

# MITIGATING MODEL ERROR VIA A MULTI-MODEL METHOD AND APPLICATION TO TROPICAL INTRASEASONAL OSCILLATIONS\*

JASON L. TORCHINSKY<sup>†</sup> AND SAMUEL N. STECHMANN<sup>‡</sup>

Published in 2023

**Abstract.** Developing a model to capture all aspects of a complex dynamical system is an immense task, and each model will have deficiencies in some areas, such as global climate models having difficulty in capturing tropical intraseasonal variability such as the Madden–Julian Oscillation. Besides complex models, it is possible to create simplified, low-dimensional models to capture specific phenomena while ignoring many aspects of the full system. Here, we propose a strategy to allow complex models to communicate with simplified models throughout a simulation. The communication allows one to leverage the strengths of each model, without needing to change their dynamics, to mitigate model error. Furthermore, to ensure ease of implementation in complex systems, the strategy is based on common data assimilation techniques that are normally used to combine models and real-world data. This strategy is investigated here in a test case that is non-linear, non-Gaussian, and high-dimensional (approximately  $10^5$  degrees of freedom), and the multiple models have different state spaces. In particular, it is an idealized tropical climate model in three spatial dimensions. The multi-model communication strategy is seen to mitigate model error and reproduce statistical features akin to those of the truth model, when the communication is sufficiently frequent. In these tests, the low-dimensional model contributes only two degrees of freedom, which suggests that, in some systems, large amounts of model error can possibly be reduced by focusing on a small set of model components.

**Key words.** multi-model, filtering, data assimilation

**AMS subject classifications.** 68Q25, 68R10, 68U05

**1. Introduction.** Complex dynamical systems arise in many contexts in science and engineering, from the interaction of cosmic dust and planetary atmospheres to the germination of fungal spores. Models of such systems are based on an incomplete understanding of the system or rely on simplifying assumptions, e.g., [42, 43]. Even with a complete understanding and minimal assumptions, solving the model numerically can introduce errors and skew any predictions we make based on their output. As such, any model is prone to deficiencies in at least some areas.

Improving a complex model is a difficult task, and can lead to new or worse deficiencies. For example, attempts to improve the strongest signals of intraseasonal and interannual variability in the tropics of global climate models has been accompanied by worse climatological mean [3, 27]. The difficulty of this task is part of the inspiration for developing simplified, low-dimensional models that sacrifice capturing many aspects of the full system to focus on particular phenomena, such as, for instance, tropical intraseasonal variability [8, 35, 55], convectively-coupled equatorial waves [25, 56], or the El Niño–Southern Oscillation [59, 70], as a small subset of

---

\*Submitted: September 28, 2022. Published online: November 3, 2023.

**Funding:** The research of SNS is partially supported by the US Office of Naval Research (ONR) grant N00014-19-1-2421. Funding for JLT comes from the DOE Computational Science Graduate Fellowship, under grant number DE-SC0020347.

<sup>†</sup>Department of Mathematics, University of Wisconsin-Madison, Madison, WI (jason.torchinsky@wisc.edu, <https://jasonltorchinsky.github.io/>).

<sup>‡</sup>Department of Mathematics, University of Wisconsin-Madison, Madison, WI and Department of Atmospheric and Oceanic Sciences, University of Wisconsin-Madison, Madison, WI (stechmann@wisc.edu, <https://people.math.wisc.edu/~stechmann/>).

examples of challenging phenomena from one example discipline of atmospheric and oceanic sciences.

The main purpose of the present paper is to investigate a multi-model framework that allows multiple models to evolve together and share information (see Figure 1 for a schematic illustration), with the aim of leveraging the models' different strengths and mitigating their weaknesses. Different strategies could potentially be devised for such a task, and some options are described below. Here, we follow the approach of [9] and move further to utilize nonlinear and non-Gaussian models and three spatial dimensions and to allow more extensive communication between models.

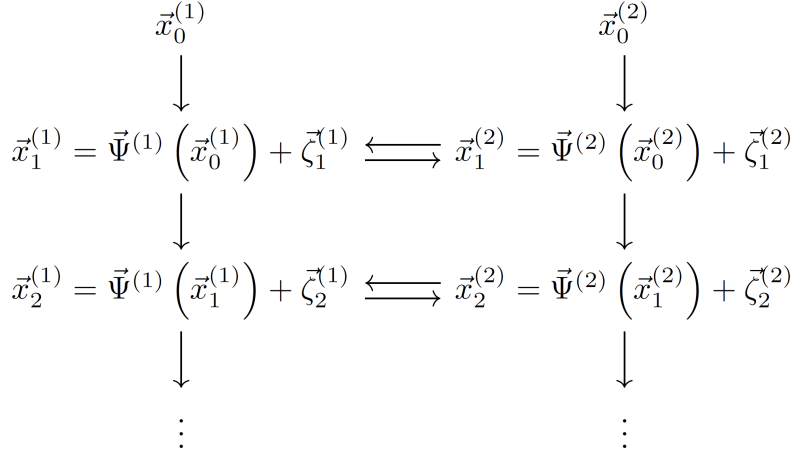


FIG. 1. Schematic diagram for the multi-model strategy with two models. The states of model 1 and model 2 are given by  $\vec{x}_k^{(1)}$  and  $\vec{x}_k^{(2)}$ , respectively, with dynamical operators  $\vec{\Psi}^{(1)}$  and  $\vec{\Psi}^{(2)}$  as well as stochastic noise  $\vec{\zeta}_k^{(1)}$  and  $\vec{\zeta}_k^{(2)}$ . The downward arrows indicate evolution in time, and the leftward and rightward arrows indicate inter-model exchange of data.

Some of the advantageous features and design principles of the multi-model method here and of [9] are as follows. Perhaps most importantly, the inter-model communication is formulated using ideas from data assimilation [20, 30, 33, 47]. While data assimilation usually involves the assimilation of observational data into a model, here it is also used for allowing one model to assimilate information from another model. The data assimilation framework then brings many benefits. First, the approach is non-invasive in the sense that the evolution equations in the individual model codes are not substantially modified; instead, when inter-model communication is needed, the data assimilation algorithms can be called on as external subroutines that adjust the state of each model. Hence, the non-invasive property is advantageous for complex computer codes that are difficult to modify. Second, many computer codes are already available for data assimilation [1, 37] and can be repurposed for multi-model communication. Third, a level of conceptual understanding is gained from using data assimilation ideas for the inter-model communication, since data assimilation methods are widely known in some scientific and engineering communities. Fourth, the approach is quite general in the sense that the collection of multiple models can be built from a wide variety of physics-based or data-driven models. Fifth, the quantification of uncertainties is an inherent aspect of data assimilation and is then naturally inherited by the multi-model communication method; it allows, for example, the different amounts of model error to be quantified and utilized as part of the multi-model

method.

Other approaches to multi-model methods have also been studied, such as the multi-model ensemble mean, a post-processing approach without coupling of the dynamics of the models [6, 17, 24]; the multi-model superensemble which uses weighted averages to combine the output of several models [28, 29, 68]; the supermodel approach based on the ideas of synchronization of dynamical systems [11, 12, 40, 54]; the multi-model ensemble Kalman filter [66, 67]; multi-scale methods such as superparameterization that couple macro-scale and micro-scale models [16, 18, 45, 58]; and a variety of multi-fidelity methods that utilize multiple models [41] and have been used in a wide range of areas, including wing-design optimization [48], robotic learning [10], and impedance cardiography [46]. These approaches are helping to provide frameworks for utilizing multiple models together in different ways in science and engineering applications. One aspect of the present approach that is appealing is its foundation of a data assimilation framework, as described in a previous paragraph. The present study also provides one of few investigations of coupling the dynamics of high- and low-dimensional models together, which brings the challenge of mapping between different state spaces, and provides the possibility of computational efficiency.

Besides multi-model methods, another approach to mitigating model error is to learn, and possibly correct for, the model error or bias of an individual model. Many techniques could be used for such an approach, such as data assimilation techniques [36, 49] or machine learning techniques [31, 53]. While not explicitly utilized in the present paper, such approaches could be used, for instance, to mitigate model error for the individual models of a multi-model method. All of these approaches are contributing to the broad community effort toward improved predictive methods using ideas from dynamical systems, data-driven modeling, data assimilation, machine learning, and data science [2, 4, 7, 13, 14, 15, 19, 21, 22, 34, 39, 44, 50, 51, 52, 62, 63].

The remainder of the paper is organized as follows. In Section 2, we describe the formulation of our multi-model strategy in detail. Section 3 provides an example application in a low-dimensional setting, using systems of non-linear stochastic differential equations (SDEs), as an illustration of the basic ideas. Section 4 presents an investigation in a high-dimensional setting with spatiotemporal variations with three spatial dimensions. Conclusions are in Section 5, and many details of the methods are presented in the appendices to streamline the main text.

**2. Multi-model communication strategy.** In this section we describe the general form of our multi-model communication strategy. Recall the setting, as described in Section 1: A complex dynamical system of interest is identified, but the evolution equations of the complex system are unknown or computationally intractable. Instead, several imperfect models are known and tractable, where each imperfect model has its own strengths and weaknesses. The goal then is to develop a strategy for allowing the multiple models to share information with each other, in hopes that the collection of imperfect models can better represent the true system than any individual imperfect model.

We begin by introducing the form of the truth model and each imperfect model in the multi-model ensemble.

Let  $\vec{x}_k$  be the state at discrete time  $k$  of a stochastic system whose dynamics may be non-linear, representing the “truth”. In particular, let  $\mathcal{S}$  be the  $d$ -dimensional state-space of the system, and its dynamics given by the discrete equation

$$(2.1) \quad \vec{x}_{k+1} = \vec{\Psi}(\vec{x}_k) + \vec{\zeta}_{k+1},$$

where  $\vec{\zeta}_{k+1} \in \mathbb{R}^d$  is unbiased noise with covariance  $\Sigma \in \mathbb{R}^{d \times d}$ . The ideas can potentially be extended to a continuum setting, although the discrete-time and finite-state-space case is described here for simplicity, as is common. The discrete model can potentially arise from the discretization of a continuum model such as deterministic or stochastic partial differential equations.

The goal is to approximate this truth with a collection of  $M$  imperfect models, indexed by  $m = 1, \dots, M$ , whose state vectors  $\vec{x}_{k|k}^{(m)}$  may be in distinct state-spaces  $\mathcal{S}^{(m)}$  of size  $d^{(m)}$ . The subscript “ $k|k$ ” denotes that  $\vec{x}_{k|k}^{(m)}$  is an *a posteriori* estimate, i.e., the estimate for  $\vec{x}_k$  by model  $m$  after the data assimilation and/or multi-model communication has occurred in the previous time-step. The dynamics of each model may also be non-linear, and are of the following discrete form

$$(2.2) \quad \vec{x}_{k+1|k}^{(m)} = \vec{\Psi}^{(m)} \left( \vec{x}_{k|k}^{(m)} \right) + \vec{\zeta}_{k+1}^{(m)}, \quad m = 1, \dots, M,$$

where  $\vec{\Psi}^{(m)}$  is the dynamics operator of model  $m$ , and  $\vec{\zeta}_{k+1}^{(m)} \in \mathbb{R}^{d^{(m)}}$  is unbiased noise with covariance  $\Sigma^{(m)} \in \mathbb{R}^{d^{(m)} \times d^{(m)}}$ . The subscript “ $k+1|k$ ” denotes that  $\vec{x}_{k+1|k}^{(m)}$  is an *a priori* estimate, i.e., the estimate for  $\vec{x}_{k+1}$  by model  $m$  before the data assimilation and/or multi-model communication has occurred.

Our multi-model communication strategy is based on ideas from data assimilation, and among the different data assimilation methods that are in use, here the presentation will most closely resemble the three-dimensional variational assimilation (3D-Var) method and ensemble Kalman filter methods. These data assimilation methods are chosen here because they are relatively simple and because they provide the foundation for other, more complex methods, which could also be considered in future work. Each of these methods is used to assimilate observations of real-world data into a simulation to improve accuracy and reliability. For a thorough exposition of the 3D-Var, ensemble Kalman filter, and related data assimilation strategies, we refer the reader to [33, 69].

One of the key equations of these data assimilation methods is

$$(2.3) \quad \vec{x}_{k+1|k+1} = \vec{x}_{k+1|k} - \mathbf{K}_{k+1} \left( \mathbf{H} \vec{x}_{k+1|k} - \vec{z}_{k+1} \right)$$

which describes how the observation is assimilated into the model state. Specifically,  $\vec{x}_{k+1|k+1}$  is the *a posteriori* estimate resulting from assimilating an observation. The matrix  $\mathbf{K}_{k+1}$  is the *gain matrix* which governs the strength of assimilation of the observation into the state of the model. The gain matrix is chosen to be constant in the 3D-Var method and evolves over time in the Kalman filter [33, 69]. Meanwhile,  $\mathbf{H}$  is the *observation matrix* which maps the unknown true state to the observation  $\vec{z}_{k+1}$  that is assimilated. The  $\mathbf{H} \vec{x}_{k+1|k}$  term may be understood as “what would be observed if the model state  $\vec{x}_{k+1|k}$  matched the true state”.

The assimilation step of our multi-model communication strategy largely resembles this equation. Instead of assimilating an observation of the truth as in data assimilation methods, we define a *pseudo-truth* for each model  $m$ . We then make an observation (a *pseudo-observation*) of the the pseudo-truth for each model, and assimilate that into the state of model  $m$ .

To generate the pseudo-truth signal for model  $m$ , we use the *a priori* estimate of every model as the input for a map  $\vec{\Gamma}^{(m)} : \prod_{n=1}^M \mathcal{S}^{(n)} \rightarrow \mathcal{S}^{(m)}$ . This map, which we call the *pseudo-truth map for model  $m$* , is used to produce the *pseudo-truth signal for*

model  $m$ ,

$$(2.4) \quad \bar{y}_{k+1}^{(m)} = \bar{\Gamma}^{(m)} \left( \bar{x}_{k+1|k}^{(1)}, \dots, \bar{x}_{k+1|k}^{(M)} \right) + \bar{\xi}_{k+1}^{(m)}, \quad m = 1, \dots, M,$$

where  $\bar{\xi}_{k+1}^{(m)} \in \mathbb{R}^{d^{(m)}}$  is unbiased noise with covariance  $\mathbf{R}^{(m)} \in \mathbb{R}^{d^{(m)} \times d^{(m)}}$ .

The pseudo-truth map for model  $m$  should be chosen to contain information to supplement the deficient aspects of model  $m$ . For example, the pseudo-truth signal for a climate model with a poor representation of the Madden-Julian Oscillation should contain information about the Madden-Julian Oscillation.

Following the ideas of data assimilation, we utilize ‘observations’ of the pseudo-truth instead of the pseudo-truth itself. We refer to these ‘observations’ as the *pseudo-observations for model  $m$* . We consider a linear pseudo-observation model

$$(2.5) \quad \bar{z}_{k+1}^{(m)} = \mathbf{H}^{(m)} \bar{y}_{k+1}^{(m)} + \bar{\lambda}_{k+1}^{(m)}, \quad m = 1, \dots, M,$$

where  $\mathbf{H}^{(m)} \in \mathbb{R}^{q^{(m)} \times d^{(m)}}$  (typically with  $q^{(m)} \leq d^{(m)}$ ) is the *pseudo-observation matrix for model  $m$* , and  $\bar{\lambda}_{k+1}^{(m)} \in \mathbb{R}^{q^{(m)}}$  is unbiased noise with covariance  $\mathbf{\Lambda}^{(m)} \in \mathbb{R}^{q^{(m)} \times q^{(m)}}$ .

In the 3D-Var method and the original Kalman filter, the prediction of the filter is clear as there is only a single model. A related data assimilation strategy is the ensemble Kalman filter, which uses multiple instances of the same model to obtain both a prediction and the uncertainty of that prediction (see [33, 60, 61, 69] for more on the ensemble Kalman filter). The prediction of the ensemble Kalman filter is generally taken to be the mean of the states of each ensemble member and has uncertainty reflected by the covariance of the ensemble members [33, 69]. However, there might not be such a straightforward way to determine the prediction of a multi-model ensemble in our strategy. For example, if some models in the ensemble contains a thermal variable and the rest do not, what is the prediction for the thermal variable of the ensemble? In later sections, we examine the output of each member individually. This approach may not be practical for ensembles with a large number of models and may require aggregate statistics to form a singular output.

Each model will have a gain matrix  $\mathbf{K}^{(m)}$  as part of the assimilation / communication step, and one contribution to it is from a background error covariance  $\mathbf{B}^{(m)} \in \mathbb{R}^{d^{(m)} \times d^{(m)}}$  for each model. The matrix  $\mathbf{B}^{(m)}$  should reflect the uncertainty or lack of confidence in the state of model  $m$ —i.e., larger entries on the diagonal of  $\mathbf{B}^{(m)}$  imply lesser trust in model  $m$  than in the pseudo-observation of a given state variable. This is similar to the background error covariance in the 3D-Var method, in which the gain matrix is chosen to be constant [33, 69]. We will pre-select  $\mathbf{B}^{(m)}$  here for simplicity, and in future work we hope to evolve the background error covariance throughout a simulation by a weighted comparison of the states of the members of the ensemble.

