

$$|\partial^\gamma m(\xi)| \lesssim |\xi|^{-\beta+|\gamma|(\alpha-1)} \tag{2}$$

for every multiindex γ with $|\gamma| \leq [\frac{d}{2}] + 1$.

A natural weakening of the Mikhlin condition $|\partial^\gamma m(\xi)| \lesssim |\xi|^{-|\gamma|}$ allowing more singular multipliers was considered by Miyachi in the 1980s.

Let $\alpha, \beta \in \mathbb{R}$ be given and suppose m is supported on $\{|\xi|^\alpha \geq 1\}$ and satisfies

$$|\partial^\gamma m(\xi)| \lesssim |\xi|^{-\beta+|\gamma|(\alpha-1)} \tag{2}$$

for every multiindex γ with $|\gamma| \leq [\frac{d}{2}] + 1$.

Examples:

- $m(\xi) = m_{\alpha,\beta}(\xi) := |\xi|^{-\beta} e^{i|\xi|^\alpha}$, studied by Hirschman, Stein, Wainger, Fefferman, Miyachi...

A natural weakening of the Mikhlin condition $|\partial^\gamma m(\xi)| \lesssim |\xi|^{-|\gamma|}$ allowing more singular multipliers was considered by Miyachi in the 1980s.

Let $\alpha, \beta \in \mathbb{R}$ be given and suppose m is supported on $\{|\xi|^\alpha \geq 1\}$ and satisfies

$$|\partial^\gamma m(\xi)| \lesssim |\xi|^{-\beta+|\gamma|(\alpha-1)} \tag{2}$$

for every multiindex γ with $|\gamma| \leq [\frac{d}{2}] + 1$.

Examples:

- $m(\xi) = m_{\alpha,\beta}(\xi) := |\xi|^{-\beta} e^{i|\xi|^\alpha}$, studied by Hirschman, Stein, Wainger, Fefferman, Miyachi...
- If $K(x) = e^{i|x|^3}$ then $m = \widehat{K}$ satisfies (2) with $\alpha = 3/2$ and $\beta = d/4$.

A natural weakening of the Mikhlin condition $|\partial^\gamma m(\xi)| \lesssim |\xi|^{-|\gamma|}$ allowing more singular multipliers was considered by Miyachi in the 1980s.

Let $\alpha, \beta \in \mathbb{R}$ be given and suppose m is supported on $\{|\xi|^\alpha \geq 1\}$ and satisfies

$$|\partial^\gamma m(\xi)| \lesssim |\xi|^{-\beta+|\gamma|(\alpha-1)} \tag{2}$$

for every multiindex γ with $|\gamma| \leq [\frac{d}{2}] + 1$.

Examples:

- $m(\xi) = m_{\alpha,\beta}(\xi) := |\xi|^{-\beta} e^{i|\xi|^\alpha}$, studied by Hirschman, Stein, Wainger, Fefferman, Miyachi...
- If $K(x) = e^{i|x|^3}$ then $m = \widehat{K}$ satisfies (2) with $\alpha = 3/2$ and $\beta = d/4$.

It will be helpful to formulate a Hörmander-style weakening of Miyachi's condition...

A Miyachi–Hörmander condition

A Miyachi–Hörmander condition

Definition (α-subdyadic ball)

Let $\alpha \in \mathbb{R}$. A (euclidean) ball $B \subseteq \mathbb{R}^d$ is α-*subdyadic* if $\operatorname{dist}(B, 0)^\alpha \geq 1$ and

$$\operatorname{diam}(B) \sim \operatorname{dist}(B, 0)^{1-\alpha}.$$

Definition (α-subdyadic ball)

Let $\alpha \in \mathbb{R}$. A (euclidean) ball $B \subseteq \mathbb{R}^d$ is α-*subdyadic* if $\mathrm{dist}(B,0)^\alpha \geq 1$ and

$$\mathrm{diam}(B) \sim \mathrm{dist}(B,0)^{1-\alpha}.$$

The Miyachi condition

$$|\partial^\gamma m(\xi)| \lesssim |\xi|^{-\beta+|\gamma|(\alpha-1)}$$

of course makes no explicit reference to the subdyadic balls.

A Miyachi–Hörmander condition

Definition (α-subdyadic ball)

Let $\alpha \in \mathbb{R}$. A (euclidean) ball $B \subseteq \mathbb{R}^d$ is α*-subdyadic* if $\text{dist}(B, 0)^\alpha \geq 1$ and

$$\text{diam}(B) \sim \text{dist}(B, 0)^{1-\alpha}.$$

The Miyachi condition

$$|\partial^\gamma m(\xi)| \lesssim |\xi|^{-\beta+|\gamma|(\alpha-1)}$$

of course makes no explicit reference to the subdyadic balls.

However, it may be weakened to the Hörmander-style condition:

$$|B|^{-1/2} \|m\Psi_B\|_{\dot{H}^\sigma} \lesssim \text{dist}(B, 0)^{-\beta+(\alpha-1)\sigma}$$

for some $s > d/2$ and all $0 \leq \sigma \leq s$; here Ψ_B is a normalised bump function adapted to B.

A Miyachi–Hörmander condition

Definition (α-subdyadic ball)

Let $\alpha \in \mathbb{R}$. A (euclidean) ball $B \subseteq \mathbb{R}^d$ is *α-subdyadic* if $\mathrm{dist}(B,0)^\alpha \geq 1$ and

$$\mathrm{diam}(B) \sim \mathrm{dist}(B,0)^{1-\alpha}.$$

The Miyachi condition

$$|\partial^\gamma m(\xi)| \lesssim |\xi|^{-\beta+|\gamma|(\alpha-1)}$$

of course makes no explicit reference to the subdyadic balls.

