SUPPLEMENTARY MATERIALS: DISCONTINUOUS FRONTS AS EXACT SOLUTIONS TO PRECIPITATING QUASI-GEOSTROPHIC EQUATIONS*

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SM1. Saturation Water Vapor as a Function of Temperature. The variables of the PQG system are derived and presented in the paper with the assumption that the saturation water vapor q_{vs} is merely a function of height, i.e., $q_{vs} = q_{vs}(z)$. One may additionally, for physical reasons, want the saturation water vapor q_{vs} to be different in the saturated and unsaturated regions. Namely, we may wish to allow higher temperature regions to hold more water vapor than lower temperature regions before saturation [SM1] to control the total amount of water in each region. This, however, requires a slight re-interpretation of the saturated buoyancy variable b_s from that presented in (6). As shown below, once the saturated buoyancy is suitably updated, all conclusions and results of the current paper remain essentially unchanged.

For specificity, we consider the case where q_{vs} is a linear function of the potential temperature θ:

$$q_{vs} = q_{vs,0} + q_{vs,1}\theta, \tag{SM1}$$

where $q_{vs,0}$ and $q_{vs,1}$ are constants. For a saturation water vapor of the form (SM1), the definition of the saturated buoyancy (6) is no longer strictly correct as this buoyancy no longer satisfies the evolution equation (2c). Explicitly, the saturated buoyancy's extension into the unsaturated region is no longer correct and must therefore be modified. To remedy this matter, we return to the conservation of equivalent potential temperature equation [SM2, eq. 22b] and re-construct the saturated buoyancy from this point of view. Namely, the equivalent potential temperature satisfies

$$\frac{D_H\theta_e}{Dt} + w\frac{d\tilde{\theta}_e}{dz} = 0 \tag{SM2}$$

throughout the domain in the PQG system and this serves as our basis for defining the saturated buoyancy. To construct the saturated buoyancy we note that the definition of the equivalent potential temperature $\theta_e = \theta + \frac{L_v}{c_p}q_v$ gives

$$\theta_e = \begin{cases} \theta + \frac{L_v}{c_p}q_v = c_1\theta + \frac{L_v}{c_p}\left(q_v - q_{vs}\right) + \frac{L_v}{c_p}q_{vs,0} & \text{for } q_t < q_{vs} \text{ (unsaturated)} \\ \theta + \frac{L_v}{c_p}q_{vs} = c_1\theta + \frac{L_v}{c_p}q_{vs,0} & \text{for } q_t \geq q_{vs} \text{ (saturated)} \end{cases}, \tag{SM3}$$

where $c_1 = 1 + \frac{L_v}{c_p}q_{vs,1}$. Thus, from the linearity of the evolution equation (SM2), the quantity $\theta_e - \frac{L_v}{c_p}q_{vs,0}$ also satisfies (SM2) and

$$b_s = \frac{g}{\theta_0}\frac{1}{c_1}\left(\theta_e - \frac{L_v}{c_p}q_{vs,0}\right) \tag{SM4}$$

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gives the natural definition of the saturated buoyancy from the fact that $b_s = g\theta/\theta_0$ in the saturated side; compare equation (SM4) with the right-hand equation in (1c). Therefore, we define the saturated buoyancy to be

$$b_s = \frac{g}{\theta_0}\left(\theta + H_u \frac{L_v}{c_p}\frac{1}{c_1}(q_t - q_{vs})\right), \tag{SM5}$$

where this saturated buoyancy (SM5) satisfies the evolution equation

$$c_1 \frac{D_H b_s}{Dt} + N_s^2 w = 0. \tag{SM6}$$

Lastly, we note that this modification to the PQG system does not change our results since this transformation merely changes the parameter choices already within the system. Namely, the system remains unchanged if we scale the the saturated buoyancy by c_1 and replace the term $1/N_s^2$ wherever it multiplies the temperature by c_1/N_s^2. Concretely, the jump system (14b) is transformed as follows: in A_ω replace $1/N_s^2$ with c_1/N_s^2 and in $\boldsymbol{c}$ replace $[[B_s]]$ with $c_1[[B_s]]$. For example, the horizontal front speed, written in (19) with the potential temperature substituted away, is now

$$\sigma_H = \overline{V} - V_T \frac{\alpha_z}{\alpha}\left(\frac{1}{1 + C_{\alpha,1}[[q_v - q_{vs}]]/[[q_r]]}\right), \tag{SM7}$$

where $C_{\alpha,1} = (\alpha^2 N_u^2 + f^2)/(\alpha^2 N_s^2 + c_1 f^2)$. Notice that in this case only the constant C_α is replaced with $C_{\alpha,1}$.