With these definitions in mind, we may now state our multi-model communication strategy in full:

**Forecast Step.** Given the posterior state  $\bar{x}_{k|k}^{(m)}$  at time-step  $k$  for each model, we calculate the *a priori* estimate  $\bar{x}_{k+1|k}^{(m)}$  for each model via its dynamical evolution equation (Eqn. 2.2).

**Communication Step.** Upon calculating the *a priori* estimate and generating a pseudo-truth signal  $\bar{y}_{k+1}^{(m)}$  (Eqn. 2.4) for each model, we generate a pseudo-observation

$\vec{z}_{k+1}^{(m)}$  (Eqn. 2.5). Each ensemble member is then updated to

$$(2.6) \quad \vec{x}_{k+1|k+1}^{(m)} = \vec{x}_{k+1|k}^{(m)} - \mathbf{K}^{(m)} \left( \mathbf{H}^{(m)} \vec{x}_{k+1|k}^{(m)} - \vec{z}_{k+1}^{(m)} \right),$$

where the gain matrix  $\mathbf{K}^{(m)} \in \mathbb{R}^{d^{(m)} \times q^{(m)}}$  is given by

$$(2.7) \quad \mathbf{K}^{(m)} = \mathbf{B}^{(m)} \left( \mathbf{H}^{(m)} \right)^T \left( \mathbf{I} + \mathbf{H}^{(m)} \mathbf{B}^{(m)} \left( \mathbf{H}^{(m)} \right)^T \right)^{-1}$$

and  $\mathbf{I}$  is the appropriately-sized identity matrix. The assimilation/communication equation of our multi-model communication strategy is the same as that of the 3D-Var and Kalman filter methods (Eqn. 2.3) with the appropriate matrix substitutions.

To summarize, the dynamics of each model of the multi-model ensemble in our multi-model communication strategy are

$$(2.8a) \quad \vec{x}_{k+1|k}^{(m)} = \vec{\Psi}^{(m)} \left( \vec{x}_{k|k}^{(m)} \right) + \vec{\zeta}_{k+1}^{(m)},$$

$$(2.8b) \quad \vec{y}_{k+1}^{(m)} = \vec{\Gamma}^{(m)} \left( \vec{x}_{k+1|k}^{(1)}, \dots, \vec{x}_{k+1|k}^{(M)} \right) + \vec{\xi}_{k+1}^{(m)}, \quad \vec{z}_{k+1}^{(m)} = \mathbf{H}^{(m)} \vec{y}_{k+1}^{(m)} + \vec{\lambda}_{k+1}^{(m)},$$

$$(2.8c) \quad \vec{x}_{k+1|k+1}^{(m)} = \vec{x}_{k+1|k}^{(m)} - \mathbf{K}^{(m)} \left( \mathbf{H}^{(m)} \vec{x}_{k+1|k}^{(m)} - \vec{z}_{k+1}^{(m)} \right)$$

for  $m = 1, \dots, M$ .

**3. Low-dimensional test case of the multi-model methodology.** In this section, we investigate a non-linear stochastic differential equation (SDE). It serves as a low-dimensional illustration of the multi-model methodology. The low-dimensional test case is useful because it allows each individual piece of the multi-model strategy to be examined. In Section 4, a high-dimensional case will be considered.

As an example of a non-linear, non-Gaussian SDE, we investigate a modified version of the model of the Madden–Julian Oscillation (MJO) introduced in 2014 by Chen, Majda, and Giannakis in [8], which we refer to as the mCMG14 system. In our modified version, the stochastic phase shift  $\omega$  was eliminated and the seasonal forcing  $v_f(t)$  was absorbed into the climate signal variable  $v$ , to give a simpler system with nearly all the same nontrivial features. The mCMG14 system is given by

$$(3.1a) \quad du_1 = (-d_u u_1 + \gamma v u_1 - a u_2) dt + \sigma_u dW_{u_1}(t),$$

$$(3.1b) \quad du_2 = (-d_u u_2 + \gamma v u_2 + a u_1) dt + \sigma_u dW_{u_2}(t),$$

$$(3.1c) \quad dv = (-d_v v - \gamma (u_1^2 + u_2^2) + v_f(t)) dt + \sigma_v dW_v(t),$$

where  $W_{u_1}(t)$ ,  $W_{u_2}(t)$ , and  $W_v(t)$  are independent Wiener processes, with time-periodic forcing

$$(3.2) \quad v_f(t) = d_v f_0 + f_t (d_v \sin(\omega_f t + \phi) - \omega_f \cos(\omega_f t + \phi))$$

to capture the active phase of the boreal winter MJO and the inactive summer season in the seasonal cycle. The apparent additional complexity of  $v_f(t)$  here versus that

in [8] is due to the change-of-variables to incorporate the time-periodic damping into  $v$ .

The variables  $u_1$  and  $u_2$  represent the time-evolution of the two primary spatial modes of the MJO, referred to as the MJO indices [8, 23, 64, 65]. Meanwhile,  $v$  represents a background climate signal and acts as a (stochastically evolving) damping/growth rate for  $u_1$  and  $u_2$ . The other parameters, along with their values given in [8], are in Table 1.

| Parameter  | Description                         | Truth                    | DTCM               | MJOM               |
|------------|-------------------------------------|--------------------------|--------------------|--------------------|
| $d_u$      | Decay Parameter for $u_1, u_2$      | $0.9 \text{ m}^{-1}$     | –                  | –                  |
| $d_v$      | Decay Parameter for $v$             | $0.9 \text{ m}^{-1}$     | –                  | –                  |
| $a$        | Background Phase of $u_1, u_2$      | $3 \text{ m}^{-1}$       | $0 \text{ m}^{-1}$ | –                  |
| $\sigma_u$ | Amplitude of Noise for $u_1, u_2$   | $0.3 \text{ m}^{-1/2}$   | –                  | –                  |
| $\sigma_v$ | Amplitude of Noise for $v$          | $1 \text{ m}^{-1/2}$     | –                  | –                  |
| $\gamma$   | Strength of Non-Linear Interactions | $0.3 \text{ m}^{-1}$     | –                  | –                  |
| $\omega_f$ | Frequency of $v_f(t)$               | $2\pi/12 \text{ m}^{-1}$ | –                  | $0 \text{ m}^{-1}$ |
| $\phi$     | Phase of $v_f(t)$                   | –1                       | –                  | 0                  |
| $f_0$      | Mean of $v_f(t)$                    | 1                        | –                  | 0                  |
| $f_t$      | Amplitude of $v_f(t)$               | 4.9                      | –                  | 0                  |

TABLE 1

Parameter values for the truth model (which match the optimal parameter values given in [8]), deficient tropical climate model (DTCM), and the MJO model (MJOM) for the low-dimensional test case. We use ‘–’ to denote that the parameter value matches that of the truth model. Note that the temporal unit  $m$  is months.

As a point of comparison for the performance of the ensemble, we first choose a configuration of the model to represent the truth. The truth model has dynamics given by the mCMG14 system (Eqs. 3.1) and uses the optimal parameter values given in [8], which are repeated here in Table 1. A sample output of the truth model is given in Figure 2a, and the probability density function (PDF) of  $u_1$  and  $u_2$  is given in Figure 2b.

The active periods of  $u_1, u_2$  variability are driven by the seasonality of the climate signal  $v$ . The active seasons correspond to peaks in  $v$ . In the original CMG14 system,  $u_1$  and  $u_2$  oscillate with varying frequency during the active season due to the stochastic phase shift. As the stochastic phase shift was eliminated in the mCMG14 system here,  $u_1$  and  $u_2$  oscillate with consistent frequency during the active season. The mCMG14 system still maintains variability in MJO strength, i.e., magnitude in  $u_1$  and  $u_2$ , in different active seasons.

The dynamics of both members of the multi-model ensemble were also given by Eqs. 3.1. The parameters of each member were chosen so that they were deficient in some way. The first model was deficient in representing the MJO and served the role of the deficient tropical climate model (DTCM). The second model lacked a meaningful climate signal and served the role of the MJO model (MJOM). This scenario is meant to mirror, in a low-dimensional setting, the difficulties faced by actual climate models in reproducing MJO variability, and the availability of simplified MJO models that are dedicated to MJO variability without detailed treatments of the rest of the climate system. Next, we describe the dynamics of each of these models in more detail. In Subsection 3.1, we describe the mappings and parameters used for multi-model communication, as well as the performance of the ensemble.

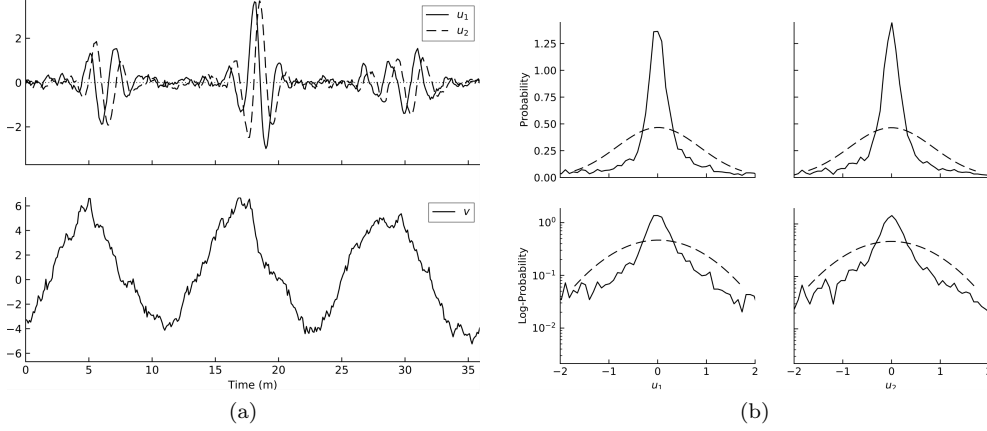


FIG. 2. (a) Sample output of the truth model, with dynamics given by Eqs. 3.1 and parameter values given in Table 1. The MJO indices  $u_1$ ,  $u_2$  undergo recurrent periods inactivity during the quiescent season followed by oscillations during the active season, e.g., the quiescent season during months 10 to 15 and the active season from months 15 to 21. (b) The probability density functions (PDFs) for  $u_1$ ,  $u_2$  (solid), along with Gaussian fits for these distributions (dashed). The PDFs for  $u_1$ ,  $u_2$  were calculated numerically from 500 independent runs of 480 months of the truth model.

The parameters for the DTCM were chosen such that the climate signal  $v$  has an evolution equation that is the same as the truth, but the dynamics of the MJO indices  $u_1$  and  $u_2$  were deficient. The background phase of  $u_1$  and  $u_2$  was zero, meaning that  $u_1$  and  $u_2$  did not oscillate during the active seasons—i.e., the model has no MJO. This is illustrated in Figure 3a, which shows a sample output of the DTCM. Despite this deficiency, the marginal PDFs of  $u_1$  and  $u_2$  (Figure 3b) strongly resemble those of the truth model.

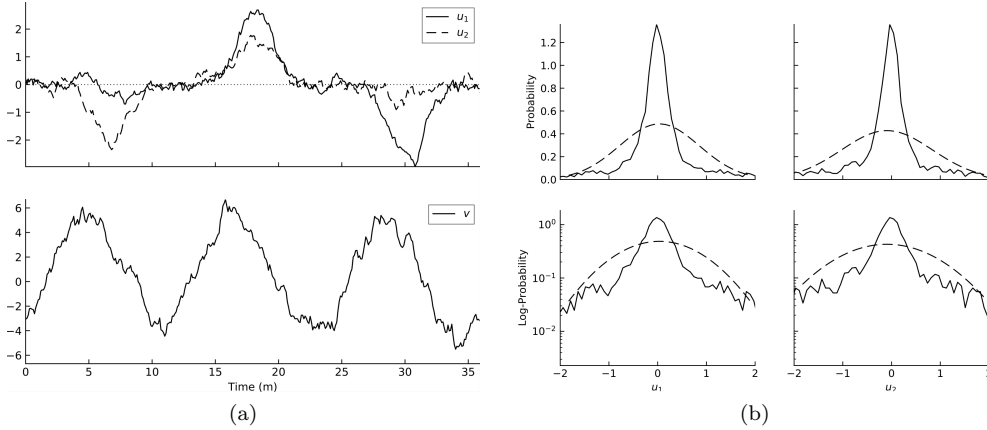


FIG. 3. (a) Sample output of the deficient tropical climate model (DTCM), which lacks intraseasonal oscillations of the MJO, with dynamics given by Eqs. 3.1 and parameter values given in Table 1. The MJO indices  $u_1$ ,  $u_2$  undergo recurrent periods of inactivity during the quiescent MJO season followed by oscillations during the active MJO season, e.g., the quiescent season during months 10 to 15 and the active season from months 15 to 21. (b) The marginal probability density functions (PDFs) for  $u_1$ ,  $u_2$  (solid), along with Gaussian fits for these distributions (dashed). Note that despite the deficiency of the DTCM, these PDFs resemble those of the truth model (Figure 2b). The PDFs for  $u_1$ ,  $u_2$  were calculated numerically from 500 independent runs of 480 months of the DTCM.

The parameters for the MJOM were chosen conversely; the dynamics of the MJO indices  $u_1$  and  $u_2$  matched those of the truth model, but the time-periodic forcing of the climate signal  $v$  was eliminated entirely. As the largest mode of the climate signal  $v$  was absent, it was very unlikely for the model to enter into an active MJO season. Were it to enter an active MJO season,  $u_1$  and  $u_2$  would behave similarly to those in the truth model. Figure 4a shows an example trajectory of the MJOM, highlighting the impact of eliminating the time-periodic forcing. Further, for small  $v$  the non-linear coupling terms in the mCMG14 system (Eqs. 3.1) are always small and the system resembles an Ornstein-Uhlenbeck process. This is reflected by the fact that the marginal PDFs of  $u_1$  and  $u_2$  closely match a normal distribution (Figure 4b).

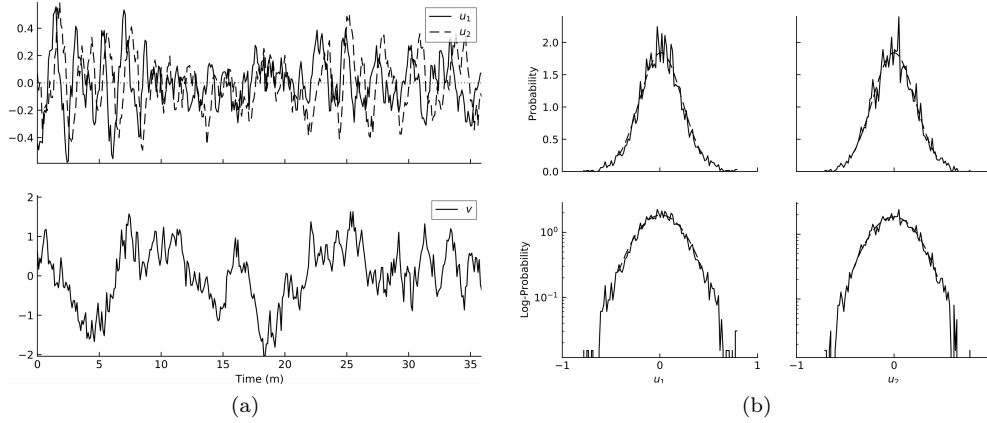


FIG. 4. (a) Sample output of the MJO model (MJOM), with dynamics given by Eqs. 3.1 and parameter values given in Table 1. Note the lack of significant activity in  $v$ , which results in the lack of distinct periods of activity versus inactivity for the  $u_1$  and  $u_2$ . (b) The marginal probability density functions (PDFs) for  $u_1$ ,  $u_2$  (solid), along with Gaussian fits for these distributions (dashed). These distributions were calculated numerically from 500 independent runs of 480 months of the MJOM.

**3.1. Multi-model communication and performance.** In this subsection, we discuss the choice of mappings and communication parameters in the multi-model communication strategy for the low-dimensional test case.

In this case, the deficient tropical climate model (DTCM) and MJO model (MJOM) share the same state space and their deficiencies are clear. Hence, the choice of pseudo-truth map for the DTCM and the MJOM is straightforward. The pseudo-truth signal for the DTCM should have information about the dynamics of the MJO, while the pseudo-truth signal for the MJOM should have contained information about the climate signal. For simplicity, we chose the same pseudo-truth map for the DTCM and the MJOM,

$$\vec{\Gamma}^{(DTCM)} \left( \vec{x}_{k+1|k}^{(DTCM)}, \vec{x}_{k+1|k}^{(MJOM)} \right) = \vec{\Gamma}^{(MJOM)} \left( \vec{x}_{k+1|k}^{(DTCM)}, \vec{x}_{k+1|k}^{(MJOM)} \right) = \begin{bmatrix} u_{1,k+1|k}^{(MJOM)} \\ u_{2,k+1|k}^{(MJOM)} \\ v_{k+1|k}^{(DTCM)} \end{bmatrix}, \quad (3.3)$$

where we utilize the notation from Section 2. The superscript  $(DTCM)$  refers to the deficient tropical climate model, and  $(MJOM)$  refers to the MJO model. In words, the pseudo-truth map for both models uses the MJO indices  $u_1$ ,  $u_2$  from the MJOM

and the climate signal  $v$  from the DTCM.

