

Supplemental information: Angular-spatial hp -adaptivity for radiative transfer with discontinuous Galerkin spectral element methods

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SI-1. Spectral bases visualization

Each spatial-angular hp -element has an associated set of quadrature nodes and weights, as well as spectral basis in each spatial and the angular dimension. In the spatial dimensions, the quadrature nodes and weights are given by the Legendre–Gauss–Lobatto quadrature [1, 2]. In the angular dimension, the quadrature nodes and weights are given by the Legendre–Gauss quadrature [1, 2]. The basis functions in each dimension are the Lagrange polynomials interpolating the quadrature nodes. The spatial and angular basis functions for five nodes are shown in Figure SI-1.

SI-2. Spatial mesh refinement visualization

Here, we show a visualization of the process of performing a spatial h - (Figure SI-2) and p -refinement (Figure SI-3) on a spatial hp -element, as described in Subsection 3.2.

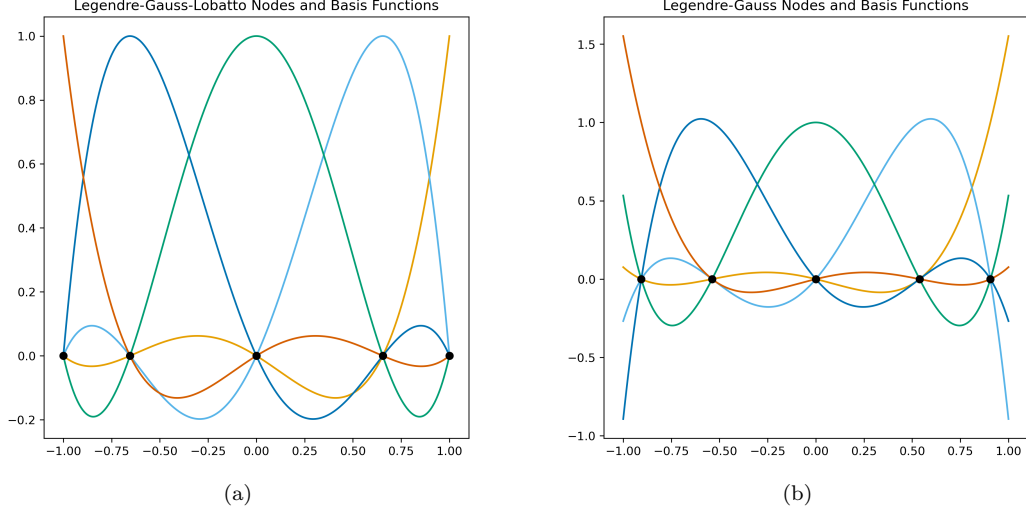


Figure SI-1: The nodes given by the (a) Legendre–Gauss–Lobatto and (b) Legendre–Gauss quadratures for five quadrature points, along with the Lagrange polynomials interpolating these nodes. These constitute the quadrature nodes and basis functions in the spatial dimensions and the angular dimension, respectively.