However, it may be weakened to the Hörmander-style condition:

$$|B|^{-1/2} \|m\Psi_B\|_{\dot{H}^\sigma} \lesssim \mathrm{dist}(B,0)^{-\beta+(\alpha-1)\sigma}$$

for some $s > d/2$ and all $0 \leq \sigma \leq s$; here Ψ_B is a normalised bump function adapted to B.

These bounds are assumed to be uniform over all α-subdyadic balls.

The controlling maximal functions

Theorem (Beltran–B 2015)

Let $\alpha, \beta \in \mathbb{R}$. *Suppose that* $m : \mathbb{R}^d \to \mathbb{C}$ *is supported in* $\{|\xi|^\alpha \geq 1\}$ *and satisfies*

$$|\partial^\gamma m(\xi)| \lesssim |\xi|^{-\beta+|\gamma|(\alpha-1)}$$

for all $|\gamma| \leq [\frac{d}{2}] + 1$ *(or the weaker Hörmander alternative). Then*

$$\int_{\mathbb{R}^d} |T_m f(x)|^2 w(x) dx \lesssim \int_{\mathbb{R}^d} |f(x)|^2 M^4 \mathcal{M}_{\alpha,\beta} M^4 w(x) dx,$$

where

$$\mathcal{M}_{\alpha,\beta} w(x) = \sup_{(y,r) \in \Gamma_\alpha(x)} \frac{1}{|B(y,r)|^{1-2\beta/d}} \int_{B(y,r)} w$$

and $\Gamma_\alpha(x) := \{(y,r) \in \mathbb{R}^d \times \mathbb{R}_+ : 0 < r^\alpha \leq 1,\ |y - x| \leq r^{1-\alpha}\}$.

The controlling maximal functions

Theorem (Beltran–B 2015)

Let $\alpha, \beta \in \mathbb{R}$. *Suppose that* $m : \mathbb{R}^d \to \mathbb{C}$ *is supported in* $\{|\xi|^\alpha \geq 1\}$ *and satisfies*

$$|\partial^\gamma m(\xi)| \lesssim |\xi|^{-\beta+|\gamma|(\alpha-1)}$$

for all $|\gamma| \leq [\frac{d}{2}] + 1$ *(or the weaker Hörmander alternative). Then*

$$\int_{\mathbb{R}^d} |T_m f(x)|^2 w(x) dx \lesssim \int_{\mathbb{R}^d} |f(x)|^2 M^4 \mathcal{M}_{\alpha,\beta} M^4 w(x) dx,$$

where

$$\mathcal{M}_{\alpha,\beta} w(x) = \sup_{(y,r)\in\Gamma_\alpha(x)} \frac{1}{|B(y,r)|^{1-2\beta/d}} \int_{B(y,r)} w$$

and $\Gamma_\alpha(x) := \{(y, r) \in \mathbb{R}^d \times \mathbb{R}_+ : 0 < r^\alpha \leq 1,\ |y - x| \leq r^{1-\alpha}\}$.

Remark: $\mathcal{M}_{\alpha,\beta}$ is closely related to a Nikodym-like maximal function –

$$\mathcal{M}_{\alpha,\beta} w(x) \gtrsim \mathcal{N}_{\alpha,\beta} w(x) := \sup_{0<r^\alpha\leq 1} \sup_{T\ni x} \frac{r^{2\beta}}{|T|} \int_T |f|,$$

where the supremum over tubes T in $\mathbb{R}^d$ of width r and length $r^{1-\alpha}$, containing x.

The controlling maximal functions

Theorem (Beltran–B 2015)

Let $\alpha, \beta \in \mathbb{R}$. *Suppose that* $m : \mathbb{R}^d \to \mathbb{C}$ *is supported in* $\{|\xi|^\alpha \geq 1\}$ *and satisfies*

$$|\partial^\gamma m(\xi)| \lesssim |\xi|^{-\beta+|\gamma|(\alpha-1)}$$

for all $|\gamma| \leq [\frac{d}{2}] + 1$ *(or the weaker Hörmander alternative). Then*

$$\int_{\mathbb{R}^d} |T_m f(x)|^2 w(x)dx \lesssim \int_{\mathbb{R}^d} |f(x)|^2 M^4 \mathcal{M}_{\alpha,\beta} M^4 w(x)dx,$$

where

$$\mathcal{M}_{\alpha,\beta} w(x) = \sup_{(y,r)\in\Gamma_\alpha(x)} \frac{1}{|B(y,r)|^{1-2\beta/d}} \int_{B(y,r)} w$$

and $\Gamma_\alpha(x) := \{(y,r) \in \mathbb{R}^d \times \mathbb{R}_+ : 0 < r^\alpha \leq 1,\ |y-x| \leq r^{1-\alpha}\}$.

Remark: $\mathcal{M}_{\alpha,\beta}$ is closely related to a Nikodym-like maximal function –

$$\mathcal{M}_{\alpha,\beta} w(x) \gtrsim \mathcal{N}_{\alpha,\beta} w(x) := \sup_{0<r^\alpha\leq 1} \sup_{T\ni x} \frac{r^{2\beta}}{|T|} \int_T |f|,$$

where the supremum over tubes T in $\mathbb{R}^d$ of width r and length $r^{1-\alpha}$, containing x.
(See also Nagel–Stein 1985, B–Carbery–Soria–Vargas 2006, B–Harrison 2012, B 2014.)

Oscillatory kernels

Oscillatory kernels

Stationary phase leads to statements on the kernel side.

Oscillatory kernels

Stationary phase leads to statements on the kernel side.

Example:

Corollary

Given $a > 0$, $a \neq 1$ *and* $b \geq d(1 - \frac{a}{2})$, *consider the kernels* $K_{a,b} : \mathbb{R}^d \to \mathbb{C}$ *given by*

$$K_{a,b}(x) = \frac{e^{i|x|^a}}{(1+|x|)^b}.$$

Then

$$\int_{\mathbb{R}^d} |K_{a,b} * f|^2 w \lesssim \int_{\mathbb{R}^d} |f|^2 M^4 \mathcal{M}_{\alpha,\beta} M^4 w,$$

where $\alpha = \frac{a}{a-1}$ *and* $\beta = \frac{da/2-d+b}{a-1}$.