The covariance matrices for the noise in the pseudo-truth signals reflect the noise for  $u_1$ ,  $u_2$ , and  $v$  in each model. For simplicity, we choose the same covariance for noise in the pseudo-truth signal for both models:

$$(3.4) \quad \mathbf{R}^{(DTCM)} = \mathbf{R}^{(MJOM)} = \begin{bmatrix} 0.1 \sigma_u & 0 & 0 \\ 0 & 0.1 \sigma_u & 0 \\ 0 & 0 & 0.1 \sigma_v \end{bmatrix}.$$

The pseudo-observation matrices ensured that the DTCM observes only the MJO indices and the MJOM observes only the climate signal:

$$(3.5) \quad \mathbf{H}^{(DTCM)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{H}^{(MJOM)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

The covariance matrices for the pseudo-observation models, similarly to that of the pseudo-truth signal, reflect the noise strength in the dynamics of  $u_1$ ,  $u_2$ , and  $v$ :

$$(3.6) \quad \mathbf{\Lambda}^{(DTCM)} = \begin{bmatrix} 0.01 \sigma_u & 0 \\ 0 & 0.01 \sigma_u \end{bmatrix}, \quad \mathbf{\Lambda}^{(MJOM)} = [0.01 \sigma_v].$$

The background error covariance matrices are chosen so that the DTCM strongly assimilated only  $u_1$ ,  $u_2$  and the MJOM strongly assimilated only  $v$ :

$$(3.7) \quad \mathbf{B}^{(DTCM)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}, \quad \mathbf{B}^{(MJOM)} = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Inter-model communication occurs at every time-step, for simplicity in this illustration.

Using these mappings and matrices, we find that the multi-model communication strategy produces a proper MJO in both of its constituent models, the DTCM and MJOM. In particular, the MJO indices  $u_1$ ,  $u_2$  in the DTCM with communication (DTCM-C) oscillate during the active seasons. The climate signal  $v$  in the MJOM with communication (MJOM-C) undergoes a seasonal cycle, and the active seasons result in the growth of an MJO. In addition to the dynamics, the probability density functions (PDFs) of  $u_1$ ,  $u_2$  in both the DTCM-C and MJOM-C strongly resemble that of the truth model (Figure 2). The frequent, strong inter-model communication lead to the similarity of the dynamics and PDFs of the DTCM-C and MJOM-C.

A sample output of the DTCM-C along with PDFs of  $u_1$ ,  $u_2$  are shown in Figures 5a and 5b. Those for the MJOM-C are shown in Figures 5c and 5d.

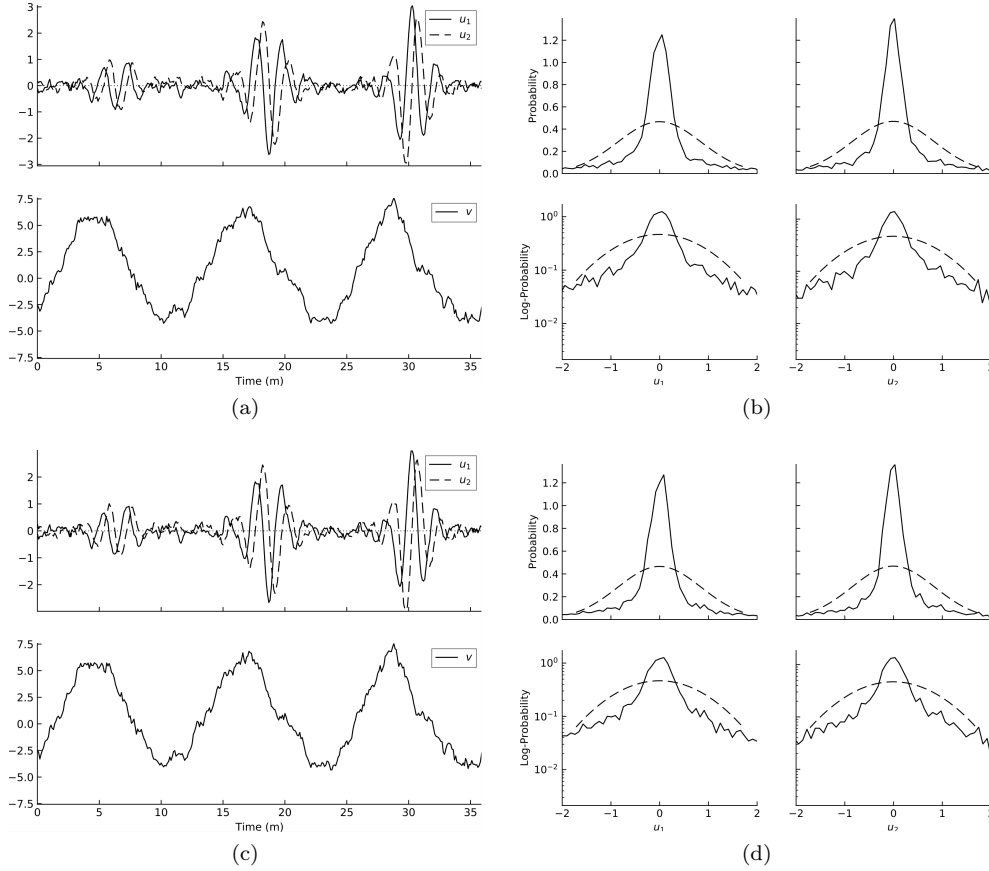


FIG. 5. (Left) Sample output of the deficient tropical climate model with communication (DTCM-C; a) and MJO model with communication (MJOM-C; c) for the low-dimensional test case, with communication mappings and parameters given throughout Subsection 3.1. Due to the frequency and strength of communication, the trajectory of both DTCM-C and MJOM-C were largely similar. The dynamics of each member of the ensemble strongly resembled that of the truth model (Figure 2a), amending the apparent deficiencies in the DTCM and MJOM (Figures 3a and 4a). (Right) The probability density functions (PDFs) for  $u_1$ ,  $u_2$  for the DTCM-C (b) and MJOM-C (d) also strongly resemble those of the truth model (Figure 2b). These distributions were calculated numerically from 500 independent runs of 480 months of the ensemble.

**4. High-dimensional test case of the multi-model methodology.** In this section, we investigate a deterministic, non-linear partial differential equation, in three spatial dimensions. It serves as a high-dimensional test case of the multi-model methodology.

As a test model that includes a variety of irregular, nonlinear waves, we investigate a model of the fluid dynamics of the tropical atmosphere introduced by Ogrosky, Stechmann, and Hottovy in 2019 in [38]. We refer to this system as the Ogrosky–Stechmann–Hottovy 2019 (OSH19) system. The work [38] explored various parameter regimes of the OSH19 system. Ogrosky, Stechmann, and Hottovy demonstrated the capability of the OSH19 system to produce phenomena in a variety of spatial and temporal scales. These phenomena included various MJO dynamics and convectively-coupled equatorial waves (CCEWs).

When describing the OSH19 system here, we re-use the variable names used in

[38] for ease of comprehension. We previously used some of these variable names for different quantities in Section 3. We take care to distinguish these meanings when, for example, we use  $v$  as the climate signal in the mCMG14 system versus  $v$  as the meridional wind velocity in the OSH19 system.

The OSH19 system consists of the moist hydrostatic incompressible Boussinesq equations with a 2-vertical-mode convective adjustment and moisture diffusion

$$(4.1a) \quad \frac{Du}{Dt} - \beta y v + \frac{\partial p}{\partial x} = -\frac{1}{\tau_u} u,$$

$$(4.1b) \quad \frac{Dv}{Dt} + \beta y u + \frac{\partial p}{\partial y} = -\frac{1}{\tau_u} v,$$

$$(4.1c) \quad \frac{\partial p}{\partial z} = \frac{g}{\theta_0} \theta,$$

$$(4.1d) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$(4.1e) \quad \frac{D\theta}{Dt} + B w = \frac{L_v}{c_p \tau_{\text{up}}} q_{\text{up}} + \frac{L_v}{c_p \tau_{\text{low}}} q_{\text{low}} - \frac{1}{\tau_\theta} \theta,$$

$$(4.1f) \quad \frac{Dq}{Dt} + v \frac{\partial q_{\text{bg}}}{\partial y} + w \frac{\partial q_{\text{bg}}}{\partial z} = -\frac{1}{\tau_{\text{up}}} q_{\text{up}} - \frac{1}{\tau_{\text{low}}} q_{\text{low}} + D_h(z) \left( \frac{\partial^2 q}{\partial^2 x} + \frac{\partial^2 q}{\partial^2 y} \right) + D_v \frac{\partial^2 q}{\partial^2 z},$$

where  $D/Dt = \partial_t + u \partial_x + v \partial_y + w \partial_z$  is the material derivative and  $u$ ,  $v$ , and  $w$  are the east-west, north-south, and vertical velocity components of winds. The spatial coordinates  $x$  and  $y$  are Cartesian approximations of longitude and latitude. Also,  $p$  is pressure anomalies from a background state,  $\theta = \theta(t, \vec{r}) = \theta(t, x, y, z)$  is potential temperature anomalies from a background state  $\theta_{\text{BG}} = \theta_0 + Bz$ , and the moisture variable  $q = q(t, \vec{r})$  represents specific humidity anomalies from a background state  $q_{\text{BG}} = q_0 + q_{\text{bg}}(y, z)$ , where the background moisture anomaly  $q_{\text{bg}}(y, z)$  is given by

$$(4.2) \quad q_{\text{bg}}(y, z) = \left( 1 - a \left( 1 - e^{-\frac{1}{2} \left( \frac{y}{L} \right)^2} \right) \right) q_{\text{bg,eq}}(z),$$

and

$$(4.3) \quad q_{\text{bg,eq}}(z) = B_{vs} H \left( \frac{e^{-\frac{H}{\tilde{s}}} - e^{-\frac{z}{\tilde{s}}}}{1 - e^{-\frac{H}{\tilde{s}}}} \right),$$

is a prescribed vertical profile of background moisture at the equator,  $0 \leq a \leq 1$  and  $\tilde{L}$  dictate the background moisture north-south decay away from the equator,  $\tilde{s}$  dictates the rate of vertical decay,  $B_{vs}$  is the mean vertical gradient of background moisture, and  $H$  is the height of the troposphere. The potential temperature and moisture at  $z = 0$  (the bottom of the model domain, which is top of the atmospheric boundary layer) are given by  $\theta_0$  and  $q_0$ , respectively. The terms proportional to  $q_{\text{up}}$  and  $q_{\text{low}}$  in Eqs. 4.1e and 4.1f are convective adjustment terms. In the 2-vertical-mode scheme used here, they are defined using only the first two baroclinic modes of moisture

$$(4.4a) \quad q_{\text{up}}(t, \vec{r}) = Q_{\text{up}} \left( \sqrt{2} \sin\left(\frac{\pi z}{H}\right) - \frac{\sqrt{2}}{2} \sin\left(\frac{2\pi z}{H}\right) \right),$$

$$(4.4b) \quad q_{\text{low}}(t, \vec{r}) = Q_{\text{low}} \left( \sqrt{2} \sin\left(\frac{\pi z}{H}\right) + \frac{\sqrt{2}}{2} \sin\left(\frac{2\pi z}{H}\right) \right),$$

and

$$(4.5a) \quad Q_{\text{up}} = \sqrt{2} q_1 \sin\left(\frac{2\pi}{3}\right) + 2\sqrt{2} q_2 \sin\left(\frac{4\pi}{3}\right),$$

$$(4.5b) \quad Q_{\text{low}} = \sqrt{2} q_1 \sin\left(\frac{\pi}{3}\right) + 2\sqrt{2} q_2 \sin\left(\frac{2\pi}{3}\right).$$

The terms  $q_1(t, \vec{r}_h) = q_1(t, x, y)$  and  $q_2(t, \vec{r}_h)$  represent the first and second baroclinic components of moisture, respectively, and are given by

$$(4.6a) \quad q_1(t, \vec{r}_h) = \frac{1}{H} \int_0^H q(t, \vec{r}) \sqrt{2} \sin\left(\frac{\pi z}{H}\right) dz,$$

$$(4.6b) \quad q_2(t, \vec{r}_h) = \frac{1}{H} \int_0^H q(t, \vec{r}) 2\sqrt{2} \sin\left(\frac{2\pi z}{H}\right) dz.$$

The diffusion coefficient  $D_h(z) = D_{h,\text{up}} + (D_{h,\text{up}} - D_{h,\text{low}}) \left(\frac{z}{H} - 1\right)$  is a linear interpolation of the lower- and upper-troposphere diffusion coefficients.

For more information on, and a thorough exploration of, the OSH19 system, see [38]. For a discussion on an earlier model that the OSH19 system is based on, see [56].

As in Section 3, we will compare the multi-model ensemble here to a model chosen to represent the truth. The dynamics of the truth model were given by the OSH19 system (Eqs. 4.1) and used the Case 5 parameters given in [38], which are repeated here in Table 2. Figure 6 includes a Hovmoller diagram (which is a space-time contour plot of data averaged over latitudes in an equatorial belt) and a Wheeler–Kiladis diagram (which is a type of power spectral density plot or plot of space-time Fourier coefficients) of the output of the truth model, which illustrate the MJO and CCEWs produced by the model. In particular, there is a strong signal at zonal wavenumbers 2, 3 in the period range of 30 d to 90 d which corresponds to the MJO [64, 65, 71], and energy in the Kelvin, equatorial Rossby, mixed Rossby-gravity, as well as eastern inertio-gravity and western inertio-gravity wave regimes. There is also a significant bias in favor of eastward propagation in the symmetric part of the spectrum, akin to the spectrum generated with real-world data in [64].

| Parameter           | Description                                   | Truth                                               |
|---------------------|-----------------------------------------------|-----------------------------------------------------|
| $H$                 | Height of troposphere                         | 16 km                                               |
| $p_Y$               | Distance from equator to channel wall         | 6000 km                                             |
| $\beta$             | Rossby parameter                              | $2.3 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$ |
| $g$                 | Acceleration due to gravity                   | $9.8 \text{ m s}^{-2}$                              |
| $c_p$               | Specific heat of dry air at constant pressure | $1006 \text{ J kg}^{-1} \text{ K}^{-1}$             |
| $L_v$               | Latent heat of vaporization                   | $2.5 \times 10^6 \text{ J kg}^{-1}$                 |
| $B$                 | $\theta_{\text{BG}}$ vertical gradient        | $3 \text{ K km}^{-1}$                               |
| $\theta_0$          | Reference potential temperature               | 300 K                                               |
| $\tau_u$            | Wind damping time-scale                       | 16 d                                                |
| $\tau_\theta$       | Potential temperature damping time-scale      | 16 d                                                |
| $\tau_{\text{up}}$  | $q$ damping time-scale (upper-troposphere)    | 1 d                                                 |
| $\tau_{\text{low}}$ | $q$ damping time-scale (lower-troposphere)    | 4/24 d                                              |
| $B_{vs}$            | Mean vertical $q_{\text{BG}}$ gradient        | $-1.2 \text{ g kg}^{-1} \text{ km}^{-1}$            |
| $a$                 | 1 - pole-to-equator $q_{\text{BG}}$ ratio     | 0.2                                                 |
| $\tilde{L}$         | $q_{\text{BG}}$ meridional decay length-scale | 3000 km                                             |
| $\tilde{s}$         | $q_{\text{bg,eq}}$ decay length-scale         | 12 km                                               |
| $D_{h,\text{up}}$   | Horizontal $q$ diffusion (upper-troposphere)  | $60.8 \text{ km}^2 \text{ s}^{-1}$                  |
| $D_{h,\text{low}}$  | Horizontal $q$ diffusion (lower-troposphere)  | $7.6 \text{ km}^2 \text{ s}^{-1}$                   |
| $D_v$               | Vertical $q$ diffusion                        | $2 \times 10^{-4} \text{ km}^2 \text{ s}^{-1}$      |

TABLE 2

Parameter values for the truth model of the high-dimensional test case. These parameters were obtained from Case 5 of [38].

The multi-model ensemble was made up of a deficient tropical climate model (DTCM) and an MJO model (MJOM). The DTCM had the same dynamics as the truth model (Eqs. 4.1) with greater  $B_{vs}$  and  $D_v$ , as well as lesser  $D_{h,\text{up}}$  and  $D_{h,\text{low}}$  (Table 3). The MJOM was based on the modified Chen–Majda–Giannakis 2014 system (Eqs. 3.1) with parameters given in Table 4.

| Parameter          | Description                                  | DTCM                                              |
|--------------------|----------------------------------------------|---------------------------------------------------|
| $B_{vs}$           | Mean vertical $q_{\text{BG}}$ gradient       | $-1.1 \text{ g kg}^{-1} \text{ km}^{-1}$          |
| $D_{h,\text{up}}$  | Horizontal $q$ diffusion (upper-troposphere) | $18.24 \text{ km}^2 \text{ s}^{-1}$               |
| $D_{h,\text{low}}$ | Horizontal $q$ diffusion (lower-troposphere) | $2.28 \text{ km}^2 \text{ s}^{-1}$                |
| $D_v$              | Vertical $q$ diffusion                       | $2.54 \times 10^{-4} \text{ km}^2 \text{ s}^{-1}$ |

TABLE 3

Parameter values for the deficient tropical climate model (DTCM) that do not match those of the truth model for the high-dimensional test case. The parameters of the truth model are given in Table 2, and were obtained from Case 5 of [38].