SM2. Relative Front Speed $\omega = 0$. In this section we discuss the possibility of the front moving at the kinematic front speed $\sigma_H = \overline{V}$. This case corresponds to the free horizontal advection of the frontal plane. As before, the row reduced matrix (15) allows us to ascertain the following properties of the front system (14a).

SM2.1. Free Variables. The front system (14a) has 2 or 3 degrees of freedom depending on the value of the non-negative rainfall speed V_T. For positive rainfall speeds the system has 2 degrees of freedom. In particular, if $V_T > 0$, then the rainwater jump satisfies $[[Q_r]] = 0$. So, notice that this case of the front speed is only realized, for non-trivial rainfall speeds, when $[[Q_r]] = 0$. That is, the rainwater in the saturated region is exactly at saturation point. Otherwise, if $V_T = 0$, an additional degree of freedom is allowed in the system since one of the pivot columns of (15) is lost.

SM2.2. Conditions for Non-Trivial Solution. If $[[Q_r]] = 0$, $[[Q_v - Q_{vs}]] = 0$, and any third jump variable of $\boldsymbol{c}$ is identically zero, e.g., $[[\Theta]] = 0$, then the front system (14a) only allows for the trivial solution.

SM2.3. Bound on the Temperature or Velocity. There is no relation between the temperature jump and water vapor as in (16b). Consequently, there is no accompanying condition on the sign of the temperature jump $[[\Theta]]$—the saturated side may be warm or cold—or bound for the along-front velocity difference or the magnitude of the velocity difference. Moreover, note that the moisture variables only arise in the definition of the buoyancy variables. So, the relative vorticity Z, vertical velocity W, and temperature $[[\Theta]]$ may be determined independently of moisture.

SM2.4. Bound on Vertical Velocity and Relative Vorticity. The vertical velocity satisfies $W = 0$. This implies that the vertical velocity is no longer singular at the interface and is in fact identically zero throughout the domain. The relative vorticity Z is given by equation (25). No inequality on the sign of the relative vorticity Z is possible, however.

SM2.5. Comments on the Potential Vorticity. Unlike in Remark 5.3, PV defined from the saturated buoyancy, $\mathcal{P}_s = Z - \alpha_z f[[B_s]]$, may indeed be singular at the front interface. This noticeable difference arises from the fact that $\omega = 0$ does not give rise to a jump condition relating water vapor and temperature as in (16b).

SM3. Relative Front Speed $\omega = -V_T\alpha_z/\alpha$. Here, we discuss the possibility of the front moving at the speed $\sigma_H = \overline{V} - V_T\alpha_z/\alpha$. We assume here that $V_T \neq 0$ since allowing $V_T = 0$ would produce the case $\omega = 0$, which was discussed in Section SM2. In this case, the row reduced matrix (15) allows for the following brief observations of the front system (14a).

SM3.1. Free Variables. The linear system (15) requires that all jump variables except $[[B_u]]$ and $[[Q_r]]$ be identically zero. That is, $Z = 0$, $W = 0$, $[[B_s]] = 0$, $[[U]] = 0$, $[[V]] = 0$, $[[Q_v - Q_{vs}]] = 0$, and $[[\Theta]] = 0$. The front system (14a) has 1 degree of freedom. Notice then that this case of the front speed is only realized when the water vapor is exactly at saturation point. The rainwater mixing ratio, on the other hand, may be freely chosen.

SM3.2. Conditions for Non-Trivial Solution. The front system (14a) has only the trivial solution if $[[Q_r]] = 0$.