Stationary phase leads to statements on the kernel side.

Example:

Corollary

Given $a > 0$, $a \neq 1$ and $b \geq d(1 - \frac{a}{2})$, consider the kernels $K_{a,b} : \mathbb{R}^d \to \mathbb{C}$ given by

$$K_{a,b}(x) = \frac{e^{i|x|^a}}{(1+|x|)^b}.$$

Then

$$\int_{\mathbb{R}^d} |K_{a,b} * f|^2 w \lesssim \int_{\mathbb{R}^d} |f|^2 M^4 \mathcal{M}_{\alpha,\beta} M^4 w,$$

where $\alpha = \frac{a}{a-1}$ and $\beta = \frac{da/2-d+b}{a-1}$.

Remarks:

- Missing point $a = 1$ corresponds to the disc multiplier and Stein's conjecture.

Oscillatory kernels

Stationary phase leads to statements on the kernel side.

Example:

Corollary

Given $a > 0$, $a \neq 1$ and $b \geq d(1 - \frac{a}{2})$, consider the kernels $K_{a,b} : \mathbb{R}^d \to \mathbb{C}$ given by

$$K_{a,b}(x) = \frac{e^{i|x|^a}}{(1 + |x|)^b}.$$

Then

$$\int_{\mathbb{R}^d} |K_{a,b} * f|^2 w \lesssim \int_{\mathbb{R}^d} |f|^2 M^4 \mathcal{M}_{\alpha,\beta} M^4 w,$$

where $\alpha = \frac{a}{a-1}$ and $\beta = \frac{da/2 - d + b}{a-1}$.

Remarks:

- Missing point $a = 1$ corresponds to the disc multiplier and Stein's conjecture.
- Controlling maximal operators optimal with regard to $L^p - L^q$ bounds.

A PDE angle

Applying our results to the specific multipliers $m_{\alpha,\beta}(\xi) := |\xi|^{-\beta} e^{i|\xi|^\alpha}$ leads to...

Corollary

$$\int_{\mathbb{R}^d} |e^{is(-\Delta)^{\alpha/2}} f(x)|^2 w(x)dx \lesssim \int_{\mathbb{R}^d} |(-\Delta)^{\beta/2} f(x)|^2 M^4 \mathfrak{M}^s_{\alpha,\beta} M^4 w(x)dx$$

where

$$\mathfrak{M}^s_{\alpha,\beta} w(x) = \sup_{(y,r)\in\Lambda^s_\alpha(x)} \frac{1}{|B(y,r)|^{1-2\beta/d}} \int_{B(y,r)} w$$

and

$$\Lambda^s_\alpha(x) := \{(y,r) \in \mathbb{R}^d \times \mathbb{R}_+ : |y-x| \leq s r^{1-\alpha}\}.$$

A PDE angle

Applying our results to the specific multipliers $m_{\alpha,\beta}(\xi) := |\xi|^{-\beta} e^{i|\xi|^\alpha}$ leads to...

Corollary

$$\int_{\mathbb{R}^d} |e^{is(-\Delta)^{\alpha/2}} f(x)|^2 w(x)dx \lesssim \int_{\mathbb{R}^d} |(-\Delta)^{\beta/2} f(x)|^2 M^4 \mathfrak{M}^s_{\alpha,\beta} M^4 w(x)dx$$

where

$$\mathfrak{M}^s_{\alpha,\beta} w(x) = \sup_{(y,r)\in\Lambda^s_\alpha(x)} \frac{1}{|B(y,r)|^{1-2\beta/d}} \int_{B(y,r)} w$$

and

$$\Lambda^s_\alpha(x) := \{(y,r) \in \mathbb{R}^d \times \mathbb{R}_+ : |y-x| \leq sr^{1-\alpha}\}.$$

Remarks:

- Power weights becomes Pitt's inequality (or Hardy's inequality).
- A local energy estimate capturing dispersive effects ($\Lambda^s_\alpha(x)$ increasing in s).

Pointwise estimates via g-functions

For $\alpha, \beta \in \mathbb{R}$ we define the square function

$$g_{\alpha,\beta}(f)(x) = \Big(\int_{\Gamma_\alpha(x)} |f * \phi_t(y)|^2 \frac{dy}{t^{(1-\alpha)d+2\beta}} \frac{dt}{t}\Big)^{1/2},$$

where, as before,

$$\Gamma_\alpha(x) := \{(y,t) \in \mathbb{R}^d \times \mathbb{R}_+ : 0 < t^\alpha \leq 1,\ |y-x| \leq t^{1-\alpha}\}.$$

Pointwise estimates via g-functions

For $\alpha, \beta \in \mathbb{R}$ we define the square function

$$g_{\alpha,\beta}(f)(x) = \Big(\int_{\Gamma_\alpha(x)} |f * \phi_t(y)|^2 \frac{dy}{t^{(1-\alpha)d+2\beta}} \frac{dt}{t}\Big)^{1/2},$$

where, as before,

$$\Gamma_\alpha(x) := \{(y,t) \in \mathbb{R}^d \times \mathbb{R}_+ : 0 < t^\alpha \leq 1,\ |y-x| \leq t^{1-\alpha}\}.$$

We also define

$$g^*_{\alpha,\beta,\lambda}(f)(x) = \Big(\int_{\mathbb{R}^d \times \mathbb{R}_+} |f * \phi_t(y)|^2 (1 + t^{\alpha-1}|x-y|)^{-d\lambda} \frac{dy}{t^{(1-\alpha)d+2\beta}} \frac{dt}{t}\Big)^{1/2}.$$