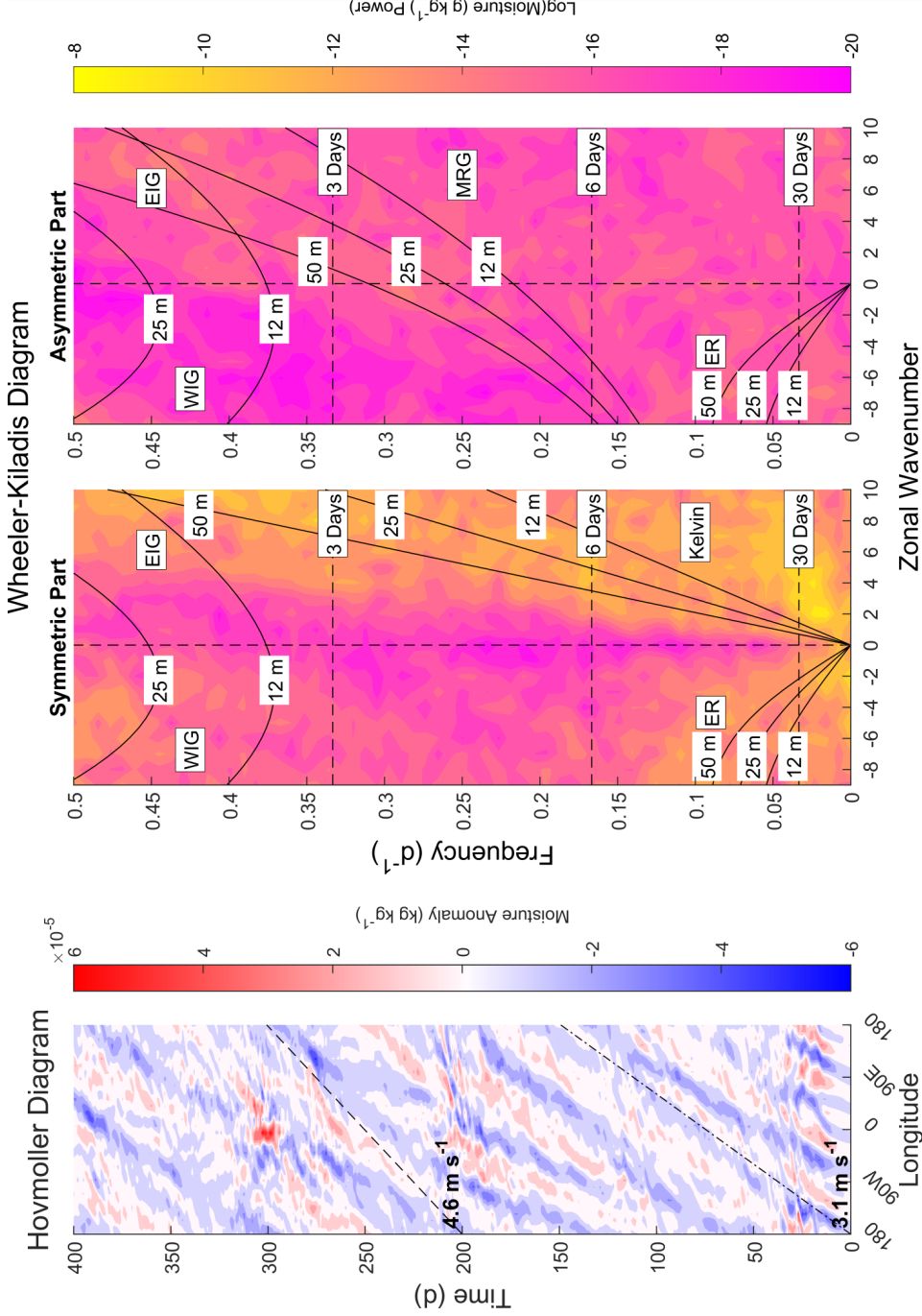


FIG. 6. Hovmöller diagram (left) and Wheeler-Kiladis diagram (right) of the truth model for the high-dimensional test case. The Hovmöller diagram displays the evolution of the moisture anomaly at the equator at an altitude of 6 km, with example velocities (dashed, dot-dashed). The Wheeler-Kiladis diagram displays a power spectrum of the moisture anomaly, with dispersion curves (solid black) for equatorial Rossby waves (ER), Kelvin waves, western inertio-gravity waves (WIG), eastern inertio-gravity waves (EIG), and mixed Rossby-gravity waves (MRG), each for equivalent depths of 12 m, 25 m, and 50 m, as well as example periods lines (horizontal, dashed) for 3 days, 6 days, and 30 days. See Appendix B for more information on the construction of the Wheeler-Kiladis diagram.

| Parameter  | Description                         | MJOM                      |
|------------|-------------------------------------|---------------------------|
| $d_u$      | Decay Parameter for $u_1, u_2$      | $0.4 \text{ m}^{-1}$      |
| $d_v$      | Decay Parameter for $v$             | $0.4 \text{ m}^{-1}$      |
| $a$        | Background Phase of $u_1, u_2$      | $-2 \text{ m}^{-1}$       |
| $\sigma_u$ | Amplitude of Noise for $u_1, u_2$   | $1.2 \text{ m}^{-1/2}$    |
| $\sigma_v$ | Amplitude of Noise for $v$          | $1 \text{ m}^{-1/2}$      |
| $\gamma$   | Strength of Non-Linear Interactions | $0.65 \text{ m}^{-1}$     |
| $\omega_f$ | Frequency of $v_f(t)$               | $2\pi/12 \text{ m}^{-1}$  |
| $\phi$     | Phase of $v_f(t)$                   | $-3\pi/12 \text{ m}^{-1}$ |
| $f_0$      | Mean of $v_f(t)$                    | 2.3                       |
| $f_t$      | Amplitude of $v_f(t)$               | 1.1                       |

TABLE 4

Parameter values for the MJO model (MJOM) of the high-dimensional test case. Note that the temporal unit  $m$  is months.

The parameters of the DTCM were chosen to weaken and slow the MJO signal, to represent deficiencies in a climate model. This also leads to the elimination of the small-scale features (i.e., the CCEWs). Figure 7 illustrates these deficiencies; the Wheeler–Kiladis diagram shows an MJO signal at zonal wave number 4 and period 90 d that is six orders of magnitude weaker than that of the truth model. Further, there was a complete lack of CCEWs. The spectrum was also only slightly biased toward eastward waves, emphasizing the weakness of the MJO signal. The MJO in the DTCM propagated at approximately half the speed as that in the truth model.

The MJOM parameters were different from those used in Section 3 to capture the properties of the MJO in the truth model. In particular, the background state phase of  $u_1$  and  $u_2$  was changed to reflect the speed of the MJO in the truth model, and the damping and time-periodic forcing adjusted to bring the  $u_1$  and  $u_2$  closer in line to the MJO indices calculated from data generated from the truth model via Principal Component Analysis (PCA) or Empirical Orthogonal Function (EOF) analysis. For an in-depth discussion on the EOF analysis used, see Appendix A. Figure 8 compares the MJO isolated from the truth data via EOF analysis to a recreation of the MJO using the spatial modes obtained from the EOF analysis and the MJO indices from the MJOM. The MJO reconstructed from the MJOM was weaker than that of the truth model and did not propagate as consistently in time, due to the seasonality of the climate signal in the model. However, its MJO signal peaked at zonal wavenumber 2 with period in the range of 30 d to 90 d, which matches that of the truth model.

**4.1. Multi-model communication and performance.** In this section, we discuss the choice of mappings and communication parameters in the multi-model communication strategy for the high-dimensional test case.

Unlike the low-dimensional test case, the deficient tropical climate model (DTCM) and MJO model (MJOM) in the high-dimensional test case did not share the same state space and their specific deficiencies were unclear. We begin our discussion by addressing the calculation of the MJO indices and climate signal from the DTCM, i.e., the pseudo-truth map for the MJOM.

It has been shown that it is possible to isolate the signal of the MJO by projecting minimally filtered daily data onto spatial patterns characteristic of the MJO [64]. In particular, we projected the zonal wind velocity and moisture anomaly onto particular meridional and vertical modes to obtain a zonally-varying signal, and isolated the spatial patterns by their variability via Empirical Orthogonal Function (EOF) analy-

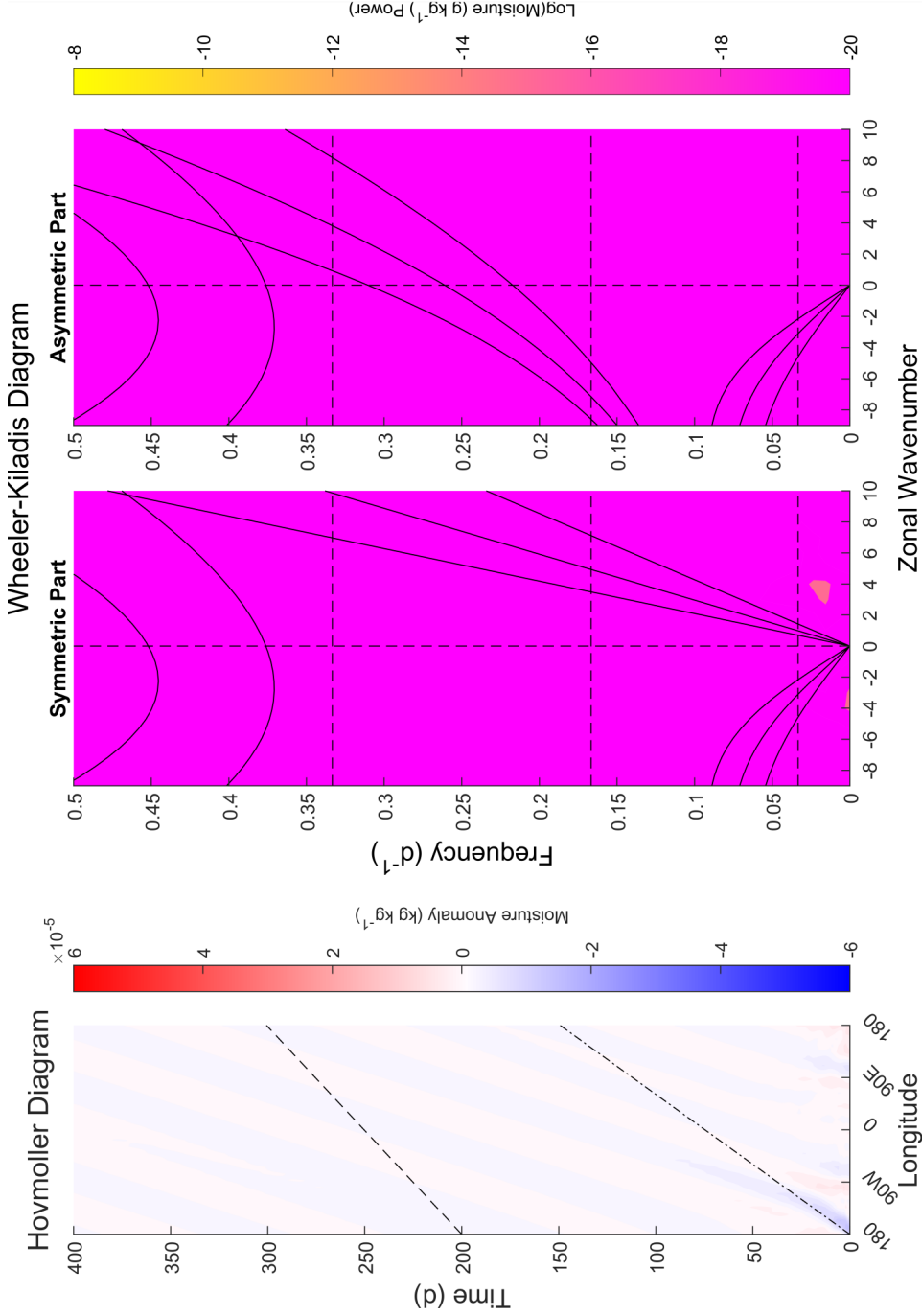


FIG. 7. Hovmöller diagram (left) and Wheeler-Kiladis diagram (right) of the deficient tropical climate model (DTCM) for the high-dimensional test case. The Hovmöller diagram displays the evolution of the moisture anomaly at the equator at an altitude of 6 km, with example velocities (dashed, dot-dashed). The Wheeler-Kiladis diagram displays a power spectrum of the moisture anomaly, with dispersion curves (solid black) for equatorial Rossby waves (ER), Kelvin waves, western inertio-gravity waves (WIG), eastern inertio-gravity waves (EIR), and mixed Rossby-gravity waves (MRG), each for equivalent depths of 12 m, 25 m, and 50 m, as well as example periods lines (horizontal, dashed) for 3 days, 6 days, and 30 days. Note the difference in color scale between the weak signals in these diagrams and the stronger signals in Figure 6. Refer to Figure 6 for the labels for the dashed and solid curves in the diagrams, and to Appendix B for more information on the construction of the Wheeler-Kiladis diagram.

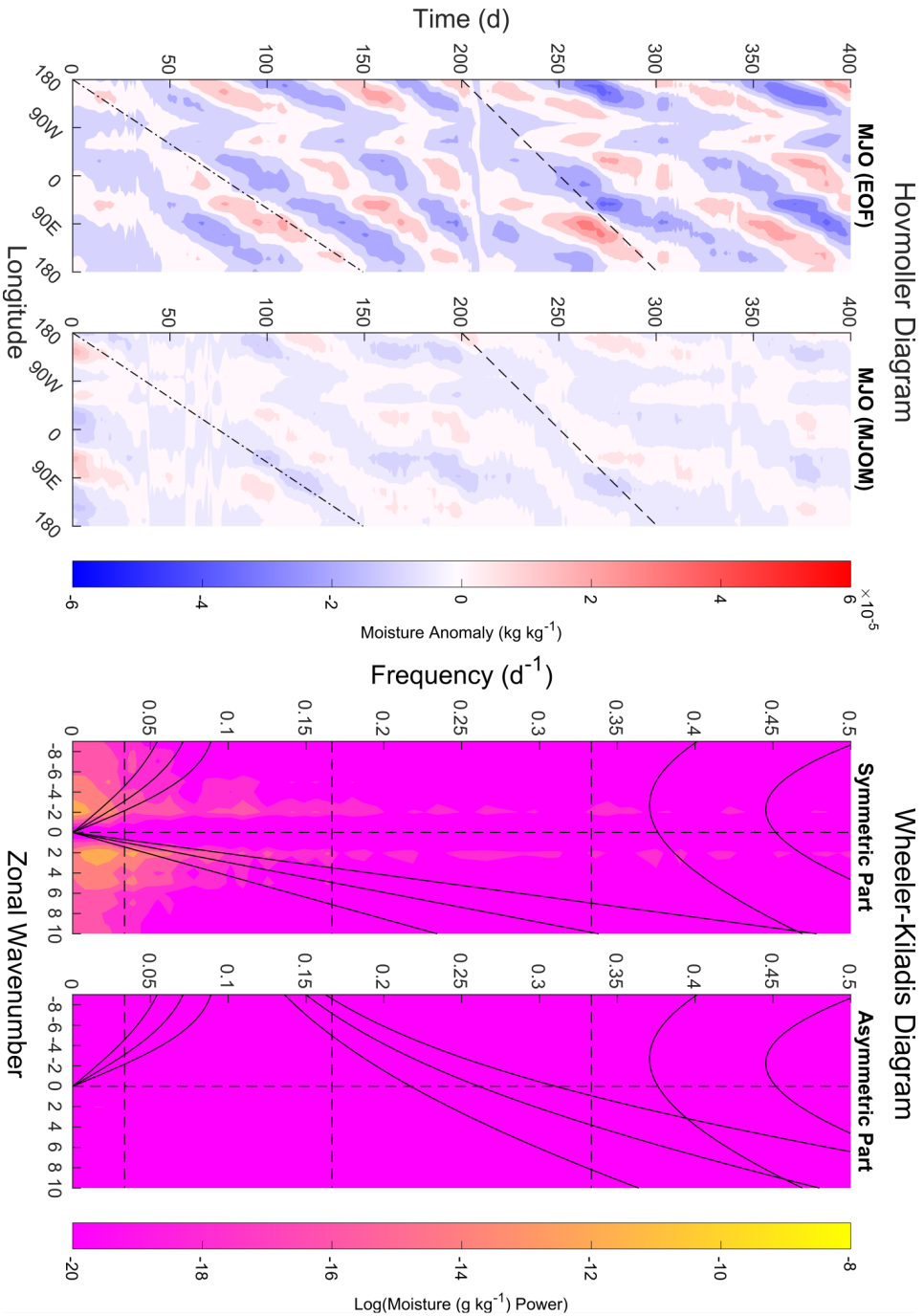


FIG. 8. (Left) Hovmöller diagrams of the Madden-Julian Oscillation (MJO) isolated from the truth data via Empirical Orthogonal Function (EOF) analysis (left) and the MJO reconstructed from the MJO model (MJOM; right). The diagrams feature the evolution of the moisture anomaly at an altitude of 6 km. Example velocities are given by dashed lines. (Right) Wheeler-Kiladis diagram of the MJO reconstructed using the MJOM, with dispersion curves (solid black) for equatorial Rossby waves (ER), Kelvin waves, western inertia-gravity waves (WIG), eastern inertia-gravity waves (EIR), and mixed Rossby-gravity waves (MRG) are shown for equivalent depths of 12 m, 25 m, and 50 m, as well as example periods lines (horizontal, dashed) are shown for 3 days, 6 days, and 30 days. Note the difference in color scale between these diagrams and those in Figure 6. Refer to Figure 6 for the labels for the dashed and solid curves in the diagrams, and to Appendix B for more information on the construction of the Wheeler-Kiladis diagram.

sis, akin to the methods in [64]. By applying this analysis to a time-series of data from the truth model, we calculated the two spatial patterns contributing the most to spatial variability, as well as their expansion coefficients (the MJO indices) which dictated how they evolve in time. We then applied this analysis to a given output of the DTCM, and calculated the MJO indices of the state. For further discussion on this analysis, see Appendix A.