SM4. Front with Saturated Region Overlaying Unsaturated Region. The propagation of the PQG front solution for $\alpha_z = -1$ is similar to that when $\alpha_z = 1$, discussed in Section 6.1 and Section SM5. Therefore, we primarily focus on the differences arising from the change in α_z. When $\alpha_z = -1$, the horizontal front speed is bounded by:

$$\overline{V} \leq \sigma_H \leq \overline{V} + \frac{V_T}{\alpha}. \tag{SM8}$$

Then, for a front in which the saturated/cold region overlays the unsaturated/warm region, it follows that:

- If $\overline{V} \geq 0$, then only cold fronts may develop.
- If $-V_T/\alpha < \overline{V} < 0$, then the solution may be a cold, warm, or stationary front.
- If $\overline{V} \leq -V_T/\alpha$, then only warm fronts may develop.

A stationary front arises only when the front speed is identically zero ($\sigma_H = 0$). Therefore, the across-front velocity must be the same sign as α_z, to counter balance the rainfall term in equation (19). Namely, for $\alpha_z = -1$, the across-front velocity must point into the saturated/cold region. As before, a zero across-front velocity $\overline{V} = 0$ will not give rise to a stationary front.

The properties of the along-front velocity, vertical velocity, and relative vorticity are nearly identical to those in Section 5.2. The bound (18) has the following implication for the along-front velocity difference. For $\alpha_z = -1$, the along-front velocity U satisfies $f[[U]] > 0$, or $U_s > U_u$ in the northern hemisphere and $U_u > U_s$ in the southern hemisphere. That is, the along-front velocity experiences a negative (positive) shear across the front in the northern (southern) hemisphere.

Lastly, the vertical velocity given by (23) satisfies the inequality $W < 0$ if $\alpha_z = -1$. So, only a downward vertical velocity constant is allowed at the front interface. In addition, (25) with $[[\Theta]] < 0$ gives the condition $fZ < 0$ if $\alpha_z = -1$ for the relative vorticity.

SM5. Further Discussion on the Effect of Mean Wind on Fronts. Here we provide additional details and extend the discussion of Section 6.1. Recall that in this case we assume that the unsaturated/warm region overlays the saturated/cold region; $\alpha_z = 1$. We begin by continuing the discussion on the effect that the mean wind has on the types of fronts which may develop.

For there to be a cold front ($\sigma_H > 0$) it is necessary that the across-front velocity of the system satisfy $\overline{V} > 0$. Namely, the across-front velocity must be directed into the unsaturated/warm region. In a physical sense, for a front with a cold poleward side, $V > 0$ implies that the across-front velocity must be equatorward for a cold front to arise. The condition $\overline{V} > 0$ does not, however, exclusively give rise to cold fronts in this setup since $0 < \overline{V} < V_T/\alpha$ also allows for warm and stationary fronts. Notably, however, no cold front is possible in the current setup if the across-front velocity points toward the saturated/cold region, $\overline{V} < 0$, since the rainfall term of (19) is negative definite.

Similarly, it follows from inequality (27) that for there to be a warm front ($\sigma_H < 0$) it is sufficient that the kinematic speed satisfy $\overline{V} < 0$. Namely, the across-front velocity is directed into the saturated/cold region. Considering the case of a front with a cold poleward side, a poleward across-front velocity guarantees a warm front. Indeed, warm fronts may form if the velocity is either poleward or equatorward; the equatorward velocity, however, must not be too large.

Stationary fronts are obtained when the front speed is identically zero ($\sigma_H = 0$). In this setup, this implies that the rainfall and kinematic speed terms in (19) balance exactly. For this to occur, it is necessary for the across-front velocity to be the same sign as α_z to counter balance the strictly negative rainfall term. Namely, since $\alpha_z = 1$, the across-front velocity must point into the unsaturated/warm region for a stationary front to arise. The kinematic front speed $\overline{V}$, moreover, must not be so large that the rainfall term cannot balance it. Considering that the across-front velocity $\overline{V}$ may be small in comparison to the value of the rainfall V_T/α, stationary fronts in this setup may require a suitably small jump in rainwater.

The along-front velocity, vertical velocity, and relative vorticity are determined independently from the across-front velocity. Their properties remain essentially unchanged from those discussed in the example of Section 5.1. So, we only briefly discuss these variables below for a generic front geometry.