Pointwise estimates via g-functions

For $\alpha, \beta \in \mathbb{R}$ we define the square function

$$g_{\alpha,\beta}(f)(x) = \Big(\int_{\Gamma_\alpha(x)} |f * \phi_t(y)|^2 \frac{dy}{t^{(1-\alpha)d+2\beta}} \frac{dt}{t}\Big)^{1/2},$$

where, as before,

$$\Gamma_\alpha(x) := \{(y,t) \in \mathbb{R}^d \times \mathbb{R}_+ : 0 < t^\alpha \leq 1,\ |y-x| \leq t^{1-\alpha}\}.$$

We also define

$$g^*_{\alpha,\beta,\lambda}(f)(x) = \Big(\int_{\mathbb{R}^d \times \mathbb{R}_+} |f * \phi_t(y)|^2 (1 + t^{\alpha-1}|x-y|)^{-d\lambda} \frac{dy}{t^{(1-\alpha)d+2\beta}} \frac{dt}{t}\Big)^{1/2}.$$

Note that $g_{0,0} = g$ and $g^*_{0,0,\lambda} = g^*_\lambda$ – the classical g-functions.

Pointwise estimates via g-functions

For $\alpha, \beta \in \mathbb{R}$ we define the square function

$$g_{\alpha,\beta}(f)(x) = \Big(\int_{\Gamma_\alpha(x)} |f * \phi_t(y)|^2 \frac{dy}{t^{(1-\alpha)d+2\beta}} \frac{dt}{t}\Big)^{1/2},$$

where, as before,

$$\Gamma_\alpha(x) := \{(y,t) \in \mathbb{R}^d \times \mathbb{R}_+ : 0 < t^\alpha \leq 1, \ |y - x| \leq t^{1-\alpha}\}.$$

We also define

$$g^*_{\alpha,\beta,\lambda}(f)(x) = \Big(\int_{\mathbb{R}^d \times \mathbb{R}_+} |f * \phi_t(y)|^2 (1 + t^{\alpha-1}|x-y|)^{-d\lambda} \frac{dy}{t^{(1-\alpha)d+2\beta}} \frac{dt}{t}\Big)^{1/2}.$$

Note that $g_{0,0} = g$ and $g^*_{0,0,\lambda} = g^*_\lambda$ – the classical g-functions.

Theorem (Pointwise estimate (Beltran–B 2015))

If a multiplier m satisfies

$$|\partial^\gamma m(\xi)| \lesssim |\xi|^{-\beta+|\gamma|(\alpha-1)}$$

for every multiindex γ with $|\gamma| \leq [\frac{d}{2}] + 1$ (or the Hörmander alternative), then for some $\lambda > 1$ we have

$$g_{\alpha,\beta}(T_m f)(x) \lesssim g^*_{\alpha,0,\lambda}(f)(x).$$

On the proof

Key ingredient. Together, $g_{\alpha,\beta}$ and $g^*_{\alpha,\beta,\lambda}$ decouple/recouple α-subdyadic frequency decompositions.

On the proof

Key ingredient. Together, $g_{\alpha,\beta}$ and $g^*_{\alpha,\beta,\lambda}$ decouple/recouple α-subdyadic frequency decompositions.

Lemma (Decoupling/recoupling)

If $\mathcal{B}$ is a collection of balls in $\mathbb{R}^d$ such that

- $\mathrm{diam}(B) \sim \mathrm{dist}(B,0)^{1-\alpha}$ *for all $B \in \mathcal{B}$,*

On the proof

Key ingredient. Together, $g_{\alpha,\beta}$ and $g^*_{\alpha,\beta,\lambda}$ decouple/recouple α-subdyadic frequency decompositions.

Lemma (Decoupling/recoupling)

If $\mathcal{B}$ is a collection of balls in $\mathbb{R}^d$ such that

- $\mathrm{diam}(B) \sim \mathrm{dist}(B,0)^{1-\alpha}$ *for all* $B \in \mathcal{B}$, *and*
- $\{\widehat{\psi}_B\}_{B\in\mathcal{B}}$ *is a regular partition of unity adapted to* $\mathcal{B}$,

On the proof

Key ingredient. Together, $g_{\alpha,\beta}$ and $g^*_{\alpha,\beta,\lambda}$ decouple/recouple α-subdyadic frequency decompositions.

Lemma (Decoupling/recoupling)

If $\mathcal{B}$ is a collection of balls in $\mathbb{R}^d$ such that

- $\mathrm{diam}(B) \sim \mathrm{dist}(B,0)^{1-\alpha}$ *for all $B \in \mathcal{B}$, and*
- $\{\widehat{\psi}_B\}_{B\in\mathcal{B}}$ *is a regular partition of unity adapted to $\mathcal{B}$,*

then

$$g_{\alpha,\beta}\Big(\sum_{B\in\mathcal{B}} f * \psi_B\Big)(x)^2 \lesssim \sum_{B\in\mathcal{B}} g^*_{\alpha,\beta,\lambda}(f * \psi_B)(x)^2 \lesssim g^*_{\alpha,\beta,\lambda}\Big(\sum_{B\in\mathcal{B}} f * \psi_B\Big)(x)^2;$$

On the proof

Key ingredient. Together, $g_{\alpha,\beta}$ and $g^*_{\alpha,\beta,\lambda}$ decouple/recouple α-subdyadic frequency decompositions.

Lemma (Decoupling/recoupling)

If $\mathcal{B}$ is a collection of balls in $\mathbb{R}^d$ such that

- $\mathrm{diam}(B) \sim \mathrm{dist}(B,0)^{1-\alpha}$ *for all* $B \in \mathcal{B}$, *and*
- $\{\widehat{\psi}_B\}_{B\in\mathcal{B}}$ *is a regular partition of unity adapted to* $\mathcal{B}$,

then

$$g_{\alpha,\beta}\Big(\sum_{B\in\mathcal{B}} f*\psi_B\Big)(x)^2 \lesssim \sum_{B\in\mathcal{B}} g^*_{\alpha,\beta,\lambda}(f*\psi_B)(x)^2 \lesssim g^*_{\alpha,\beta,\lambda}\Big(\sum_{B\in\mathcal{B}} f*\psi_B\Big)(x)^2;$$

(For this reason we refer to $g_{\alpha,\beta}$ as a *subdyadic square function*.)