Despite its deficiencies in representing the MJO and CCEWs, the DTCM represents some of the gross features of the ratio of mean vertical total moisture gradient to mean vertical  $q_{BG}$  gradient of the truth model (Figure 9). Given the impact of vertical moisture gradient on the climate [64, 65], we chose this ratio to be the non-dimensional climate signal to be communicated with the MJOM.

In summary, the output of the pseudo-truth map for the MJOM consisted of 1) the MJO indices calculated from the state of the DTCM, which corresponded to  $u_1$  and  $u_2$ , and 2) the ratio of mean vertical total moisture gradient to mean vertical  $q_{BG}$  gradient, which corresponded to  $v$ .

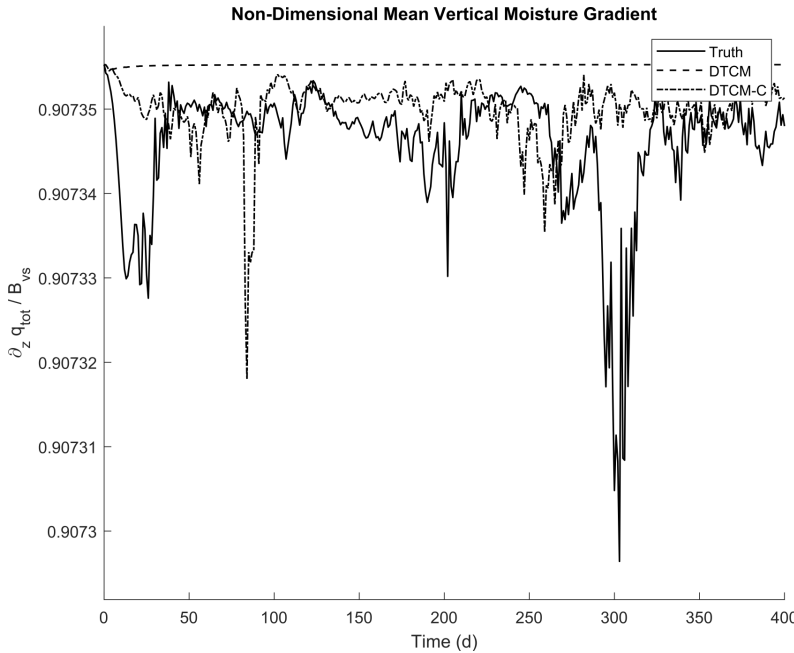


FIG. 9. The ratio of mean vertical total moisture gradient to mean vertical  $q_{BG}$  gradient of the truth model (dashed), deficient tropical climate model (DTCM; dashed), and deficient tropical climate model with communication (DTCM-C; dash-dotted), which is used as the non-dimensional climate signal for the pseudo-truth map of the MJO model.

The pseudo-truth map for the DTCM is more straightforward; we may reconstruct the MJO using the MJO indices ( $u_1$  and  $u_2$ ) from the MJOM and the aforementioned spatial patterns characteristic of the MJO. For further discussion of the communication strategy for this ensemble, including communication parameters, see Appendix C.

When using multi-model communication, the improvement of each member model was readily apparent (Figures 10 and 11). The deficient tropical climate model with communication (DTCM-C) produces a strongly asymmetric power spectra biased toward eastward propagation of energy, in agreement with that of the truth model. The

DTCM-C contains an MJO signal at zonal wavenumber 2 in the period range of 30 d to 60 d of similar strength to that in the truth model, as well as energy in the same CCEWs regions as the truth model. Further, the climate signal of the DTCM-C has greater variability than that of the DTCM, more closely matching that of the truth model.

The MJO model with communication (MJOM-C) produces a stronger MJO signal that is active year-round, in contrast to the weaker seasonal MJO of the MJOM (Figure 11). The loss of seasonality of the MJO, although unrealistic when compared to reality, matches the continual MJO present in the truth model of this example case.

**4.2. Sensitivity studies.** We now describe some sensitivity studies of the multi-model communication strategy used in the high-dimensional test case.

We first studied the effect of the frequency of communication between the deficient tropical climate model (DTCM) and the Madden-Julian Oscillation model (MJOM). Tests were performed for communication occurring every 6 hours, 18 hours, 1 day, 3 days, 7 days, 21 days, 30 days, and 60 days, in addition to the base case of every 12 hours presented in the body of this work. We found that communicating every 6 hours leads to a stronger Madden-Julian Oscillation (MJO) as well as smaller-scale and higher-frequency phenomena, but did not introduce any phenomena absent in the base case. For longer amounts of time between communication, the smaller-scale and higher-frequency phenomena diminished. The multi-model ensemble failed to produce any convectively-coupled equatorial waves when communication occurred every 3 days, and failed to produce a significant MJO when communication occurred every 7 days (Figure 12). We also observed the presence of horizontal bands of energy in the Wheeler-Kiladis diagrams for several communication frequencies, which we attributed to the effect of communication frequency and resonance. For example, the horizontal band of energy in the left-most diagram of Figure 12 occurs at a frequency of approximately once every 3 days, which is equivalent to once every four communication events.

We then studied the effect of communication strength between the DTCM and MJOM. In particular, we examined the effect of scaling down the background error covariance matrices  $\mathbf{B}^{(MJOM)}$  and  $\mathbf{B}^{(DTCM)}$  in Eqs. C.2 and C.3 by a factor of  $10^{-1}$ ,  $10^{-2}$ , and  $10^{-3}$ , in addition to the base case of unscaled background error covariance (Figure 13). The progression in degradation of the solution followed similarly to that when investigating the frequency of communication, i.e., as the strength of communication decreased, the signals for smaller-scale higher-frequency phenomena disappeared first, followed by the loss of the MJO signal.

**5. Conclusion.** In this work, we introduced (Section 2) and investigated (Sections 3 and 4) a multi-model communication strategy that allows several models of various complexities to exchange information throughout a simulation. The test case here was selected to be non-linear, non-Gaussian, and high-dimensional, in order to approach the type of setting of many complex dynamical systems in science and engineering applications.

The investigations here showed that this strategy can improve the constituent models of an ensemble, and in this particular case a variety of model deficiencies were corrected even though only a small amount of data (two degrees of freedom) is communicated between the models. In particular, communicating information to a deficient tropical climate model about the state of the MJO instilled not only an MJO into the system, but higher-frequency variability of convectively-coupled equatorial waves as well (Section 4).

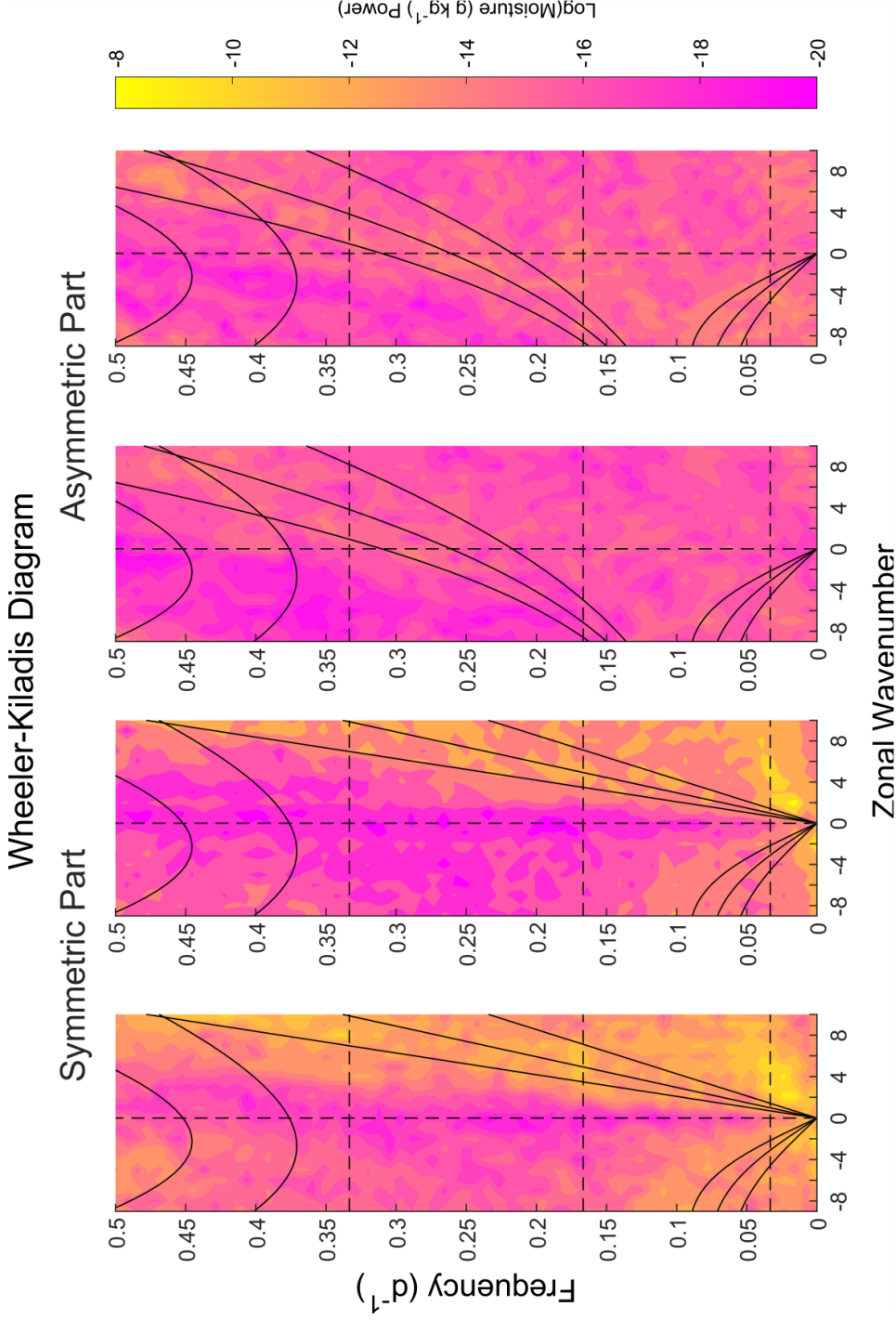


FIG. 10. Wheeler-Kiladis diagrams for the truth model (left in each pair) and the deficient tropical climate model with communication (DTCM-C; right in each pair). With communication, the DTCM-C contains both a signal for an MJO, as well as signals for Kelvin, equatorial Rossby, mixed Rossby-gravity, and inertio-gravity waves. Dispersion curves (solid black) are given for various equatorial waves and example period (horizontal, dashed) are shown for 3 days, 6 days, and 30 days. See Figure 6 for the labels for these curves. Refer to Appendix B for more information on the construction of the Wheeler-Kiladis diagram.

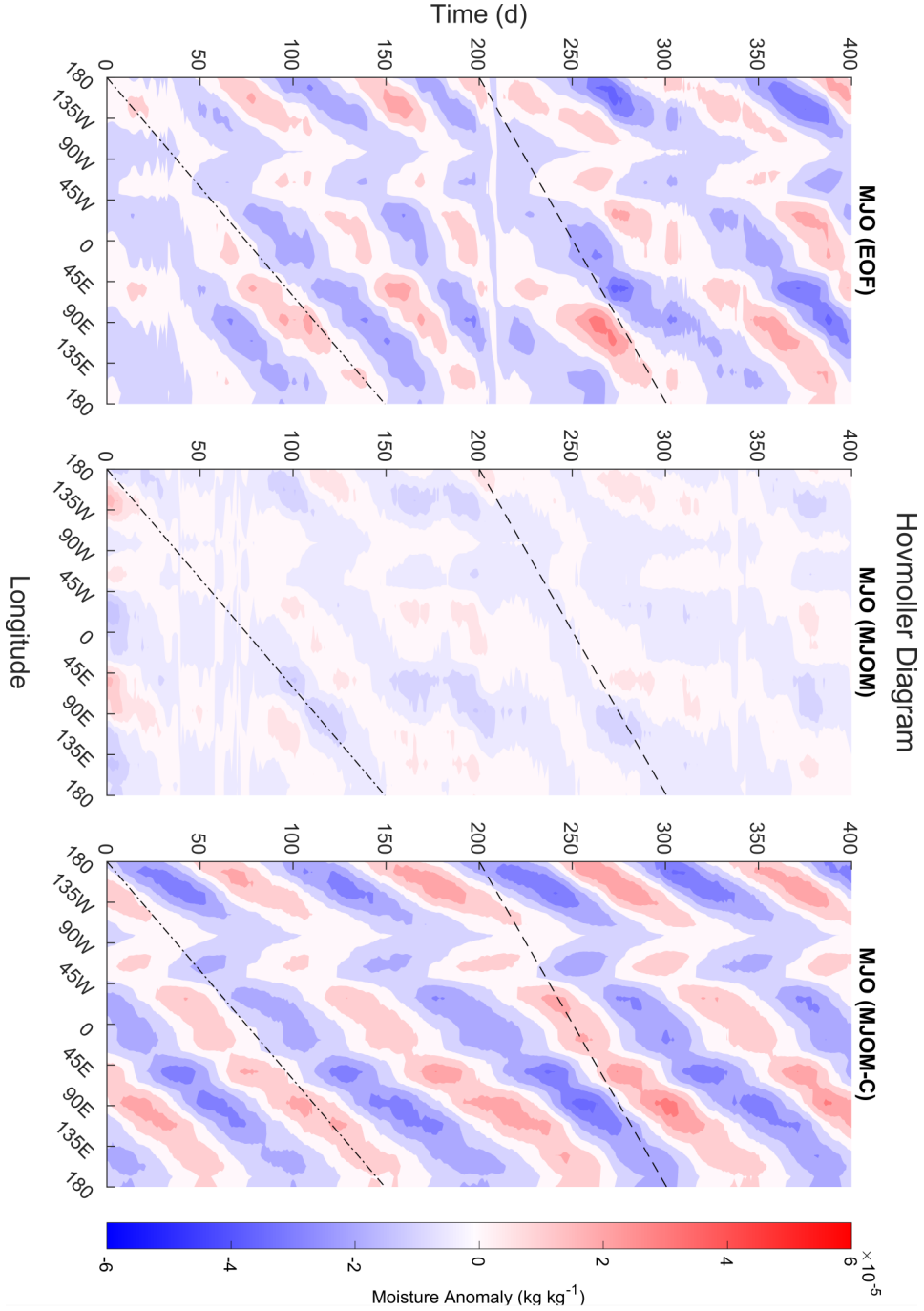


FIG. 11. Hovmöller diagrams of the MJO with spatial modes obtained from Empirical Orthogonal Function decomposition (EOF) of the truth data using (left) the expansion coefficients from the EOF decomposition to evolve the spatial modes, (center) MJO indices generated by the MJO model (MJOM), and (right) MJO indices generated by the MJO model with communication (MJOM-C). The diagrams feature the evolution of the moisture anomaly at the equator at an altitude of 6 km. Example velocities are given by dashed lines, see Figure 6 for the labels for these example velocities.