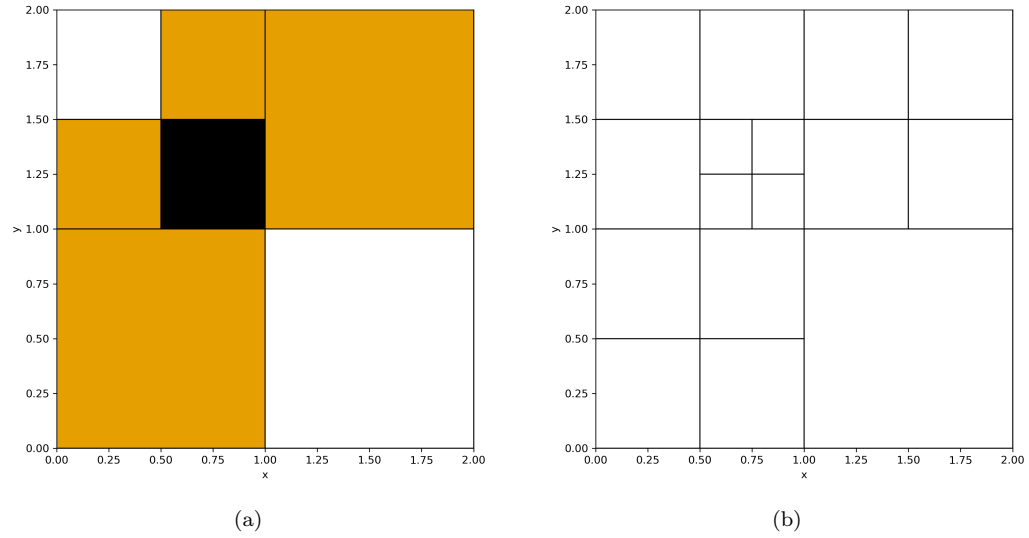


Figure SI-2: Demonstration of a spatial h -refinement of a spatial hp -element (black). (a) We first check if neighboring spatial elements (orange) must be h -refined, and (b) the resultant spatial h -mesh after spatial h -refinement has been performed.

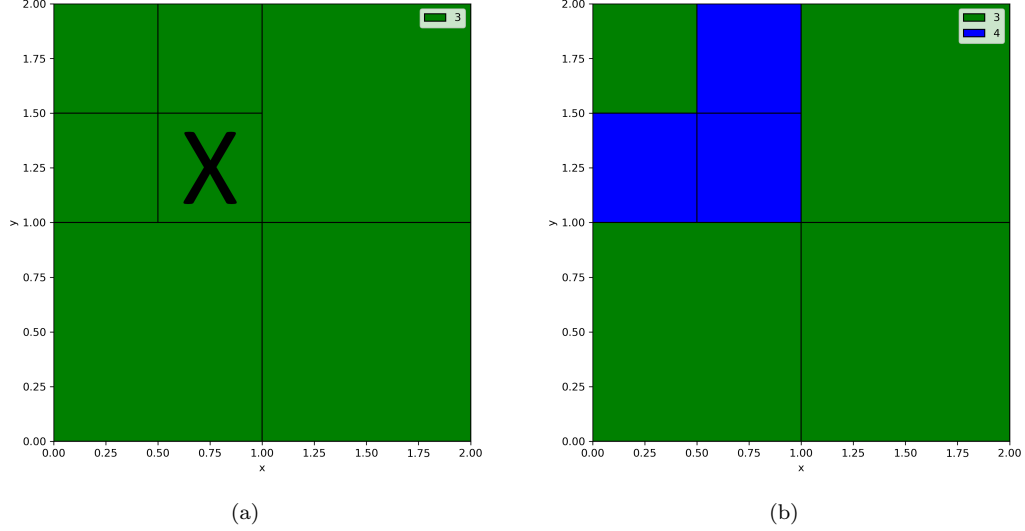


Figure SI-3: Demonstration of a p -refinement of a spatial hp -element (black ‘X’). (a) Before the p -refinement, and (b) the resultant spatial hp -mesh after p -refinement has been performed. The color of each spatial element shows its number of spatial degrees of freedom in each of the x - and y -dimensions.

SI-3. Spatial regularity estimator

For hp -steering for spatial refinement, we use the regularity estimator of [3], as derived from results from [4]. We restate the method of the regularity estimator here for completeness.

Without loss of generality, consider a function $g^{\overline{C}}(\overline{x}, \overline{y})$ defined on the reference spatial element $\overline{C}$. The reference spatial element has domain $\Omega^{\overline{C}} = [-1, 1] \times [-1, 1]$, with associated Legendre–Gauss–Lobatto quadrature nodes and weights, and basis functions denoted

$$\left\{ \left(\overline{x}_i^{\overline{C}}, w_i^{\overline{C},x}, \overline{\phi}_i^{\overline{C}}(\overline{x}) \right) \right\}_{i=0}^{n_x^{\overline{C}}-1}, \quad \text{and} \quad \left\{ \left(\overline{y}_j^{\overline{C}}, w_j^{\overline{C},y}, \overline{\psi}_j^{\overline{C}}(\overline{y}) \right) \right\}_{j=0}^{n_y^{\overline{C}}-1} \quad (1)$$

in the x - and y -dimensions, respectively. We may use the following analysis on any spatial element by utilizing pull-back and push-forward maps, as discussed in Section SI-8.