On the proof

Key ingredient. Together, $g_{\alpha,\beta}$ and $g^*_{\alpha,\beta,\lambda}$ decouple/recouple α-subdyadic frequency decompositions.

Lemma (Decoupling/recoupling)

If $\mathcal{B}$ is a collection of balls in $\mathbb{R}^d$ such that

- $\mathrm{diam}(B) \sim \mathrm{dist}(B,0)^{1-\alpha}$ *for all $B \in \mathcal{B}$, and*
- $\{\widehat{\psi}_B\}_{B\in\mathcal{B}}$ *is a regular partition of unity adapted to $\mathcal{B}$,*

then

$$g_{\alpha,\beta}\Big(\sum_{B\in\mathcal{B}} f * \psi_B\Big)(x)^2 \lesssim \sum_{B\in\mathcal{B}} g^*_{\alpha,\beta,\lambda}(f*\psi_B)(x)^2 \lesssim g^*_{\alpha,\beta,\lambda}\Big(\sum_{B\in\mathcal{B}} f*\psi_B\Big)(x)^2;$$

(For this reason we refer to $g_{\alpha,\beta}$ as a *subdyadic square function*.)

Given the lemma it suffices to prove that

$$g^*_{\alpha,\beta,\lambda}(T_m(f*\psi_B))(x) \lesssim g^*_{\alpha,0,\lambda}(f*\psi_B)(x)$$

uniformly over balls B such that $\mathrm{diam}(B) \sim \mathrm{dist}(B,0)^{1-\alpha}$.

Key ingredient. Together, $g_{\alpha,\beta}$ and $g^*_{\alpha,\beta,\lambda}$ decouple/recouple α-subdyadic frequency decompositions.

Lemma (Decoupling/recoupling)

If $\mathcal{B}$ is a collection of balls in $\mathbb{R}^d$ such that

- $\mathrm{diam}(B) \sim \mathrm{dist}(B,0)^{1-\alpha}$ *for all $B \in \mathcal{B}$, and*
- $\{\widehat{\psi}_B\}_{B\in\mathcal{B}}$ *is a regular partition of unity adapted to $\mathcal{B}$,*

then

$$g_{\alpha,\beta}\Big(\sum_{B\in\mathcal{B}} f * \psi_B\Big)(x)^2 \lesssim \sum_{B\in\mathcal{B}} g^*_{\alpha,\beta,\lambda}(f*\psi_B)(x)^2 \lesssim g^*_{\alpha,\beta,\lambda}\Big(\sum_{B\in\mathcal{B}} f*\psi_B\Big)(x)^2;$$

(For this reason we refer to $g_{\alpha,\beta}$ as a *subdyadic square function*.)

Given the lemma it suffices to prove that

$$g^*_{\alpha,\beta,\lambda}(T_m(f*\psi_B))(x) \lesssim g^*_{\alpha,0,\lambda}(f*\psi_B)(x)$$

uniformly over balls B such that $\mathrm{diam}(B) \sim \mathrm{dist}(B,0)^{1-\alpha}$.

This localised estimate can be proved very much as in the classical case $\alpha = \beta = 0$.

Bounds on the square functions

In order to deduce the weighted inequalities for the Miyachi–Hörmander multipliers, we need forward and reverse weighted inequalities for $g^*_{\alpha,\beta,\lambda}$ and $g_{\alpha,\beta}$ respectively.

In order to deduce the weighted inequalities for the Miyachi–Hörmander multipliers, we need forward and reverse weighted inequalities for $g^*_{\alpha,\beta,\lambda}$ and $g_{\alpha,\beta}$ respectively.

Theorem (Reverse estimate)

$$\int_{\mathbb{R}^d} |f(x)|^2 w(x)dx \lesssim \int_{\mathbb{R}^d} g_{\alpha,\beta}(f)(x)^2 \mathcal{M}_{\alpha,\beta} M^4 w(x)dx$$

for any weight w, where (we recall),

$$\mathcal{M}_{\alpha,\beta} w(x) = \sup_{(y,r)\in\Gamma_\alpha(x)} \frac{1}{|B(y,r)|^{1-2\beta/d}} \int_{B(y,r)} w.$$

In order to deduce the weighted inequalities for the Miyachi–Hörmander multipliers, we need forward and reverse weighted inequalities for $g^*_{\alpha,\beta,\lambda}$ and $g_{\alpha,\beta}$ respectively.

Theorem (Reverse estimate)

$$\int_{\mathbb{R}^d} |f(x)|^2 w(x)dx \lesssim \int_{\mathbb{R}^d} g_{\alpha,\beta}(f)(x)^2 \mathcal{M}_{\alpha,\beta} M^4 w(x)dx$$

for any weight w, where (we recall),

$$\mathcal{M}_{\alpha,\beta} w(x) = \sup_{(y,r)\in\Gamma_\alpha(x)} \frac{1}{|B(y,r)|^{1-2\beta/d}} \int_{B(y,r)} w.$$

Closely related to the duality between Carleson measures and nontangential maximal operators.

Bounds on the square functions

In order to deduce the weighted inequalities for the Miyachi–Hörmander multipliers, we need forward and reverse weighted inequalities for $g^*_{\alpha,\beta,\lambda}$ and $g_{\alpha,\beta}$ respectively.

Theorem (Reverse estimate)

$$\int_{\mathbb{R}^d} |f(x)|^2 w(x)dx \lesssim \int_{\mathbb{R}^d} g_{\alpha,\beta}(f)(x)^2 \mathcal{M}_{\alpha,\beta} M^4 w(x)dx$$

for any weight w, where (we recall),

$$\mathcal{M}_{\alpha,\beta} w(x) = \sup_{(y,r)\in\Gamma_\alpha(x)} \frac{1}{|B(y,r)|^{1-2\beta/d}} \int_{B(y,r)} w.$$

Closely related to the duality between Carleson measures and nontangential maximal operators.