The along-front velocity difference may be understood by means of the bound (18). Considering that $\alpha_z = 1$, the along-front velocity U satisfies $f[[U]] < 0$, or $U_s < U_u$ in the northern hemisphere and $U_u < U_s$ in the southern hemisphere. These conditions about the along-front velocity may be interpreted simply as requiring a positive (negative) velocity shear across the front in the northern (southern) hemisphere. Moreover, by (18), this velocity difference is allowed to be larger if the corresponding temperature and water vapor jump is large enough. In contrast, notice that the only constraint on the across-front velocity difference is continuity, i.e., $[[V]] = 0$.

Lastly, since $\alpha_z \omega < 0$, the vertical velocity given by (23) satisfies the inequality $W > 0$ if $\alpha_z = 1$. That is, only an upward vertical velocity constant is allowed at the interface. In addition, since $\alpha_z = 1$, equation (25) with $[[\Theta]] < 0$ gives the condition $fZ > 0$ if $\alpha_z = 1$. Thus, admissible fronts require positive relative vorticity in the northern hemisphere and negative relative vorticity in the southern hemisphere at the front interface. Again, we summarize the observations on the velocities of the last two paragraphs, for $\overline{U} = 0$, in pictorial form in Figure SM1.

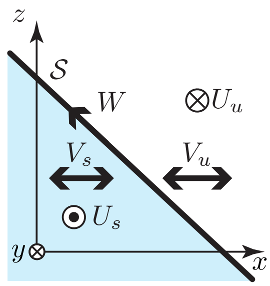

FIG. SM1. *Sketch of physical fronts in the northern hemisphere for $\overline{U} = 0$; the front interface is labeled $\mathcal{S}$ and separates the saturated (light blue) and unsaturated (clear) sides. The vertical velocity constant W must be positive on the front interface and zero elsewhere. The across-front velocity may be positive or negative, but must point in the same direction on both sides.*

SM6. On the Possibility of Discontinuous Fronts with No Phase Change. In this section we consider the possibility of a front in the PQG system (2a)–(2c) with relations (4a)–(4c) and (6) between two regions of the same phase. That is, we discuss the possibility of frontal solutions when the regions on each side of the front $\mathcal{S}$ are both unsaturated or both saturated only. In principle, since no phase change takes place, this case is analogous to the study of frontal solutions of the QG equations.

SM6.1. Unsaturated Jump. In the case when both sides of $\mathcal{S}$ are unsaturated, we obtain a set of jump conditions similar to (12a)–(12g). Since there is no jump in phase, we proceed with the understanding that the symbol $[[\varphi]]$ represents the difference of the variable φ between the two sides of $\mathcal{S}$. Indeed, most jump equations remain unchanged, since moisture only arises in our model in the buoyancy and buoyancy evolution equations. Moreover, since we are only considering unsaturated dynamics, we only need to consider the unsaturated buoyancy B_u. Namely, the evolution equation for the saturated buoyancy B_s does not contribute additional information in the unsaturated case so we do not consider this variable.

The original jump conditions (12a) and (12c)–(12e) remain unchanged in this scenario as they are independent of moisture. Jump condition (12f) for the unsaturated buoyancy, however, is changed due to our different phase change assumptions and becomes

$$[[B_u]] = \frac{1}{N_u^2}\,[[\Theta]]\,. \tag{SM9}$$

Importantly, the jump condition (12b) arising from the evolution of saturated buoyancy remains unchanged except for the fact that the rainwater term is dropped. Additionally, jump condition (12g) remains identical. Since both jump equations for the saturated buoyancy do not impose additional restrictions to the frontal jump but merely add an additional unknown quantity with each new equation, we may safely ignore these two jump equations.

Therefore, this scenario gives rise to the linear system $A_\omega \boldsymbol{c} = \boldsymbol{0}$, as in (14a), with

$$A_\omega = \begin{bmatrix} \omega & f\frac{\alpha_z}{\alpha} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & f\frac{\alpha_z}{\alpha} & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{N_u^2} \end{bmatrix} \quad \text{and} \quad \boldsymbol{c} = \begin{bmatrix} \alpha^{-1}Z \\ \alpha^{-1}W \\ [[B_u]] \\ [[U]] \\ [[V]] \\ [[\Theta]] \end{bmatrix}. \tag{SM10}$$

We may row reduce the matrix A_ω to obtain

$$A_\omega \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{\alpha}{f\alpha_z} \\ 0 & 1 & 0 & 0 & 0 & -\omega\frac{\alpha^2}{f^2\alpha_z^2} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{N_u^2} \\ 0 & 0 & 0 & 1 & 0 & -\frac{\alpha}{f\alpha_z} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega\left(\frac{\alpha^2}{f^2\alpha_z^2} + \frac{1}{N_u^2}\right) \end{bmatrix}. \tag{SM11}$$

As before, row reduced (SM11) clearly shows that the rank of A_ω is dependent on the value of ω.