The function $g^{\overline{C}}(\overline{x}, \overline{y})$ may be represented as a linear combination of

these spatial basis functions

$$g^{\overline{C}}(\overline{x}, \overline{y}) = \sum_{i=0}^{n_x^{\overline{C}}-1} \sum_{j=0}^{n_y^{\overline{C}}-1} g_{ij}^{\overline{C}} \overline{\phi}_i^{\overline{C}}(\overline{x}) \overline{\psi}_j^{\overline{C}}(\overline{y}). \quad (2)$$

To estimate the regularity of $g^{\overline{C}}(\overline{x}, \overline{y})$ based on the coefficients $g_{ij}^{\overline{C}}$, we write the Legendre polynomial expansion of $g^{\overline{C}}(\overline{x}, \overline{y})$

$$g^{\overline{C}}(\overline{x}, \overline{y}) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \gamma_{pq}^{\overline{C}} \mathcal{L}_p(\overline{x}) \mathcal{L}_q(\overline{y}) \quad (3)$$

where $\{\mathcal{L}_p\}_{p=0}^{\infty}$ is the Legendre polynomial on $[-1, 1]$. We may calculate the coefficient $\gamma_{pq}^{\overline{C}}$ by substituting the expansion Eqn. 2

$$\begin{aligned} \gamma_{pq}^{\overline{C}} &= \frac{(2p+1)(2q+1)}{4} \int_{-1}^1 \int_{-1}^1 g^{\overline{C}}(\overline{x}, \overline{y}) \mathcal{L}_p(\overline{x}) \mathcal{L}_q(\overline{y}) d\overline{x} d\overline{y} \\ &= \frac{(2p+1)(2q+1)}{4} \sum_{i=0}^{n_x^{\overline{C}}-1} \sum_{j=0}^{n_y^{\overline{C}}-1} g_{ij}^{\overline{C}} \left(\int_{-1}^1 \overline{\phi}_i^{\overline{C}}(\overline{x}) \mathcal{L}_p(\overline{x}) d\overline{x} \right) \left(\int_{-1}^1 \overline{\psi}_j^{\overline{C}}(\overline{y}) \mathcal{L}_q(\overline{y}) d\overline{y} \right). \end{aligned} \quad (4)$$

We estimate these integrals using the numerical quadrature associated with $\overline{C}$,

$$\gamma_{pq}^{\overline{C}} \approx \begin{cases} \frac{(2p+1)(2q+1)}{4} \sum_{i=0}^{n_x^{\overline{C}}-1} \sum_{j=0}^{n_y^{\overline{C}}-1} g_{ij}^{\overline{C}} (w_i^{\overline{C},x} \mathcal{L}_p(\overline{x}_i^{\overline{C}})) (w_j^{\overline{C},y} \mathcal{L}_q(\overline{y}_j^{\overline{C}})), & 0 \leq p \leq n_x^{\overline{C}}-1, \\ & 0 \leq q \leq n_y^{\overline{C}}-1, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