Optimality. Optimal Lebesgue bounds for $\mathcal{M}_{\alpha,\beta}$ imply optimal lower bounds of the form

$$\|f\|_{L^p(\mathbb{R}^d)} \lesssim \|g_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}^d)}, \quad \text{certain } p, q \geq 2.$$

Weighted *maximal* multiplier inequalities - a question

Weighted *maximal* multiplier inequalities - a question

Recall the weighted Schrödinger inequality:

Corollary

$$\int_{\mathbb{R}^d} |e^{is(-\Delta)^{\alpha/2}} f(x)|^2 w(x)dx \lesssim \int_{\mathbb{R}^d} |(-\Delta)^{\beta/2} f(x)|^2 M^4 \mathfrak{M}^{s}_{\alpha,\beta} M^4 w(x)dx$$

where

$$\mathfrak{M}^{s}_{\alpha,\beta} w(x) = \sup_{(y,r)\in\Lambda^{s}_{\alpha}(x)} \frac{1}{|B(y,r)|^{1-2\beta/d}} \int_{B(y,r)} w$$

and

$$\Lambda^{s}_{\alpha}(x) := \{(y,r) \in \mathbb{R}^d \times \mathbb{R}_+ : |y-x| \leq sr^{1-\alpha}\}.$$

Weighted *maximal* multiplier inequalities - a question

Recall the weighted Schrödinger inequality:

Corollary

$$\int_{\mathbb{R}^d} |e^{is(-\Delta)^{\alpha/2}} f(x)|^2 w(x)dx \lesssim \int_{\mathbb{R}^d} |(-\Delta)^{\beta/2} f(x)|^2 M^4 \mathfrak{M}^s_{\alpha,\beta} M^4 w(x)dx$$

where

$$\mathfrak{M}^s_{\alpha,\beta} w(x) = \sup_{(y,r)\in\Lambda^s_\alpha(x)} \frac{1}{|B(y,r)|^{1-2\beta/d}} \int_{B(y,r)} w$$

and

$$\Lambda^s_\alpha(x) := \{(y,r) \in \mathbb{R}^d \times \mathbb{R}_+ : |y-x| \leq s r^{1-\alpha}\}.$$

Since the regions $\Lambda^s_\alpha(x)$ are increasing in s, we have

$$\sup_{0\leq s\leq 1} \int_{\mathbb{R}^d} |e^{is(-\Delta)^{\alpha/2}} f|^2 w \lesssim \int_{\mathbb{R}^d} |(-\Delta)^{\beta/2} f|^2 M^2 \mathfrak{M}_{\alpha,\beta} M^4 w,$$

where $\mathfrak{M}_{\alpha,\beta} = \mathfrak{M}^1_{\alpha,\beta}$.

Weighted *maximal* multiplier inequalities - a question

Recall the weighted Schrödinger inequality:

Corollary

$$\int_{\mathbb{R}^d} |e^{is(-\Delta)^{\alpha/2}} f(x)|^2 w(x)dx \lesssim \int_{\mathbb{R}^d} |(-\Delta)^{\beta/2} f(x)|^2 M^4 \mathfrak{M}^s_{\alpha,\beta} M^4 w(x)dx$$

where

$$\mathfrak{M}^s_{\alpha,\beta} w(x) = \sup_{(y,r)\in\Lambda^s_\alpha(x)} \frac{1}{|B(y,r)|^{1-2\beta/d}} \int_{B(y,r)} w$$

and

$$\Lambda^s_\alpha(x) := \{(y,r) \in \mathbb{R}^d \times \mathbb{R}_+ : |y-x| \leq sr^{1-\alpha}\}.$$

Since the regions $\Lambda^s_\alpha(x)$ are increasing in s, we have

$$\sup_{0\leq s\leq 1} \int_{\mathbb{R}^d} |e^{is(-\Delta)^{\alpha/2}} f|^2 w \lesssim \int_{\mathbb{R}^d} |(-\Delta)^{\beta/2} f|^2 M^2 \mathfrak{M}_{\alpha,\beta} M^4 w,$$

where $\mathfrak{M}_{\alpha,\beta} = \mathfrak{M}^1_{\alpha,\beta}$. **Obvious question:** Might this be strengthened to

$$\int_{\mathbb{R}^d} \sup_{0\leq s\leq 1} |e^{is(-\Delta)^{\alpha/2}} f|^2 w \lesssim \int_{\mathbb{R}^d} |(-\Delta)^{\beta/2} f|^2 M^2 \mathfrak{M}_{\alpha,\beta} M^4 w,$$

at least for certain α, β?

Part 2: Some inequalities with *specific* weights.

Part 2: Some inequalities with *specific* weights.

Although the general weighted questions for $g \mapsto \widehat{gd\sigma}$ seem difficult, there are certain specific weights for which a quite thorough understanding is available...

Theorem (B–Carbery–Soria–Vargas 2006)

$$\int_{\mathbb{S}^1} |\widehat{gd\sigma}(R\xi)|^2 d\mu(\xi) \lesssim \frac{1}{R} \int_{\mathbb{S}^1} |g|^2 \mathcal{M}_R \mu$$

for all measures μ supported on $\mathbb{S}^1$, where

$$\mathcal{M}_R\mu(\omega) := \sup_{T||\omega} \frac{\mu(T)}{\alpha(T)};$$

here the supremum is taken over all tubes T in $\mathbb{R}^2$ with dimensions $\alpha \times \alpha^2 R$, with $R^{-2/3} \leq \alpha \leq R^{-1/2}$, parallel to ω.