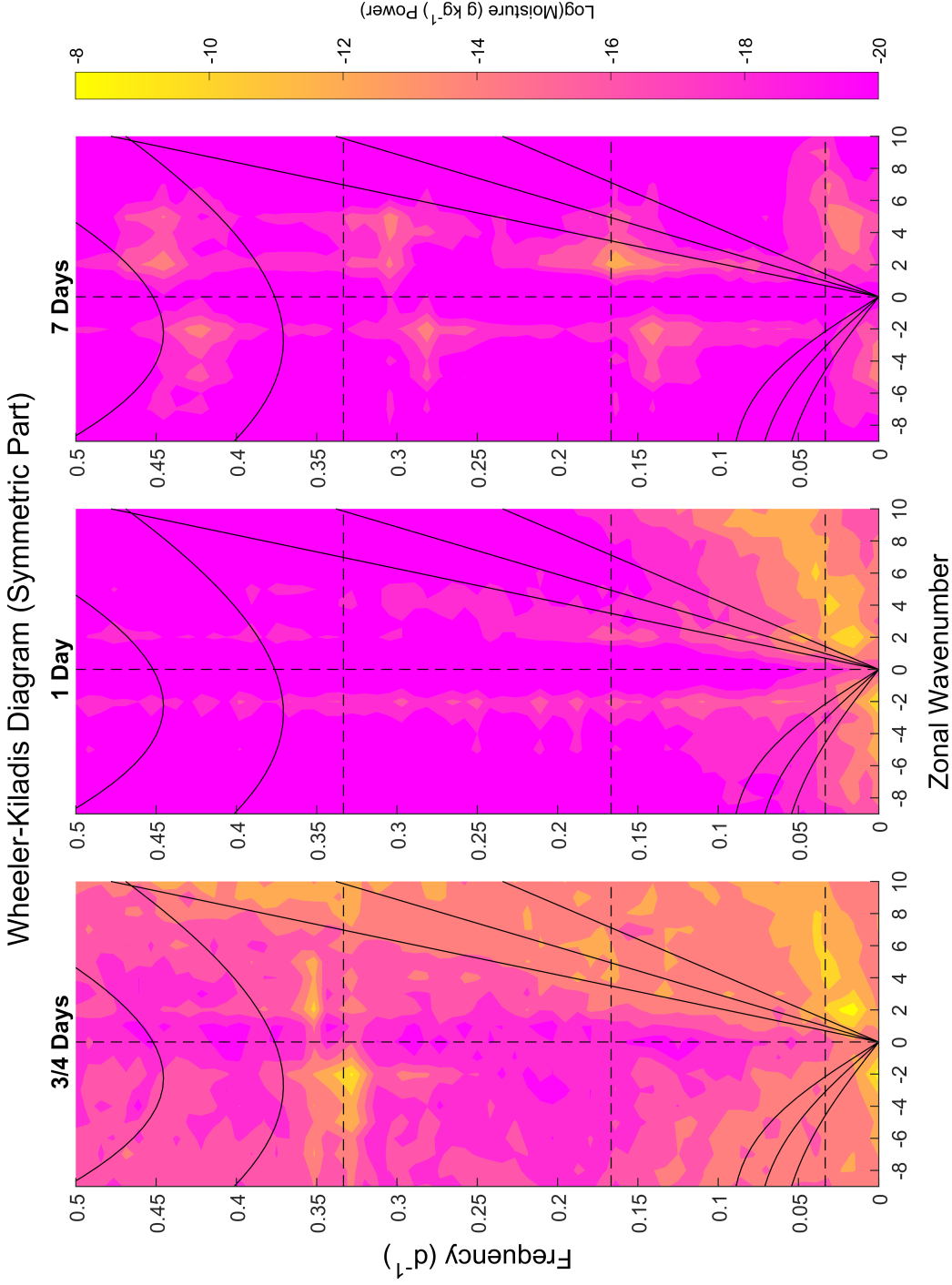


FIG. 12. The symmetric part of the Wheeler-Kiladis diagrams for the multi-model ensemble with a communication frequency of once every 18 hours (left), every day (center) and 7 days (right). For communication frequencies less than once per day, the signals for Kelvin, equatorial Rossby, mixed Rossby-gravity, and inertio-gravity waves were absent. For communication frequencies less than once per 7 days, the signal for the Madden-Julian Oscillation was very weak or totally absent. Dispersion curves (solid black) are given for various equatorial waves and example period (horizontal, dashed) are shown for 3 days, 6 days, and 30 days. See Figure 6 for the labels for these curves. Refer to Appendix B for more information on the construction of the Wheeler-Kiladis diagram.

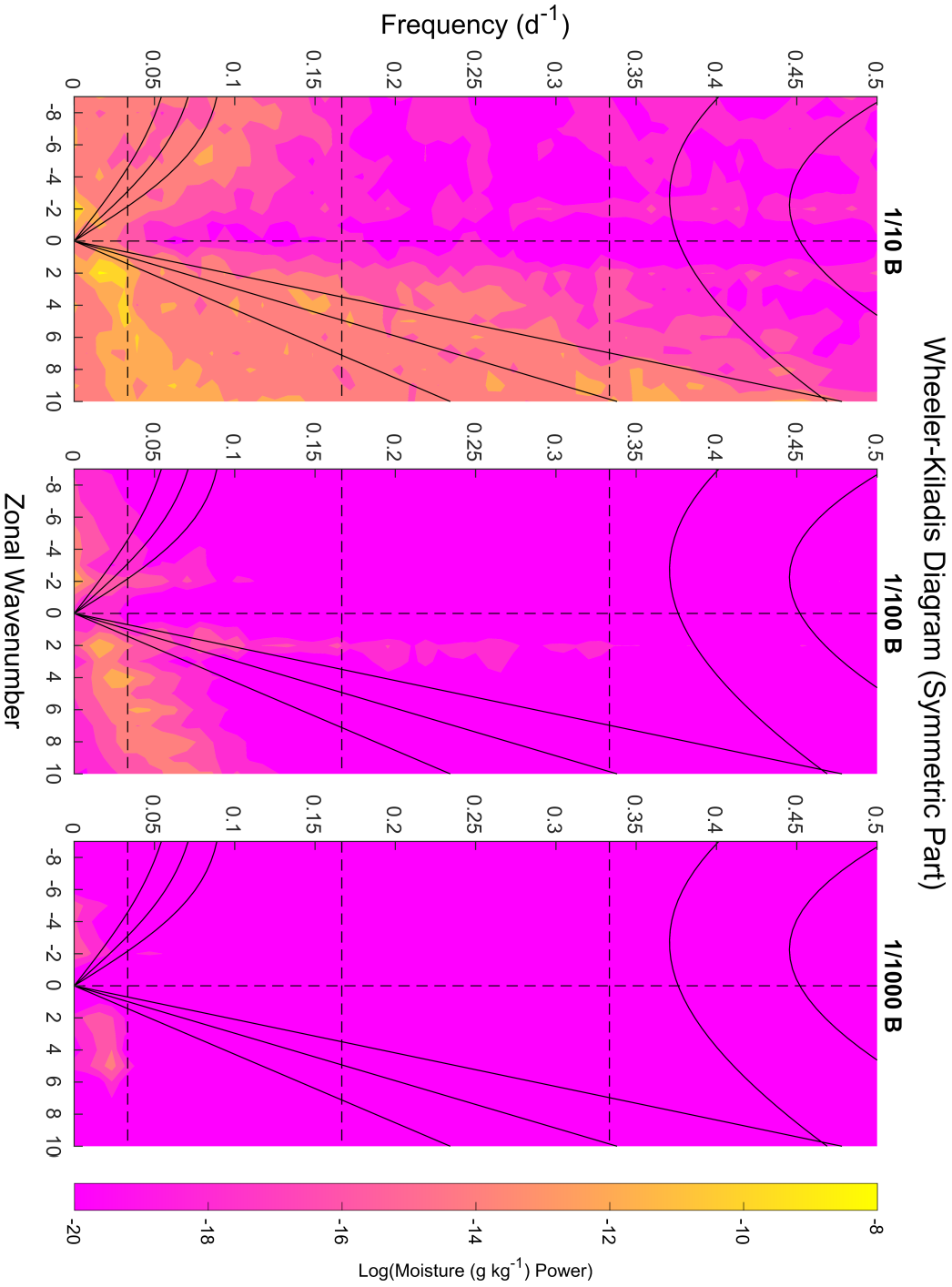


FIG. 13. The symmetric part of the Wheeler-Kiladis diagrams for the multi-model ensemble with scaled backscatter error covariance for  $1/10$  (left),  $1/100$  (center), and  $1/1000$  (right). When scaled by a factor of approximately  $1/100$  or less, the signals for Kelvin, equatorial Rossby, mixed Rossby-gravity, and inertia-gravity waves were absent. When scaled by a factor of  $1/1000$  or less, the signal for the Madden-Julian Oscillation was very weak or totally absent. Dispersion curves (solid black) are given for various equatorial waves and example period (horizontal, dashed) are shown for 3 days, 6 days, and 30 days. See Figure 6 for the labels for these curves. Refer to Appendix B for more information on the construction of the Wheeler-Kiladis diagram.

Many possible future directions could potentially be explored. For instance, further complexity could be introduced into the strategy as multi-model ensembles with a larger size, e.g., ten distinct atmosphere models, or greater model complexity, e.g., global climate models; or the background error covariance  $\mathbf{B}^{(m)}$  for each model could be allowed to evolve throughout a simulation by considering the evolving state and strengths of each model; or the multi-model communication strategy could be investigated for other applications, such as astronomical or biological; or theoretical foundations, potentially using dynamical systems ideas, could be formulated for the stability or accuracy of the multi-model communication strategy.

#### Appendix A. Empirical orthogonal function analysis.

In this appendix, we describe our use of empirical orthogonal function analysis to enable multi-model communication in the high-dimensional test case (Section 4).

Empirical Orthogonal Function (EOF) analysis is used to decompose time-series data into spatial modes of variability and describe how those modes change in time [5, 72]. The spatial modes are referred to as empirical orthogonal functions, and the evolution of each EOF is described by its respective expansion coefficients. This technique has been used extensively with observational data to define EOF representations of the Madden–Julian Oscillation (MJO) [23, 64]. Here, we utilize an EOF decomposition of the zonal wind velocity and the moisture anomaly to enable our multi-model communication strategy.

To effectively use EOF analysis, we utilized a separation of variables in each spatial dimension. We expressed the variability in each spatial dimension using three function bases. The vertical and meridional basis functions we used here were derived from those in [32, 56, 57]. The zonal basis functions were those obtained from EOF analysis. To simplify the calculations, each basis was orthonormal.

For an orthonormal vertical basis, we utilized two families of sinusoids

$$(A.1) \quad \begin{aligned} C_0(z) &= 1, & C_k(z) &= \sqrt{2} \cos\left(\frac{k\pi z}{H}\right), \\ S_k(z) &= \sqrt{2} \sin\left(\frac{k\pi z}{H}\right), & k &= 1, 2, 3, \dots \end{aligned}$$

with inner product

$$(A.2) \quad \langle f(z), g(z) \rangle_z = \frac{1}{H} \int_0^H f(z) g(z) dz.$$

The  $C$ 's and  $S$ 's form orthonormal bases on the vertical domain  $z \in [0, H]$ . These bases originated from linearizing a two-mode shallow water equations system, isolating the  $z$ -dependence using a separation of variables, and solving the resultant Sturm-Liouville problem [32, 57]. We then expressed the zonal wind velocity and the moisture anomaly as

$$(A.3) \quad u(t, \vec{r}) = \sum_{k=0}^{\infty} u_k(t, \vec{r}_h) C_k(z), \quad q(t, \vec{r}) = \sum_{k=1}^{\infty} q_k(t, \vec{r}_h) S_k(z).$$

However, we defined an upper- and lower-mode of moisture using the first two baroclinic modes of moisture

$$(A.4) \quad S_{\text{up}}(z) = \frac{1}{\sqrt{2}} (S_1(z) - S_2(z)), \quad S_{\text{low}}(z) = \frac{1}{\sqrt{2}} (S_1(z) + S_2(z)),$$

which allowed us to easily isolate moisture in the upper- and lower-troposphere while maintaining the orthonormality of the vertical basis

$$(A.5) \quad q(t, \vec{r}) = q_{\text{up}}(t, \vec{r}_h) S_{\text{up}}(z) + q_{\text{low}}(t, \vec{r}_h) S_{\text{low}}(z) + \sum_{k=3}^{\infty} q_k(t, \vec{r}_h) S_k(z).$$

These vertical bases are shown in Figure 14.

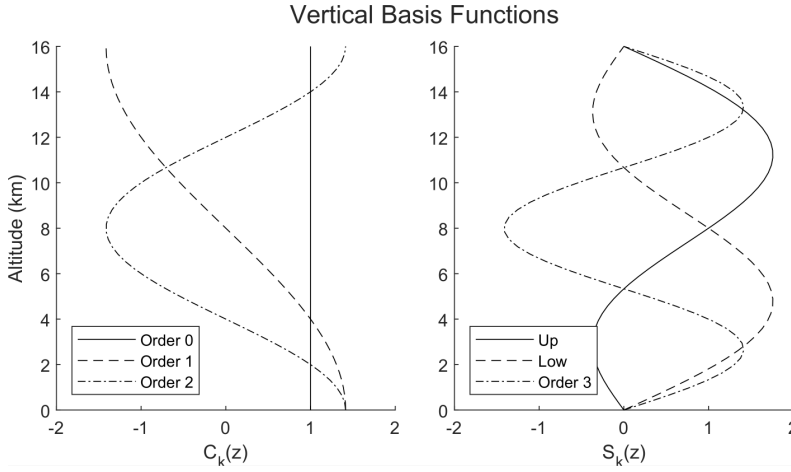


FIG. 14. Vertical orthonormal bases used for the zonal wind velocity (left) and moisture anomaly (right) utilized for the EOF analysis.

We chose a meridional basis made up of the normalized parabolic cylinder functions

$$(A.6) \quad \phi_j(y) = \frac{1}{(2\pi)^{1/4} (2^j (j!))^{1/2}} e^{-\frac{y^2}{4}} H_j\left(\frac{y}{\sqrt{2}}\right), \quad j = 0, 1, 2, \dots$$

where  $H_j(y)$  is the  $j^{\text{th}}$  Hermite polynomial

$$(A.7) \quad H_j(y) = (-1)^j e^{y^2} \frac{d^j}{dy^j} [e^{-y^2}]$$

with inner product

$$(A.8) \quad \langle f(y), g(y) \rangle_y = \int_{-\infty}^{\infty} f(y) g(y) dy.$$

The meridional basis functions are shown in Figure 15.

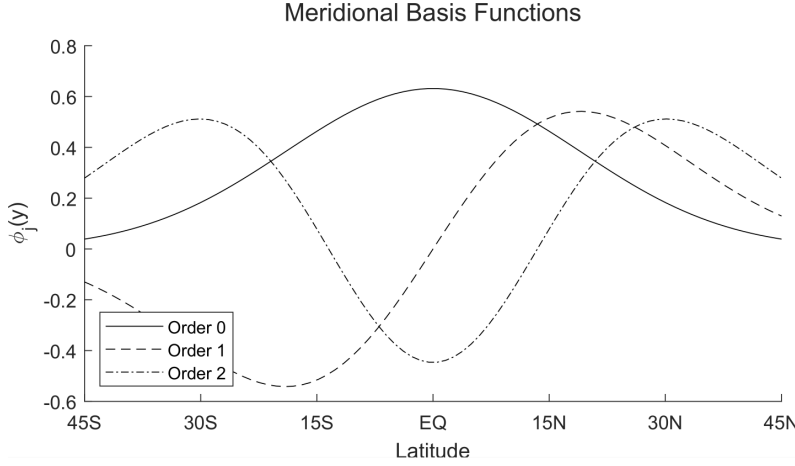


FIG. 15. Meridional basis functions utilized for the EOF analysis.

With this inner product, the normalized parabolic cylinder functions form an orthonormal basis of the meridional domain. Hence, we expressed the zonal wind velocity and the moisture anomaly as

$$(A.9) \quad u(t, \vec{r}) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_{jk}(t, x) \phi_j(y) C_k(z),$$

$$(A.10) \quad q(t, \vec{r}) = \sum_{j=0}^{\infty} \phi_j(y) \left( q_{j\text{up}}(t, x) S_{\text{up}}(z) + q_{j\text{low}}(t, x) S_{\text{low}}(z) + \sum_{k=3}^{\infty} q_{jk}(t, x) S_k(z) \right).$$

The spatial modes of the MJO correspond to the spatial modes of greatest variability of  $u_{01}(t, x)$  and  $q_{0\text{low}}(t, x)$ . In particular, we discretized the norms given by Eqs. A.2 and A.8 to obtain the time-series for  $u_{01}(t, x)$  and  $q_{0\text{low}}(t, x)$ . When taking the meridional inner product, we scaled the meridional coordinate by  $\tilde{y} = y/\tilde{L}_y$  for meridional length scale  $\tilde{L}_y = 1490 \text{ km}$  given in [57].

To obtain the EOFs of the coupled zonal wind-moisture anomaly field, we put each time-series  $u_{01}(t, x)$  and  $q_{0\text{low}}(t, x)$  into a matrix  $\mathbf{U}_{01}$  and  $\mathbf{Q}_{0\text{low}}$ , respectively. Each row of  $\mathbf{U}_{01}$  contained the value of  $u_{01}(t, x)$  at each location for a particular time, and each column of  $\mathbf{U}_{01}$  contained the value of  $u_{01}(t, x)$  at a particular location for each time. The matrix  $\mathbf{Q}_{0\text{low}}$  was structured similarly.

We filtered the long time-scale variability by removing the mean of each column of both  $\mathbf{U}_{01}$  and  $\mathbf{Q}_{0\text{low}}$ . We preserved the spatial variation of  $\mathbf{U}_{01}$  and  $\mathbf{Q}_{0\text{low}}$  by scaling each by its subsequent standard deviation, denoted  $\sigma_u$  and  $\sigma_q$ , to obtain  $\tilde{\mathbf{U}}_{01}$  and  $\tilde{\mathbf{Q}}_{0\text{low}}$ .

We then formed the covariance matrix by concatenating  $\tilde{\mathbf{U}}_{01}$  and  $\tilde{\mathbf{Q}}_{0\text{low}}$  into a single matrix  $\mathbf{A} = \begin{bmatrix} \tilde{\mathbf{U}}_{01} & \tilde{\mathbf{Q}}_{0\text{low}} \end{bmatrix}$  and calculating  $\mathbf{C} = \mathbf{A}^T \mathbf{A}$ . The EOF analysis was then performed via eigendecomposition of  $\mathbf{C}$ .