Since the Legendre–Gauss–Lobatto quadrature with N nodes and weights is exact for polynomials with degree up to $2N - 1$ [2], the approximation sign above may be replaced by true equality for $p < n_x^{\overline{C}} - 1$ and $q < n_y^{\overline{C}} - 1$. The definition of Legendre polynomials implies that $\gamma_{pq}^{\overline{C}} = 0$ when $p > n_x^{\overline{C}}$ or $q > n_y^{\overline{C}}$. With the Legendre expansion coefficients, we may use the regularity estimate of [4]. We calculate

$$\left| \gamma_p^{\overline{C},x} \right|^2 := \sum_{q=0}^{\infty} \frac{2}{2q+1} \left| \gamma_{pq}^{\overline{C}} \right|^2, \quad \left| \gamma_q^{\overline{C},y} \right|^2 := \sum_{p=0}^{\infty} \frac{2}{2p+1} \left| \gamma_{pq}^{\overline{C}} \right|^2, \quad (6a)$$

$$l_p^{\vec{x}} := \frac{1}{2} \left(\frac{\log \left(\frac{2n_x^{\bar{C}} - 1}{2|\gamma_f^{\bar{C},x}|^2} \right)}{2 \log (n_x^{\bar{C}} - 1)} + \frac{\log \left(\frac{2n_y^{\bar{C}} - 1}{2|\gamma_f^{\bar{C},y}|^2} \right)}{2 \log (n_y^{\bar{C}} - 1)} \right) \quad (6b)$$

where the subscript f denotes the final valid index for which the associated quantity is not certainly zero, e.g., $\gamma_f^{\bar{C},x} = \gamma_{n_x^{\bar{C}}-1}^{\bar{C},x}$. Then, if $l_p^{\vec{x}} - \frac{1}{2} \geq \frac{1}{2} (n_x^{\bar{C}} + n_y^{\bar{C}})$, we perform p -refinement; otherwise, we perform h -refinement.

We apply this hp -steering criterion when performing spatial refinement. Specifically, we evaluate the regularity of the angular integral

$$g(\theta) = \int_{K \in \mathcal{K}^C} u_{hp}(\vec{x}, \vec{s}) d\vec{s} \quad (7)$$

on each spatial hp -element C .

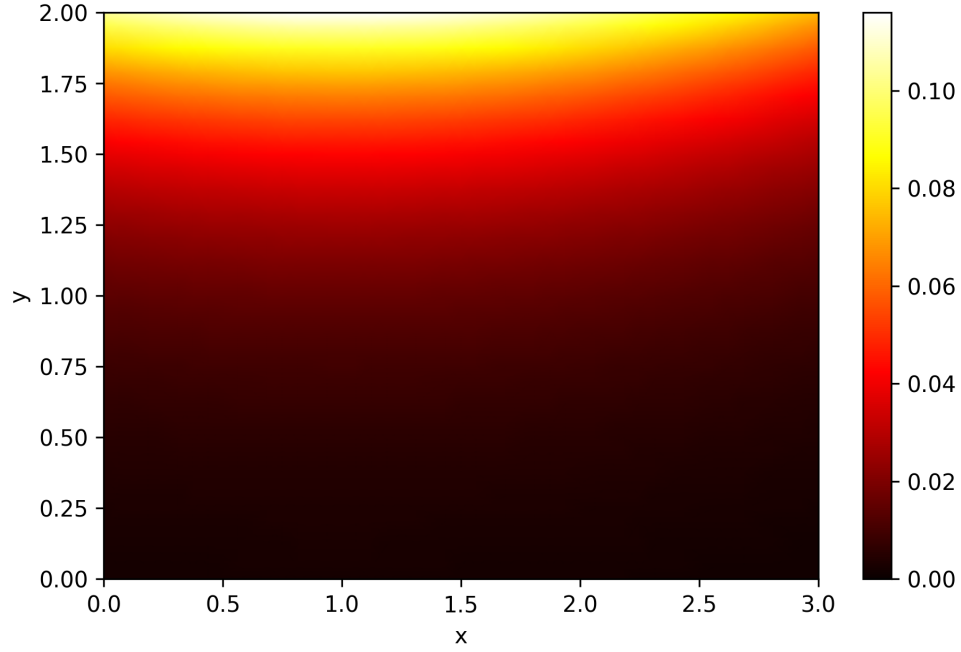
SI-4. Test case 1: Manufactured solution - Numerical solutions and mesh visualizations

We provide visualizations for each refinement strategy for the *Manufactured Solution* test case below. These visualizations include the final numerical solution for each refinement strategy and the final mesh for the adaptive refinement strategies. The primary discussion of the results of the *Manufactured Solution* test case is given in Subsection 4.1.