Weights (or densities) on spheres

Theorem (B–Carbery–Soria–Vargas 2006)

$$\int_{\mathbb{S}^1} |\widehat{gd\sigma}(R\xi)|^2 d\mu(\xi) \lesssim \frac{1}{R} \int_{\mathbb{S}^1} |g|^2 \mathcal{M}_R \mu$$

for all measures μ supported on $\mathbb{S}^1$, where

$$\mathcal{M}_R \mu(\omega) := \sup_{T || \omega} \frac{\mu(T)}{\alpha(T)};$$

here the supremum is taken over all tubes T in $\mathbb{R}^2$ with dimensions $\alpha \times \alpha^2 R$, with $R^{-2/3} \leq \alpha \leq R^{-1/2}$, parallel to ω.

Key point:

$$g \mapsto \widehat{gd\sigma}(R\cdot)\Big|_{\mathbb{S}^1} = e^{iR\cos(\cdot)} * g$$

– a *convolution* operator on $\mathbb{S}^1$...

Weights (or densities) on spheres

Theorem (B–Carbery–Soria–Vargas 2006)

$$\int_{\mathbb{S}^1} |\widehat{gd\sigma}(R\xi)|^2 d\mu(\xi) \lesssim \frac{1}{R} \int_{\mathbb{S}^1} |g|^2 \mathcal{M}_R \mu$$

for all measures μ supported on $\mathbb{S}^1$, where

$$\mathcal{M}_R\mu(\omega) := \sup_{T||\omega} \frac{\mu(T)}{\alpha(T)};$$

here the supremum is taken over all tubes T in $\mathbb{R}^2$ with dimensions $\alpha \times \alpha^2 R$, with $R^{-2/3} \leq \alpha \leq R^{-1/2}$, parallel to ω.

Key point:

$$g \mapsto \widehat{gd\sigma}(R\cdot)\Big|_{\mathbb{S}^1} = e^{iR\cos(\cdot)} * g$$

– a *convolution* operator on $\mathbb{S}^1$...

No higher-dimensional version of theorem known, although the optimal range of $L^p(\mathbb{S}^2)$ estimates for

$$g \mapsto \widehat{gd\sigma}(R\cdot)\Big|_{\mathbb{S}^2}$$

is known (B–Seeger 2009.)

Weights (or densities) on other varieties

Weights (or densities) on other varieties

Let X denote the *X-ray transform* in the plane, i.e.

$$Xf(\ell) := \int_\ell f,$$

where ℓ belongs to the manifold $\mathcal{L}$ of lines in $\mathbb{R}^2$.

Weights (or densities) on other varieties

Let X denote the *X-ray transform* in the plane, i.e.

$$Xf(\ell) := \int_\ell f,$$

where ℓ belongs to the manifold $\mathcal{L}$ of lines in $\mathbb{R}^2$.

Theorem (B–Bez–Flock–Gutiérrez–Iliopoulou 2016)

If u solves $i\partial_t u = \Delta u$ with initial datum $f \in L^2(\mathbb{R}^2)$ then

$$\|X(|u|^2)\|_{L^3_{t,\ell}(\mathbb{R}\times\mathcal{L})} \leq C\|f\|^2_{L^2(\mathbb{R}^2)},$$

with equality if and only if f is an isotropic centred gaussian.

Weights (or densities) on other varieties

Let X denote the *X-ray transform* in the plane, i.e.

$$Xf(\ell) := \int_\ell f,$$

where ℓ belongs to the manifold $\mathcal{L}$ of lines in $\mathbb{R}^2$.

Theorem (B–Bez–Flock–Gutiérrez–Iliopoulou 2016)

If u solves $i\partial_t u = \Delta u$ with initial datum $f \in L^2(\mathbb{R}^2)$ then

$$\|X(|u|^2)\|_{L^3_{t,\ell}(\mathbb{R}\times\mathcal{L})} \leq C\|f\|^2_{L^2(\mathbb{R}^2)},$$

with equality if and only if f is an isotropic centred gaussian.

Proof in non-sharp form:

$$\|X(|u|^2)\|_{L^3_{t,\ell}} \lesssim \||u|^2\|_{L^3_t L^{3/2}_x} = \|u\|_{L^6_t L^3_x} \lesssim \|f\|_2^2,$$

by the $L^{3/2} \to L^3$ bound for X (Oberlin–Stein) and the $L^2 \to L^6_t L^3_x$ Strichartz estimate.

Weights (or densities) on other varieties

Let X denote the *X-ray transform* in the plane, i.e.

$$Xf(\ell) := \int_\ell f,$$

where ℓ belongs to the manifold $\mathcal{L}$ of lines in $\mathbb{R}^2$.

Theorem (B–Bez–Flock–Gutiérrez–Iliopoulou 2016)

If u solves $i\partial_t u = \Delta u$ with initial datum $f \in L^2(\mathbb{R}^2)$ then

$$\|X(|u|^2)\|_{L^3_{t,\ell}(\mathbb{R}\times\mathcal{L})} \leq C\|f\|^2_{L^2(\mathbb{R}^2)},$$

with equality if and only if f is an isotropic centred gaussian.

Proof in non-sharp form:

$$\|X(|u|^2)\|_{L^3_{t,\ell}} \lesssim \||u|^2\|_{L^3_t L^{3/2}_x} = \|u\|_{L^6_t L^3_x} \lesssim \|f\|_2^2,$$

by the $L^{3/2} \to L^3$ bound for X (Oberlin–Stein) and the $L^2 \to L^6_t L^3_x$ Strichartz estimate. Unlikely to give the sharp form as gaussians are *not* extremisers for the X-ray estimate (Christ).