In the case that $\omega \neq 0$, we may note that only the trivial jump $\boldsymbol{c} = \boldsymbol{0}$ is possible. Therefore, the solution cannot move with a speed different from the kinematic front speed (pure advection) if there is no phase change.

In the case $\omega = 0$, it is indeed possible to have a frontal solution. Notice, however, that the vertical velocity is then forced to be identically zero throughout the domain and the motion of the discontinuity corresponds to pure horizontal advection.

Now, if in addition to $\omega = 0$, we require for the PV to be bounded, only the trivial jump solution is possible. First, requiring a bounded PV implies

$$\mathcal{P}_u = Z - \alpha_z f[[B_u]] = 0. \tag{SM12}$$

Then, using (SM10)–(SM11) allows us to rewrite

$$\mathcal{P}_u = -f\alpha_z\left(\frac{\alpha^2}{f^2} + \frac{1}{N_u^2}\right)[[\Theta]] = 0. \tag{SM13}$$

Equation (SM13) shows that a bounded PV requires $[[\Theta]] = 0$ and the jump matrix then only allows the trivial solution. So, broadly speaking, either a singular PV or a phase change is required across the front for the solutions to be non-trivial.

SM6.2. Saturated jump. We proceed in a similar manner as the previous section. In the case when both sides of $\mathcal{S}$ are saturated, we only need to consider the saturated buoyancy B_s. Jump conditions (12a)–(12e) remain mostly unchanged, as in the unsaturated case, with moisture terms being dropped. Jump condition (12g) is then

$$[[B_s]] = \left[\left[\frac{\Theta}{N_s^2}\right]\right]. \tag{SM14}$$

Again, we may write the linear matrix system $A_\omega \boldsymbol{c} = \boldsymbol{0}$ with

$$A_\omega = \begin{bmatrix} \omega & f\frac{\alpha_z}{\alpha} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & f\frac{\alpha_z}{\alpha} & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{N_s^2} \end{bmatrix} \quad \text{and} \quad \boldsymbol{c} = \begin{bmatrix} \alpha^{-1}Z \\ \alpha^{-1}W \\ [[B_s]] \\ [[U]] \\ [[V]] \\ [[\Theta]] \end{bmatrix}. \tag{SM15}$$

Note that (SM15) is identical to the system (SM10) we obtained in the unsaturated case if we replace N_s^2, B_s with N_u^2, B_u, respectively. Therefore, the same broad conclusions apply to this system. Namely, $\omega \neq 0$ allows for only the trivial jump $\boldsymbol{c} = \boldsymbol{0}$. In the case $\omega = 0$, we may have non-trivial solutions unless the PV is bounded; $\mathcal{P}_s = Z - \alpha_z f[[B_s]] = 0$ in this case.

SM7. Jump Conditions in Large Rainfall Speed Limit. The PQG equations (2a)–(2c) used in this paper are derived under the assumption that rainfall speed is of the same order as the characteristic vertical velocity. For this reason, modifications arising from the large rainfall speed limit are of some interest. In this section, we discuss how the jump conditions that arise from the PQG system using the ansatz (10c)–(10d) are modified in the case that rainfall is considered asymptotically large in comparison to the vertical velocity. We do not include a careful asymptotic justification for the leading order balances and resulting equations discussed below, but refer the interested reader to [SM2] for a systematic derivation of the PQG equations with large rainfall speed.