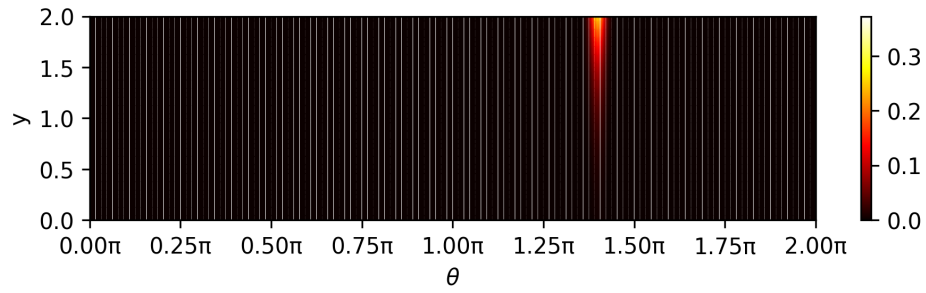
SI-5. Comparison of the analytic, high-resolution, and angular-jump error indicators

The choice of error indicator for the adaptive refinement strategy is fundamental to its efficacy. Such types of estimators exist for the simplified-PN approximation [5, 6], which are most effective when the scattering term is dominant. These estimators also exist for convection-diffusion equations [7, 8], which are most effective when scattering is negligible. To the authors' knowledge, no such error indicator exists for the full radiative transfer equation.

A spatial-jump error indicator has been utilized previously for the full radiative transfer equation [3]. Here, for adaptive angular refinements, we utilize an angular-jump error indicator (Eqn. 14). We study its efficacy by

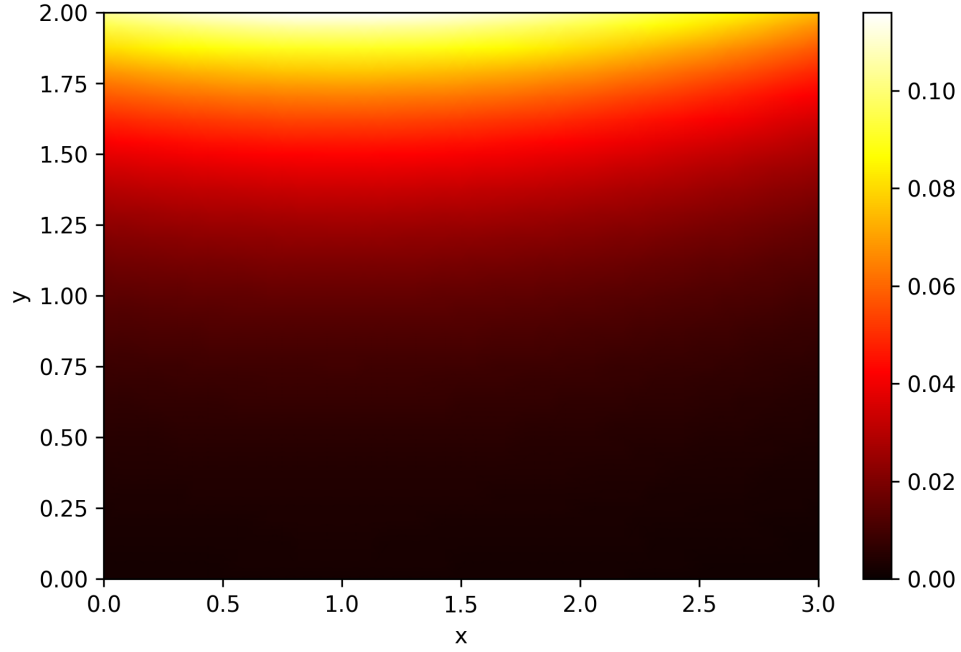


(a)