Proof. The required inequality $\|X(|u|^2)\|_{L^3_{t,\ell}(\mathbb{R}\times\mathcal{L})} \leq C\|f\|^2_{L^2(\mathbb{R}^2)}$ reduces to the "weighted" extension inequality

$$\int_{\mathbb{R}^6} |\widehat{gd\sigma}(x)|^2 \delta(\rho(x))dx \leq C\|g\|^2_{L^2(\mathbb{S}^5)}, \tag{3}$$

where

$$\rho(x) := \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Proof. The required inequality $\|X(|u|^2)\|_{L^3_{t,\ell}(\mathbb{R}\times\mathcal{L})} \leq C\|f\|^2_{L^2(\mathbb{R}^2)}$ reduces to the "weighted" extension inequality

$$\int_{\mathbb{R}^6} |\widehat{gd\sigma}(x)|^2 \delta(\rho(x))dx \leq C\|g\|^2_{L^2(\mathbb{S}^5)}, \tag{3}$$

where

$$\rho(x) := \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Since

$$\begin{aligned}\int_{\mathbb{R}^6} |\widehat{gd\sigma}|^2 \delta\circ\rho = &\int_{\mathbb{S}^5} |g(\omega)|^2 [(\widehat{\delta\circ\rho}) * d\sigma(\omega)] d\sigma(\omega) \\ &- \frac{1}{2}\int_{\mathbb{S}^5}\int_{\mathbb{S}^5} |g(\omega)-g(\omega')|^2 \widehat{\delta\circ\rho}(\omega-\omega')d\sigma(\omega)d\sigma(\omega'),\end{aligned}$$

matters reduce to the *nonnegativity* of $\widehat{\delta\circ\rho}$ and the fact that the function

$$(\widehat{\delta\circ\rho}) * d\sigma$$

is (an explicitly computable) *constant*.

Proof. The required inequality $\|X(|u|^2)\|_{L^3_{t,\ell}(\mathbb{R}\times\mathcal{L})} \leq C\|f\|^2_{L^2(\mathbb{R}^2)}$ reduces to the "weighted" extension inequality

$$\int_{\mathbb{R}^6} |\widehat{gd\sigma}(x)|^2 \delta(\rho(x))dx \leq C\|g\|^2_{L^2(\mathbb{S}^5)}, \tag{3}$$

where

$$\rho(x) := \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Since

$$\begin{aligned}\int_{\mathbb{R}^6} |\widehat{gd\sigma}|^2 \delta \circ \rho = \int_{\mathbb{S}^5} &|g(\omega)|^2 [(\widehat{\delta \circ \rho}) * d\sigma(\omega)] d\sigma(\omega) \\ &- \frac{1}{2} \int_{\mathbb{S}^5} \int_{\mathbb{S}^5} |g(\omega) - g(\omega')|^2 \widehat{\delta \circ \rho}(\omega - \omega') d\sigma(\omega) d\sigma(\omega'),\end{aligned}$$

matters reduce to the *nonnegativity* of $\widehat{\delta \circ \rho}$ and the fact that the function

$$(\widehat{\delta \circ \rho}) * d\sigma$$

is (an explicitly computable) *constant*. Clearly (3) holds with equality if g is constant – generating the claimed gaussian extremisers.

Proof. The required inequality $\|X(|u|^2)\|_{L^3_{t,\ell}(\mathbb{R}\times\mathcal{L})} \leq C\|f\|^2_{L^2(\mathbb{R}^2)}$ reduces to the "weighted" extension inequality

$$\int_{\mathbb{R}^6} |\widehat{gd\sigma}(x)|^2\delta(\rho(x))dx \leq C\|g\|^2_{L^2(\mathbb{S}^5)}, \tag{3}$$

where

$$\rho(x) := \det\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Since

$$\begin{aligned}\int_{\mathbb{R}^6} |\widehat{gd\sigma}|^2\delta\circ\rho = &\int_{\mathbb{S}^5} |g(\omega)|^2[(\widehat{\delta\circ\rho}) * d\sigma(\omega)]d\sigma(\omega) \\ &- \frac{1}{2}\int_{\mathbb{S}^5}\int_{\mathbb{S}^5} |g(\omega)-g(\omega')|^2\widehat{\delta\circ\rho}(\omega-\omega')d\sigma(\omega)d\sigma(\omega'),\end{aligned}$$

matters reduce to the *nonnegativity* of $\widehat{\delta\circ\rho}$ and the fact that the function

$$(\widehat{\delta\circ\rho}) * d\sigma$$

is (an explicitly computable) *constant*. Clearly (3) holds with equality if g is constant – generating the claimed gaussian extremisers. Uniqueness of gaussian extremisers follows by a routine analysis of functional equations...

Proof. The required inequality $\|X(|u|^2)\|_{L^3_{t,\ell}(\mathbb{R}\times\mathcal{L})} \leq C\|f\|^2_{L^2(\mathbb{R}^2)}$ reduces to the "weighted" extension inequality

$$\int_{\mathbb{R}^6} |\widehat{gd\sigma}(x)|^2 \delta(\rho(x))dx \leq C\|g\|^2_{L^2(\mathbb{S}^5)}, \tag{3}$$

where

$$\rho(x) := \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Since

$$\begin{aligned}\int_{\mathbb{R}^6} |\widehat{gd\sigma}|^2 \delta \circ \rho = \int_{\mathbb{S}^5} & |g(\omega)|^2 [(\widehat{\delta \circ \rho}) * d\sigma(\omega)] d\sigma(\omega) \\ & - \frac{1}{2} \int_{\mathbb{S}^5}\int_{\mathbb{S}^5} |g(\omega) - g(\omega')|^2 \widehat{\delta \circ \rho}(\omega - \omega') d\sigma(\omega) d\sigma(\omega'),\end{aligned}$$

matters reduce to the *nonnegativity* of $\widehat{\delta \circ \rho}$ and the fact that the function

$$(\widehat{\delta \circ \rho}) * d\sigma$$

is (an explicitly computable) *constant*. Clearly (3) holds with equality if g is constant – generating the claimed gaussian extremisers. Uniqueness of gaussian extremisers follows by a routine analysis of functional equations...

Thanks for listening!