What the A_2 theorem has taught us about singular integrals

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May 2016

Integral to the field of harmonic analysis are maximal functions and singular integrals. Maximal functions allow us to study behavior of functions through behavior of averages, such as the standard Hardy-Littlewood maximal function: $Mf(x) := \sup_{Q \ni x} \fint_Q |f| dx$.

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A classic example to keep in mind is the Hilbert transform, which we can understand in the principal value sense as:

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$Hf(x) := p.v. \frac{1}{\pi} \int_{\varepsilon < |x-y| < R} \frac{f(y)}{x-y} dy$. The Hilbert transform belongs to an important class of singular integrals called Calderón-Zygmund Operators.

An important question was the boundedness of maximal functions and singular integral operators in more general contexts than Euclidean space with Lebesgue measure. For instance, if μ is a measure such that $d\mu = w(x)dx$ do these L^p bounds still hold?

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Theorem (Muckenhoupt 1972)

We have

$$\|Mf\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$$

if and only if w *is an* A_p *weight, that is*

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Today I will discuss a framework for obtaining bounds for CZOs, started by Lerner, that I extended to a more general setting. This led to a wide range of surprising applications such as sharp estimates in weighted norm inequalities, reverse Hölder extrapolation, and pointwise control of CZOs.

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Specifically, a space of homogeneous type (SHT) is a triple $(X, \rho, |\cdot|)$ where X is a set, ρ is a quasimetric, and the positive measure $|\cdot|$ is doubling, that is

$$0 < |B(x_0, 2r)| \leq C_d |B(x_0, r)| < +\infty.$$

These arise frequently in applications, such as elliptic PDE and fractals [23, 25].

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5. $\mathbb{Z}$ with counting measure
6. Homogeneous space from algebra/Riemann surface theory (A group G acting continuously and transitively on a topological space X), is a particular case

Definition

We'll say that $K : X \times X \setminus \{x = y\} \to R$ is a Calderón-Zygmund kernel if there exist $\eta > 0$ and $C < \infty$ such that for all $x_0 \neq y \in X$ and $x \in X$ it satisfies the decay condition:

$$|K(x_0, y)| \leq \frac{C}{|B(x_0, \rho(x_0, y))|} \tag{1}$$

and the smoothness condition for $\rho(x_0, x) \leq \eta\rho(x_0, y)$:

$$|K(x, y) - K(x_0, y)| \leq \left(\frac{\rho(x, x_0)}{\rho(x_0, y)}\right)^{\eta} \frac{1}{|B(x_0, \rho(x_0, y))|}, \tag{2}$$

$$|K(y, x) - K(y, x_0)| \leq \left(\frac{\rho(x, x_0)}{\rho(x_0, y)}\right)^{\eta} \frac{1}{|B(x_0, \rho(x_0, y))|}.$$

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Here is our result on how to treat a CZO:

Theorem (Lerner ($\mathbb{R}^n$), TCA and Vagharshakyan (SHT))

Let T be a CZO. Then we have

$$\|Tf\|_X \leq c \sup_{D,S} \|T^S|f|\|_X$$

where the T^S are sparse operators and X is a Banach function space, such as $L^2(w)$.

I'll discuss the supremum and sparse operators next.

T.C. Anderson, and A. Vagharshakyan. *A simple proof of the sharp weighted estimate for Calderón–Zygmund operators on homogeneous spaces*. Journal of Geometric Analysis.

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We work with this dyadic system:

Theorem (Hytönen and Kairema)

There exists a family of sets $D = \cup_{k \in \mathbb{Z}} D_k$, called a dyadic decomposition of X, constants $0 < C, \epsilon < \infty$, and a corresponding family of points $\{x_c(Q)\}_{Q \in D}$ such that:

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1. $X = \bigcup_{Q \in D_k} Q$, *for all* $k \in \mathbb{Z}$.
2. *If* $Q_1 \cap Q_2 \neq \emptyset$, *then* $Q_1 \subseteq Q_2$ *or* $Q_2 \subseteq Q_1$.
3. *For every* $Q \in D_k$ *there exists at least one child cube* $Q_c \in D_{k-1}$ *such that* $Q_c \subseteq Q$.
4. *For every* $Q \in D_k$ *there exists exactly one parent cube* $Q_p \in D_{k+1}$ *such that* $Q \subseteq Q_p$.
5. *If* Q_2 *is a child of* Q_1 *then* $\mu(Q_2) \geq \epsilon \mu(Q_1)$.
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A distinct dyadic system can contain a finite number of grids.

Recall:

Theorem (Lerner ($\mathbb{R}^n$), TCA and Vagharshakyan (SHT))

We have

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A sparse family S is a collection of dyadic cubes such that for $Q \in S$

$$\mu\left(\bigcup_{Q' \subsetneq Q, Q' \in S} Q'\right) \leq \frac{\mu(Q)}{2}.$$

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Definition

We say a weight $w \in A_2$ if

$$\sup_{r>0} \fint_{B(x,r)} w\,d\mu \left(\fint_{B(x,r)} w^{-1}\,d\mu \right) = [w]_{A_2} < \infty.$$

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We can also define this with respect to dyadic cubes.

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Theorem (Hytönen (2010 - $\mathbb{R}^n$), Lerner (2012 - $\mathbb{R}^n$), TCA and Vagharshakyan (2012 - SHT))

Let T be a Calderon-Zygmund operator and X a SHT. Then for any $w \in A_2$,

$$\|T\|_{L^2(w)} \leq C(T,X)[w]_{A_2}.$$

Technology developed in SHT allowed extensions of other results to SHT and simplified their proofs even in the Euclidean setting.

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$$\int_{\mathbb{R}^n} (Tf)^p w dx \leq C \int_{\mathbb{R}^n} f^p v dx$$

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Anderson, Theresa C. and Damián, Wendolín. *Calderón-Zygmund operators and commutators in spaces of homogeneous type: weighted inequalities. To appear in Journal of Math Inequalities.*

Theresa C. Anderson. *A new sufficient two-weighted bump assumption for L^p boundedness of Calderón-Zygmund operators. Proceedings of the AMS.*

Recently, there have been exciting developments that sharpen the bound in norm of a CZO by sparse operators to a pointwise bound! This was work by Conde-Alonso and Rey, Nazarov and Lerner, and Lacey. Even more recently, this type of decomposition has been extended to more general operators than CZOs by Bernicot, Frey and Petermichl.

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Thank you!

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