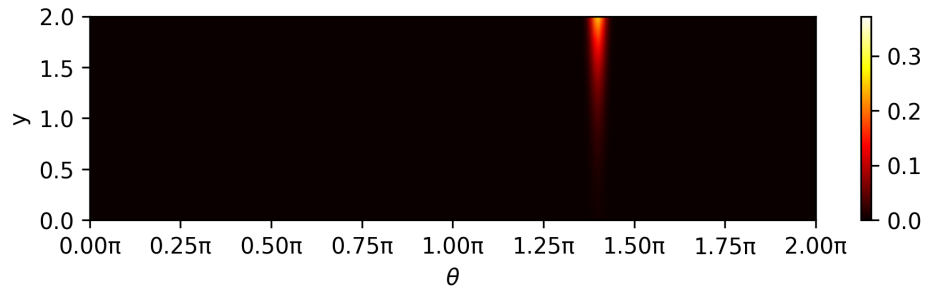


(b)

Figure SI-4: Visualizations of the final numerical solution for the uniform angular h -refinement strategy (h -Unif. Ang.) for the *Manufactured Solution* test case. The angular (a) and horizontal (b) integral of $u_{hp}(x, y, \theta)$.

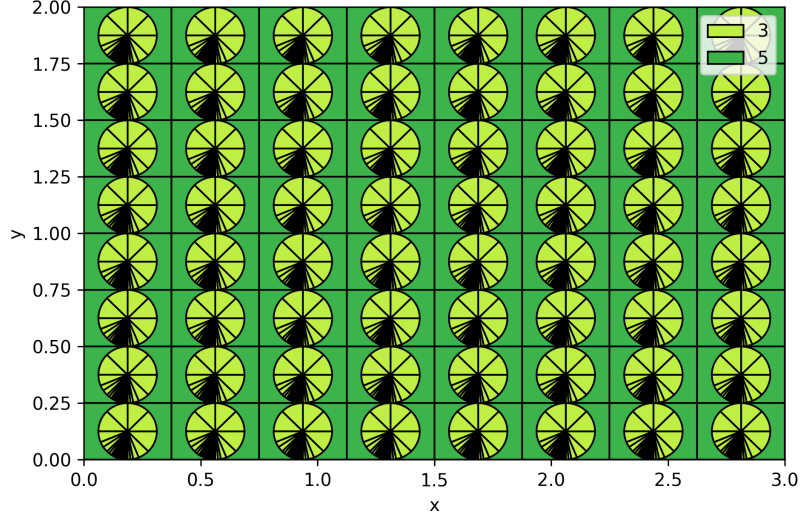


(a)

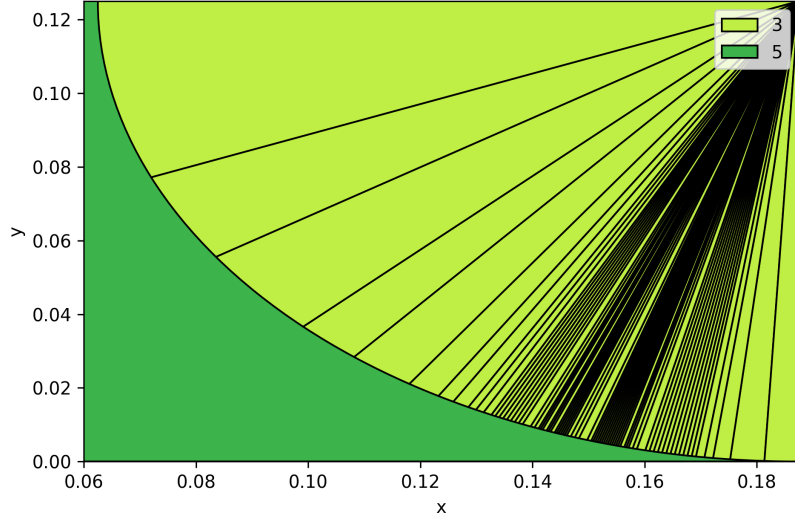


(b)

Figure SI-5: Visualizations of the final numerical solution for the uniform angular p -refinement strategy (p -Unif. Ang.) for the *Manufactured Solution* test case. The angular (a) and horizontal (b) integral of $u_{hp}(x, y, \theta)$.