The EOFs were given by the eigenvectors  $\vec{\mathcal{F}}_i$  of  $\mathbf{C}$ , and the expansion coefficients of  $\vec{\mathcal{F}}_i$   $\vec{\mathcal{M}}_i$  were given by  $\vec{\mathcal{M}}_i = \mathbf{C} \vec{\mathcal{F}}_i$ . These EOFs form an orthonormal basis for the

space of functions on the zonal part of the discretized domain with the typical vector norm [5, 64, 72].

In particular, given that the time-series data contained  $n_x$  points zonally at  $n_t$  times, we obtained  $2n_x$  EOFs of size  $2n_x \times 1$  and  $2n_x$  expansion coefficients of size  $n_t \times 1$ . Each EOF may be decomposed into two halves as

$$(A.11) \quad \vec{\mathcal{F}}_i = \begin{bmatrix} \vec{\mathcal{F}}_i^u \\ \vec{\mathcal{F}}_i^q \end{bmatrix}$$

where  $\vec{\mathcal{F}}_i^u$  corresponded to the zonal wind velocity while  $\vec{\mathcal{F}}_i^q$  corresponded to the moisture anomaly.

The EOFs corresponding to the largest eigenvalues of  $\mathbf{C}$  correspond to spatial patterns of greatest variability. Using a descending ordering of eigenvalues, the EOFs  $\vec{\mathcal{F}}_1$  and  $\vec{\mathcal{F}}_2$  corresponded to the MJO [64]. The leading two EOFs of data generated by the truth model of the high-dimensional test case (Section 4), re-dimensionalized via multiplying them by  $\sigma_u$  and  $\sigma_q$ , respectively, is given in Figure 16.

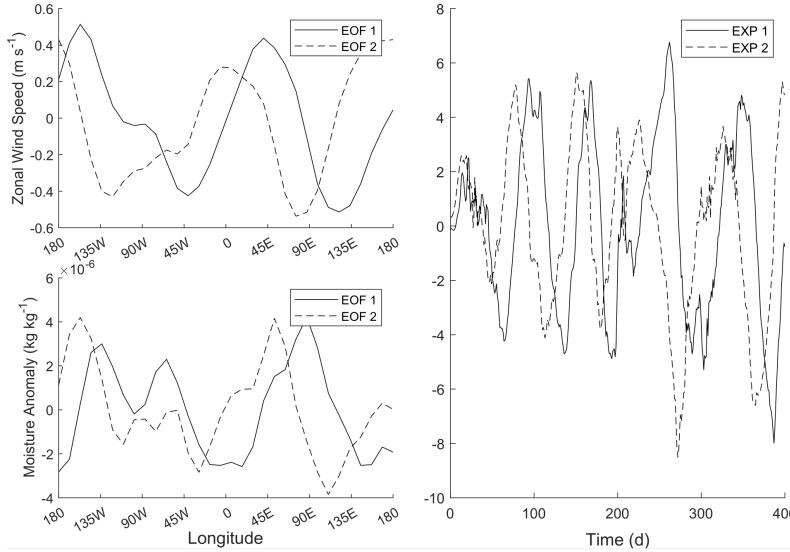


FIG. 16. The leading two Empirical Orthogonal Functions (EOFs; left) and expansion coefficients (right) of a 400-day time-series generated by the truth model in the high-dimensional test case (Section 4). The EOFs have been re-dimensionalized via multiplying them by  $\sigma_u$  and  $\sigma_q$ , respectively.

We then reconstructed the time-series for the MJO part of the time-series data by the following algorithm:

1. For  $i = 1, 2$ , re-dimensionalize  $\vec{\mathcal{F}}_i^u$  and  $\vec{\mathcal{F}}_i^q$  via multiplication by  $\sigma_u$  and  $\sigma_q$ , respectively, to obtain dimensional spatial maps.
2. Multiply each dimensional spatial map for zonal wind velocity by  $\phi_0(\tilde{y})$  and  $C_1(z)$  to obtain  $\vec{u}_{\text{MJO1}}(\vec{r})$  and  $\vec{u}_{\text{MJO2}}(\vec{r})$ . Similarly, multiply each dimensional spatial map for moisture anomaly by  $\phi_0(\tilde{y})$  and  $S_{\text{low}}(z)$  to obtain  $\vec{q}_{\text{MJO1}}(\vec{r})$  and  $\vec{q}_{\text{MJO2}}(\vec{r})$ .
3. The zonal wind velocity part of the MJO at time-step  $n$  is then given by  $\vec{\mathcal{M}}_1^{(n)} \vec{u}_{\text{MJO1}}(\vec{r}) + \vec{\mathcal{M}}_2^{(n)} \vec{u}_{\text{MJO2}}(\vec{r})$ , where  $(n)$  denotes the  $n^{\text{th}}$  entry of the

vector. Similarly, the moisture anomaly part of the MJO at time-step  $n$  is given by  $\vec{M}_1^{(n)} \vec{q}_{\text{MJO1}}(\vec{r}) + \vec{M}_2^{(n)} \vec{q}_{\text{MJO1}}(\vec{r})$ .

Using the EOFs obtained from the truth data, we may always write the zonal wind velocity and moisture anomaly as an MJO part and a non-MJO part

$$\begin{aligned}
 u(t, \vec{r}) &= \sum_{i=1}^{2n_x} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_{ijk}(t) \vec{F}_i^u \phi_j(y) C_k(z) \\
 &= \left( u_{101}(t) \vec{F}_1^u + u_{201}(t) \vec{F}_2^u \right) \phi_0(y) C_1(z) + u'(t, \vec{r}) \\
 (A.12) \quad &= u_{\text{MJO}}(t, \vec{r}) + u'(t, \vec{r}),
 \end{aligned}$$

$$\begin{aligned}
 q(t, \vec{r}) &= \sum_{i=1}^{2n_x} \sum_{j=0}^{\infty} \vec{F}_i^q \phi_j(y) \left( q_{i\text{Jup}}(t) S_{\text{up}}(z) + q_{i\text{Jlow}}(t) S_{\text{low}}(z) \right. \\
 &\quad \left. + \sum_{k=3}^{\infty} q_{ijk}(t) S_k(z) \right) \\
 &= \left( q_{10\text{low}}(t) \vec{F}_1^q + q_{20\text{low}}(t) \vec{F}_2^q \right) \phi_0(y) S_1(z) + q'(t, \vec{r}) \\
 (A.13) \quad &= q_{\text{MJO}}(t, \vec{r}) + q'(t, \vec{r}).
 \end{aligned}$$

### Appendix B. Construction of Wheeler–Kiladis diagrams.

In this appendix, we describe the construction of the Wheeler–Kiladis diagrams presented throughout Section 4.

Wheeler–Kiladis diagrams are wavenumber-frequency power spectra of tropical cloudiness data (or moisture) that illustrate the presence of different equatorial waves and other phenomena in a system [64, 65]. For example, the Madden–Julian Oscillation (MJO) is characterized by a spectral peak at a period of 30–90 d over zonal wavenumbers  $k_x = 1, 2, 3$  [64, 65, 71].

The curves on the Wheeler–Kiladis diagram correspond to dispersion relations for classes of zonally-propagating waves from the linearized shallow water equations. These dispersion relations are given by the solutions to the second-order parital differential equation

$$(B.1) \quad \frac{\partial^2 \hat{v}}{\partial^2 y} + \left( \frac{\omega^2}{g h_e} - k_x^2 - \frac{k_x}{\omega} \beta - \frac{\beta^2 y^2}{g h_e} \right) \hat{v} = 0$$

where  $\hat{v}$  is the Fourier transform of the meridional wind velocity,  $g$  is the acceleration due to gravity,  $h_e$  is the equivalent depth of the atmosphere,  $\beta$  is the variation of Coriolis parameter with altitude,  $\omega$  is frequency, and  $k_x$  is zonal wave number [26].

Solutions which decay away from the equator require that the constant part of the coefficient in parentheses obeys the cubic

$$(B.2) \quad \frac{\sqrt{g h_e}}{\beta} \left( \frac{\omega^2}{g h_e} - k_x^2 - \frac{k_x}{\omega} \beta \right) = 2n + 1, \quad n = 0, 1, 2, \dots,$$

which lead to a class of waves for each root: eastern inertio-gravity (EIG) waves, western inertio-gravity (WIG) waves, and equatorial Rossby (ER) waves [26].

The  $n = 0$  solution corresponds to the mixed Rossby-gravity (MRG) waves. The Kelvin wave solution is not covered by Eqn. B.2 but rather comes from the solution for with  $\hat{v} = 0$  for all  $y$ , which corresponds to  $n = -1$  for Eqn. B.2.

The curves on the Wheeler-Kiladis diagram correspond to solutions of Eqn. B.2 with  $h_e = 12\text{ m}$ ,  $25\text{ m}$ ,  $50\text{ m}$  and  $n = -1, 0, 1$ . Note that the zonal wavenumber and frequency must be scaled due to dimensionality of the equations

$$(B.3) \quad \omega \rightarrow 2\pi\omega, \quad k_x \rightarrow \frac{2\pi}{2\pi P_E} k_x$$

where  $P_E$  is the radius of the Earth.

To obtain the actual power spectra in the Wheeler-Kiladis diagram, we used only the center half  $-P_Y/2 \leq y \leq P_Y/2$  of the moisture anomaly data with its global mean removed. We presented the moisture anomaly at only the specified altitude  $q(t, \vec{r}_h)$ . We then obtained the symmetric part and asymmetric part of the moisture anomaly using the parabolic cylinder functions and meridional inner product defined in Appendix A

$$(B.4) \quad q_{\text{sym}}(t, x) = \langle q(t, \vec{r}_h), \phi_0(\tilde{y}) \rangle_y, \quad q_{\text{asym}}(t, x) = \langle q(t, \vec{r}_h), \phi_1(\tilde{y}) \rangle_y.$$

For each, we removed the running 128-day mean as well as the first 128 days of data (which is considered to be the spin-up phase), and separated them into 128-day windows with 38-day overlap. The first and last ten percent of each window is tapered with a cosine function. We obtained the power in Fourier space by taking a two-dimensional Fourier transform of each and calculating their respective complex moduli.

### Appendix C. High-dimensional test case: multi-model communication strategy.

In this appendix, we give greater detail on the multi-model communication strategy for the high-dimensional test case (Section 4).

The greatest challenge for having the deficient tropical climate model (DTCM) and the Madden-Julian Oscillation model (MJOM) communicate in this test case was the difference in their state spaces. As such, we defined a mapping between their state spaces by using Empirical Orthogonal Function (EOF) analysis, discussed in Appendix A.

Using the decomposition introduced in Appendix A, we directly obtained the MJO indices of the DTCM at any given time. The MJO indices actually appear in the decompositions Eqs. A.12 and A.13 as  $u_{101}$ ,  $u_{201}$ ,  $q_{10\text{low}}$  and  $q_{20\text{low}}$ .

To calculate these MJO indices, we began by considering the zonal wind velocity  $u(\vec{r})$  and moisture anomaly  $q(\vec{r})$  at a given time in the DTCM. By taking the inner product of these values with the appropriate vertical and meridional basis functions, we obtained the discrete data  $\vec{u}_{01}$  and  $\vec{q}_{0\text{low}}$ . We non-dimensionalized these vectors by dividing them by  $\sigma_u$  and  $\sigma_q$ , respectively, and concatenated them into a single vector  $\vec{a}$ .

The vector  $\vec{a}$  is in the vector space spanned by the EOFs  $\vec{\mathcal{F}}_i$  obtained from the truth time-series data. Taking the standard dot product of  $\vec{a}$  and  $\vec{\mathcal{F}}_1$  gave the first Madden-Julian Oscillation (MJO) index, and with  $\vec{\mathcal{F}}_2$  the second. The first MJO index is the  $u_{101}$  and  $q_{10\text{low}}$  coefficients in Eqs. A.12 and A.13, while the second MJO index is  $u_{201}$  and  $q_{20\text{low}}$ .

We chose a non-dimensionalization of the mean vertical moisture gradient in the DTCM to correspond to climate signal  $v$  of the MJOM. In particular, for the total

moisture  $q_{\text{tot}}(t, \vec{r}) = q(t, \vec{r}) + q_{BG}(\vec{r})$  we defined  
(C.1)

$$v^{(DTCM)}(t) = \frac{1}{H} \int_0^z \left[ \frac{1}{2P_Y} \int_{-P_Y}^{P_Y} \left[ \frac{1}{2\pi P_E} \int_0^{2\pi P_E} \frac{1}{B_{vs}} \frac{\partial}{\partial z} [q_{\text{tot}}(t, \vec{r})] dx \right] dy \right] dz,$$

where  $v^{(DTCM)}(t)$  is the climate signal from the DTCM. The pseudo-truth signal for the MJOM was then given by  $[u_{101} \ u_{201} \ v^{(DTCM)}]^T$ . The other matrices associated with the MJOM-side of the communication were

$$(C.2) \quad \begin{aligned} \mathbf{R}^{(MJOM)} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & \mathbf{H}^{(MJOM)} &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \\ \mathbf{B}^{(MJOM)} &= \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, & \mathbf{\Lambda}^{(MJOM)} &= \begin{bmatrix} 0.1 \end{bmatrix}. \end{aligned}$$

When assimilating data from the MJOM, we did not wish to adjust the entirety of the zonal wind velocity nor the entirety of the moisture anomaly, but only the MJO-part. To this end, we performed data assimilation on the MJO indices of the DTCM state. This not only adjusted only the MJO-part of the DTCM state, but required far fewer computations when implemented.

Let the pseudo-truth signal for the DTCM be the triple  $[u_1 \ u_2 \ v]$  from the MJOM. We adjusted the triple  $[u_{101} \ u_{201} \ v^{(DTCM)}]$  toward the former triple using our multi-model communication strategy. For the other communication parameters, we used

$$(C.3) \quad \begin{aligned} \mathbf{R}^{(DTCM)} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & \mathbf{H}^{(DTCM)} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\ \mathbf{B}^{(DTCM)} &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, & \mathbf{\Lambda}^{(DTCM)} &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}. \end{aligned}$$

Once the MJO indices  $u_{101}$  and  $u_{201}$  (which were equal to  $q_{10\text{low}}$  and  $q_{20\text{low}}$ , respectively) were been adjusted, we reconstructed the MJO using the first term in the second line of Eqs. A.12 and A.13, which we denote  $u'_{\text{MJO}}$  and  $q'_{\text{MJO}}$ , respectively. The zonal wind velocity and moisture anomaly of the DTCM were then changed to

$$(C.4) \quad u(t, \vec{r}) \rightarrow u'_{\text{MJO}}(t, \vec{r}) + u'(t, \vec{r}), \quad q(t, \vec{r}) \rightarrow q'_{\text{MJO}}(t, \vec{r}) + q'(t, \vec{r}).$$

**Acknowledgments.** We would like to thank H. R. Ogrosky for the original Matlab code to solve the OSH19 system. This research was performed using the compute resources and assistance of the University of Wisconsin-Madison Center For High Throughput Computing (CHTC) in the Department of Computer Sciences. The CHTC is supported by the University of Wisconsin-Madison, the Advanced Computing Initiative, the Wisconsin Alumni Research Foundation, the Wisconsin Institutes for Discovery, and the National Science Foundation, and is an active member of the Open Science Grid, which is supported by the National Science Foundation and the U.S. Department of Energy's (DOE) Office of Science. This research also used resources of the National Energy Research Scientific Computing Center (NERSC), a U.S. Department of Energy Office of Science User Facility located at Lawrence Berkeley National Laboratory, operated under Contract No. DE-AC02-05CH11231.