Much of the structure of our current PQG system remains unchanged in the large rainfall speed limit. In practical terms, the large rainfall speed limit essentially entails replacing the evolution equation (2b) for the unsaturated buoyancy, which arises from a leading order balance of zero vertical velocity, with a leading order balance between the vertical velocity and the rainfall speed. Namely, equation (2b) is replaced by the equation

$$w\frac{d\tilde{q}_t}{dz} = V_T \frac{\partial q_r}{\partial z}. \tag{SM16}$$

which becomes

$$W = \gamma \alpha_z \frac{V_T}{N_u^2}[[Q_r]] \tag{SM17}$$

for the ansatz (10c)–(10d), where $d\tilde{q}_t/dz$ is the gradient of background total water and $\gamma = N_u^2 / \left(\frac{gL_v}{\theta_0 c_p} \frac{d\tilde{q}_t}{dz} \right)$. Therefore, our system of jump equations is essentially unchanged and we proceed to only highlight the key differences.

In complete analogy to the work in the Section 4, we may write a linear system for the jump variables in terms of the front geometry. This system, with constraints (14d) imposed by the moisture, takes the form

$$A_\omega \boldsymbol{c} = \boldsymbol{0}, \tag{SM18a}$$

where
(SM18b)

$$A_\omega = \begin{bmatrix} \omega & f\frac{\alpha_z}{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\gamma V_T\frac{\alpha_z}{\alpha} & 0 & 0 \\ 0 & 1 & 0 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f\frac{\alpha_z}{\alpha} & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{N_u^2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -\frac{1}{N_s^2} \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} \alpha^{-1}Z \\ \alpha^{-1}W \\ [[B_u]] \\ [[B_s]] \\ [[U]] \\ [[V]] \\ \frac{1}{N_u^2}[[Q_r]] \\ \frac{1}{N_s^2}[[Q_v - Q_{vs}]] \\ [[\Theta]] \end{bmatrix}.$$

The 8×9 matrix A_ω may be row reduced to echelon form as:

$$A_\omega \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha}{f\alpha_z} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega\frac{\alpha^2}{f^2\alpha_z^2} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{N_u^2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -\frac{1}{N_s^2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{\alpha}{f\alpha_z} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma V_T\frac{\alpha_z}{\alpha} & 0 & -\omega\frac{\alpha^2}{f^2\alpha_z^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & \omega\left(\frac{\alpha^2}{f^2\alpha_z^2} + \frac{1}{N_s^2}\right) \end{bmatrix}. \tag{SM19}$$

Note that the row reduced system (SM19) is nearly identical to our original system (15) obtained for rainfall speed comparable to the vertical velocity. Comparing these two systems, only the equation relating the rainwater with the temperature jump is different. That is, the only equation that is different is that which gives the front speed of the system.

The equation for the rainwater and temperature that replaces (16a) is now

$$\gamma V_T \frac{\alpha_z}{\alpha}\frac{f^2}{N_u^2}[[Q_r]] - \omega\alpha^2[[\Theta]] = 0. \tag{SM20}$$

Solving explicitly for the front speed and using the equation for the water vapor, which is identical to (16b), we find in analogy to the front speed (19) that in the large rainfall speed limit the horizontal front speed is

$$\sigma_H = \overline{V} - \gamma V_T \frac{\alpha_z}{\alpha}\left(\frac{\alpha^2 N_s^2 + f^2}{\alpha^2 N_u^2}\right)\frac{[[q_r]]}{[[q_v - q_{vs}]]}. \tag{SM21}$$

Note interestingly that in this case the horizontal front speed $\sigma_H = \sigma/\alpha$ is only bounded above or below by the kinematic front speed $\overline{V}$ and is not bounded in any form by the rainfall speed, in contrast to (20). Namely, either

$$\overline{V} < \sigma_H \quad \text{or} \quad \sigma_H < \overline{V} \tag{SM22}$$

depending on the sign of α_z. So, the front speed in the large rainfall speed limit is now unbounded.

REFERENCES

[SM1] R. R. Rogers and M. K. Yau, *A short course in cloud physics*, Butterworth-Heinemann, 3rd ed., 1989.

[SM2] L. M. Smith and S. N. Stechmann, *Precipitating quasigeostrophic equations and potential vorticity inversion with phase changes*, J. Atmos. Sci., 74 (2017), pp. 3285–3303, https://doi.org/10.1175/JAS-D-17-0023.1.