## REFERENCES

- [1] J. ANDERSON, T. HOAR, K. RAEDER, H. LIU, N. COLLINS, R. TORN, AND A. AVELLANO, *The data assimilation research testbed: A community facility*, Bulletin of the American Meteorological Society, 90 (2009), pp. 1283–1296.
- [2] H. ARBABI AND I. MEZIC, *Ergodic theory, dynamic mode decomposition, and computation of spectral properties of the koopman operator*, SIAM Journal on Applied Dynamical Systems, 16 (2017), pp. 2096–2126.
- [3] J. BENEDICT, E. MALONEY, A. SOBEL, D. FRIERSON, AND L. DONNER, *Tropical intraseasonal variability in version 3 of the GFDL atmosphere model*, Journal of Climate, 26 (2013), pp. 426–449, <https://doi.org/10.1175/JCLI-D-12-00103.1>.
- [4] T. BERRY, J. R. CRESSMAN, Z. GREGURIC-FERENCSEK, AND T. SAUER, *Time-scale separation from diffusion-mapped delay coordinates*, SIAM Journal on Applied Dynamical Systems, 12 (2013), pp. 618–649.
- [5] H. BJOÖRNSSON AND S. VENEGAS, *A manual for EOF and SVD analyses of climatic data*. Electronic., 1997, <http://www.ccpo.odu.edu/~klinck/Reprints/PaperWebPages/bjornssonEOF1997.html>.
- [6] M. BRANICKI AND A. J. MAJDA, *An information-theoretic framework for improving imperfect dynamical predictions via multi-model ensemble forecasts*, J. Nonlin. Sci., 25 (2015), pp. 489–538.
- [7] N. CHEN, Y. LI, AND H. LIU, *Conditional gaussian nonlinear system: A fast preconditioner and a cheap surrogate model for complex nonlinear systems*, Chaos: An Interdisciplinary Journal of Nonlinear Science, 32 (2022), p. 053122.
- [8] N. CHEN, A. MAJDA, AND D. GIANNAKIS, *Predicting the cloud patterns of the Madden-Julian Oscillation through a low-order nonlinear stochastic model*, Geophys. Res. Lett., 41 (2014), pp. 5612–5619, <https://doi.org/10.1002/2014GL060876>.
- [9] Y. CHEN AND S. STECHMANN, *Multi-model communication and data assimilation for mitigating model error and improving forecasts*, Chinese Annals of Mathematics Series B, 40 (2019), pp. 689–720, <https://doi.org/10.1007/s11401-019-0157-1>.
- [10] M. CUTLER, T. WALSH, AND J. HOW, *Reinforcement learning with multi-fidelity simulators*, in 2014 IEEE International Conference on Robotics and Automation (ICRA), Institute of Electrical and Electronics Engineers, Sep. 2014, pp. 3888–3895, <https://doi.org/10.1109/ICRA.2014.6907423>.
- [11] G. DUANE, J. TRIBBIA, AND J. WEISS, *Synchronicity in predictive modelling: A new view of data assimilation*, Nonlin. Processes Geophys., 13 (2006), pp. 601–612, <https://doi.org/10.5194/npg-13-601-2006>.
- [12] G. DUANE, W. WIEGERINCK, F. SELTEN, M. SHEN, AND N. KEENLYSIDE, *Supermodeling: Synchronization of alternative dynamical models of a single objective process*, Advances in Nonlinear Geosciences, (2018), pp. 101–121, [https://doi.org/10.1007/978-3-319-58895-7\\_5](https://doi.org/10.1007/978-3-319-58895-7_5).
- [13] D. GIANNAKIS, *Dynamics-adapted cone kernels*, SIAM Journal on Applied Dynamical Systems, 14 (2015), pp. 556–608.
- [14] D. GIANNAKIS, *Data-driven spectral decomposition and forecasting of ergodic dynamical systems*, Applied and Computational Harmonic Analysis, 47 (2019), pp. 338–396.
- [15] G. A. GOTTFELD AND S. REICH, *Supervised learning from noisy observations: Combining machine-learning techniques with data assimilation*, Physica D: Nonlinear Phenomena, 423 (2021), p. 132911.
- [16] W. W. GRABOWSKI AND P. K. SMOLARKIEWICZ, *CRCP: a Cloud Resolving Convection Parameterization for modeling the tropical convecting atmosphere*, Physica D: Nonlinear Phenomena, 133 (1999), pp. 171–178.
- [17] R. HAGEDORN, F. J. DOBLAS-REYES, AND T. PALMER, *The rationale behind the success of multi-model ensembles in seasonal forecasting—i. basic concept*, Tellus A, 57 (2005), pp. 219–233.
- [18] W. HANNAH, C. JONES, B. HILLMAN, M. NORMAN, D. BADER, M. TAYLOR, L. LEUNG, M. PRITCHARD, M. BRANSON, G. LIN, K. PRESSEL, AND J. LEE, *Initial results from the super-parametrized E3SM*, Journal of Advances in Modeling Earth Systems, 12 (2020), pp. 1–19, <https://doi.org/10.1029/2019MS001863>.
- [19] J. HARLIM, S. W. JIANG, S. LIANG, AND H. YANG, *Machine learning for prediction with missing dynamics*, Journal of Computational Physics, 428 (2021), p. 109922.
- [20] E. KALNAY, *Atmospheric Modeling, Data Assimilation and Predictability*, Cambridge University Press, 2003.
- [21] M. KAMB, E. KAISER, S. L. BRUNTON, AND J. N. KUTZ, *Time-delay observables for koopman: Theory and applications*, SIAM Journal on Applied Dynamical Systems, 19 (2020), pp. 886–917.

- [22] K. KASHINATH, M. MUSTAFA, A. ALBERT, J. WU, C. JIANG, S. ESMAEILZADEH, K. AZIZZADE-NESHELI, R. WANG, A. CHATTOPADHYAY, A. SINGH, ET AL., *Physics-informed machine learning: case studies for weather and climate modelling*, Philosophical Transactions of the Royal Society A, 379 (2021), p. 20200093.
- [23] W. KESSLER, *EOF representations of the Madden-Julian Oscillation and its connection with ENSO*, Journal of Climate, 14 (2001), pp. 3055–3061, [https://doi.org/10.1175/1520-0442\(2001\)014<3055:EROTMJ>2.0.CO;2](https://doi.org/10.1175/1520-0442(2001)014<3055:EROTMJ>2.0.CO;2).
- [24] V. V. KHARIN AND F. W. ZWIERS, *Climate predictions with multimodel ensembles*, Journal of Climate, 15 (2002), pp. 793–799.
- [25] B. KHOUDER AND A. J. MAJDA, *A simple multicloud parameterization for convectively coupled tropical waves. part ii: Nonlinear simulations*, Journal of the atmospheric sciences, 64 (2007), pp. 381–400.
- [26] G. KILADIS, M. WHEELER, P. HAERTEL, K. STRAUB, AND P. ROUNDY, *Convectively coupled equatorial waves*, Rev. Geophys., 47 (2009), pp. 1–42, <https://doi.org/10.1029/2008RG000266>.
- [27] D. KIM, A. SOBEL, E. MALONEY, D. FRIERSON, AND I. KANG, *A systematic relationship between intraseasonal variability and mean state bias in AGCM simulations*, Journal of Climate, 24 (2011), pp. 5506–5520, <https://doi.org/10.1175/2011JCLI4177.1>.
- [28] T. KRISHNAMURTI, C. KISHTAWAL, T. LAROW, D. BACHIOCHI, Z. ZHANG, C. WILLIFORD, S. GADGIL, AND S. SURENDRAN, *Improved weather and seasonal climate forecasts from multimodel superensemble*, Science, 285 (1999), pp. 1548–1550, <https://doi.org/10.1126/science.285.5433.1548>.
- [29] T. KRISHNAMURTI, V. KUMAR, A. SIMON, A. BHARDWAJ, T. GHOSH, AND R. ROSS, *A review of multimodel superensemble forecasting for weather, seasonal climate, and hurricanes*, Rev. Geophys., 54 (2016), pp. 336–377, <https://doi.org/10.1002/2015RG000513>.
- [30] K. LAW, A. STUART, AND K. ZYGALAKIS, *Data Assimilation: A Mathematical Introduction*, Springer, 2015.
- [31] M. E. LEVINE AND A. M. STUART, *A framework for machine learning of model error in dynamical systems*, arXiv preprint arXiv:2107.06658, (2021).
- [32] A. MAJDA, *Introduction to PDEs and Waves for the Atmosphere and Ocean*, American Mathematical Society, 2003.
- [33] A. MAJDA AND J. HARLIM, *Filtering Complex Turbulent Systems*, Cambridge University Press, Cambridge, UK, 1 ed., 2012.
- [34] A. J. MAJDA AND D. QI, *Strategies for reduced-order models for predicting the statistical responses and uncertainty quantification in complex turbulent dynamical systems*, SIAM Review, 60 (2018), pp. 491–549.
- [35] A. J. MAJDA AND S. N. STECHMANN, *The skeleton of tropical intraseasonal oscillations*, Proceedings of the National Academy of Sciences, 106 (2009), pp. 8417–8422.
- [36] B. E. MAPES AND J. T. BACMEISTER, *Diagnosis of tropical biases and the MJO from patterns in the MERRA analysis tendency fields*, Journal of Climate, 25 (2012), pp. 6202–6214.
- [37] L. NERGER, Q. TANG, AND L. MU, *Efficient ensemble data assimilation for coupled models with the parallel data assimilation framework: example of awi-cm (awi-cm-pdaf 1.0)*, Geoscientific Model Development, 13 (2020), pp. 4305–4321.
- [38] H. OGROSKY, S. STECHMANN, AND S. HOTTOVY, *Instability and nonlinear dynamics of the MJO in a tropical channel model with vertically varying convective adjustment*, Theor. Comput. Fluid Dyn., 33 (2019), pp. 307–323, <https://doi.org/10.1007/s00162-019-00495-x>.
- [39] H. R. OGROSKY, S. N. STECHMANN, N. CHEN, AND A. J. MAJDA, *Singular spectrum analysis with conditional predictions for real-time state estimation and forecasting*, Geophysical Research Letters, 46 (2019), pp. 1851–1860.
- [40] L. PECORA AND T. CARROLL, *Synchronization in chaotic systems*, Phys. Rev. Lett., 64 (1990), pp. 821–825, <https://doi.org/10.1103/PhysRevLett.64.821>.
- [41] B. PEHERSTORFER, K. WILLCOX, AND M. GUNZBURGER, *Survey of multifidelity methods in uncertainty propagation, inference, and optimization*, SIAM Review, 60 (2018), pp. 550–591.
- [42] M. PELEG AND M. NORMAND, *Modeling of fungal and bacterial spore germination under static and dynamics conditions*, Appl. Environ. Microbiol., 79 (2013), pp. 6765–6775, <https://doi.org/10.1128/AEM.02521-13>.
- [43] P. POKORNÝ, D. VOKROUHLICKÝ, D. NESVORNÝ, M. CAMPBELL-BROWN, AND P. BROWN, *Dynamical model for the toroidal sporadic meteors*, The Astrophysical Journal, 789 (2014), pp. 1–20, <https://doi.org/10.1088/0004-637X/789/1/25>.
- [44] D. QI AND A. J. MAJDA, *Using machine learning to predict extreme events in complex systems*, Proceedings of the National Academy of Sciences, 117 (2020), pp. 52–59.

- [45] D. RANDALL, M. KHAIROUTDINOV, A. ARAKAWA, AND W. GRABOWSKI, *Breaking the cloud parameterization deadlock*, Bull. Amer. Meteor. Soc., 84 (2003), pp. 1547–1564.
- [46] S. RANFTL, G. MELITO, V. BADELI, A. REINBACHER-KÖSTINGER, K. ELLERMANN, AND W. VON DER LINDEN, *Bayesian uncertainty quantification with multi-fidelity data and Gaussian processes for impedance cardiography of aortic dissection*, Entropy, 22 (2019), pp. 1–14, <https://doi.org/10.3390/e22010058>.
- [47] S. REICH AND C. COTTER, *Probabilistic Forecasting and Bayesian Data Assimilation*, Cambridge University Press, 2015.
- [48] T. ROBINSON, K. WILLCOX, M. ELDRED, AND R. HAIMES, *Multifidelity optimization for variable-complexity design*, in 11th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference, American Institute of Aeronautics and Astronautics, Sep. 2006, pp. 1–18, <https://doi.org/10.2514/6.2006-7114>.
- [49] M. RODWELL AND T. PALMER, *Using numerical weather prediction to assess climate models*, Quarterly Journal of the Royal Meteorological Society, 133 (2007), pp. 129–146.
- [50] S. RUDY, A. ALLA, S. L. BRUNTON, AND J. N. KUTZ, *Data-driven identification of parametric partial differential equations*, SIAM Journal on Applied Dynamical Systems, 18 (2019), pp. 643–660.
- [51] H. SCHAEFFER, G. TRAN, AND R. WARD, *Extracting sparse high-dimensional dynamics from limited data*, SIAM Journal on Applied Mathematics, 78 (2018), pp. 3279–3295.
- [52] T. SCHNEIDER, S. LAN, A. STUART, AND J. TEIXEIRA, *Earth system modeling 2.0: A blueprint for models that learn from observations and targeted high-resolution simulations*, Geophysical Research Letters, 44 (2017), pp. 12–396.
- [53] T. SCHNEIDER, A. M. STUART, AND J.-L. WU, *Learning stochastic closures using ensemble kalman inversion*, Transactions of Mathematics and Its Applications, 5 (2021), p. tnab003.
- [54] M. SHEN, N. KEENLYSIDE, F. SELTEN, W. WIEGERINCK, AND G. DUANE, *Dynamically combining climate models to ‘supermodel’ the tropical Pacific*, Geophys. Res. Lett., 43 (2016), pp. 359–366, <https://doi.org/10.1002/2015GL066562>.
- [55] J. P. STACHNIK, D. E. WALISER, A. J. MAJDA, S. N. STECHMANN, AND S. THUAL, *Evaluating mjo event initiation and decay in the skeleton model using an rmm-like index*, Journal of Geophysical Research: Atmospheres, 120 (2015), pp. 11–486.
- [56] S. STECHMANN AND S. HOTTOVY, *Unified spectrum of tropical rainfall and waves in simple stochastic model*, Geophys. Res. Lett., 44 (2017), pp. 10713–10724, <https://doi.org/10.1002/2017GL075754>.
- [57] S. STECHMANN, A. MAJDA, AND B. KHOUIDER, *Nonlinear dynamics of hydrostatic internal gravity waves*, Theor. Comput. Fluid Dyn., 22 (2008), pp. 407–432, <https://doi.org/10.1007/s00162-008-0080-7>.
- [58] S. N. STECHMANN, *Multiscale eddy simulation for moist atmospheric convection: Preliminary investigation*, J. Comput. Phys., 271 (2014), pp. 99–117.
- [59] S. THUAL, A. J. MAJDA, N. CHEN, AND S. N. STECHMANN, *Simple stochastic model for el niño with westerly wind bursts*, Proceedings of the National Academy of Sciences, 113 (2016), pp. 10245–10250.
- [60] X. TONG, A. MAJDA, AND D. KELLY, *Nonlinear stability and ergodicity of ensemble based Kalman filters*, Nonlinearity, 29 (2016), pp. 657–691, <https://doi.org/10.1088/0951-7715/29/2/657>.
- [61] X. TONG, A. MAJDA, AND D. KELLY, *Nonlinear stability the ensemble Kalman filter with adaptive covariance inflation*, Commun. Math. Sci., 14 (2016), pp. 1283–1313, <https://doi.org/10.1088/0951-7715/29/2/657>.
- [62] P. R. VLACHAS, W. BYEON, Z. Y. WAN, T. P. SAPSIS, AND P. KOUMOUTSAKOS, *Data-driven forecasting of high-dimensional chaotic systems with long short-term memory networks*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 474 (2018), p. 20170844.
- [63] P. R. VLACHAS, J. PATHAK, B. R. HUNT, T. P. SAPSIS, M. GIRVAN, E. OTT, AND P. KOUMOUTSAKOS, *Backpropagation algorithms and reservoir computing in recurrent neural networks for the forecasting of complex spatiotemporal dynamics*, Neural Networks, 126 (2020), pp. 191–217.
- [64] M. WHEELER AND H. HENDON, *An all-season real-time multivariate MJO index: Development of an index for monitoring and prediction*, Monthly Weather Review, 132 (2004), pp. 1917–1932, [https://doi.org/10.1175/1520-0493\(2004\)132<1917:AARMMI>2.0.CO;2](https://doi.org/10.1175/1520-0493(2004)132<1917:AARMMI>2.0.CO;2).
- [65] M. WHEELER AND G. KILADIS, *Convectively coupled equatorial waves: Analysis of clouds and temperature in the wavenumber-frequency domain*, Journal of the Atmospheric Sciences, 56 (1999), pp. 374–399, [https://doi.org/10.1175/1520-0469\(1999\)056<0374:CCEWAO>2.0.CO;2](https://doi.org/10.1175/1520-0469(1999)056<0374:CCEWAO>2.0.CO;2).

- [66] L. XUE, *Application of the multimodel ensemble Kalman filter method in groundwater system*, Water, 7 (2015), pp. 528–545, <https://doi.org/10.3390/w7020528>.
- [67] L. XUE AND D. ZHANG, *A multimodel data assimilation framework via the ensemble Kalman filter*, Water Resour. Res., 50 (2014), pp. 4197–4219, <https://doi.org/10.1002/2013WR014525>.
- [68] W. YUN, L. STEFANOVA, A. MITRA, V. KUMAR, W. DEWAR, AND T. KRISHNAMURTI, *A multimodel superensemble algorithm for seasonal climate prediction using DEMETER forecasts*, Tellus, 57A (2005), pp. 280–289, <https://doi.org/10.3402/tellusa.v57i3.14699>.
- [69] P. ZARCHAN AND H. MUSOFF, *Fundamentals of Kalman Filtering: A Practical Approach*, American Institute of Aeronautics and Astronautics, Inc., Reston, VA, 3 ed., 2009.
- [70] S. E. ZEBIAK AND M. A. CANE, *A model El Niño–Southern Oscillation*, Monthly Weather Review, 115 (1987), pp. 2262–2278.
- [71] C. ZHANG, *Madden–Julian Oscillation*, Rev. Geophys., 43 (2005), pp. 1–36, <https://doi.org/10.1029/2004RG000158>.
- [72] Z. ZHANG AND J. MOORE, *Mathematical and Physical Fundamentals of Climate Change*, Elsevier, Oxford, UK, 2015, ch. 6, pp. 161–198.