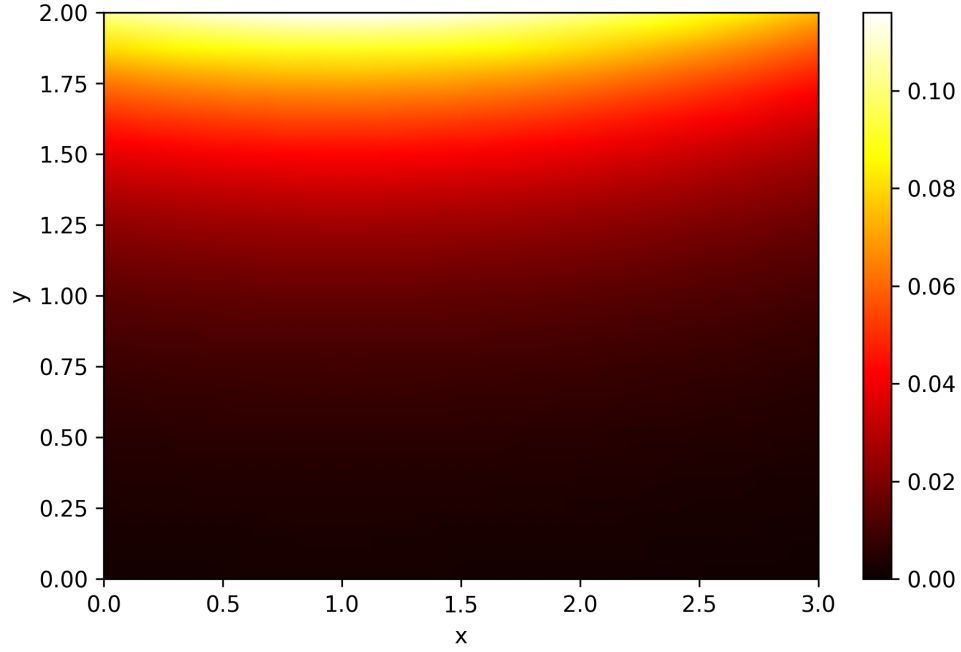


(a)

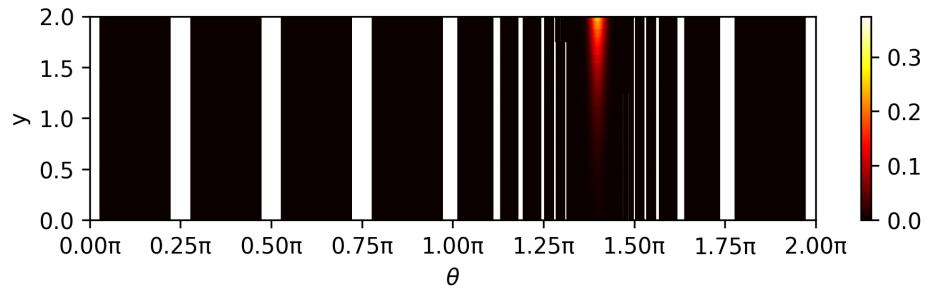


(b)

Figure SI-6: (a) The final spatial-angular hp -mesh for the adaptive angular h -refinement strategy (h -Adap. Ang.) for the *Manufactured Solution* test case. (b) A small section of the final spatial-angular hp -mesh highlighting the large number of h -refinements in the direction of the radiation field propagation. The spatial hp -elements (rectangles) are colored according to the number of degrees of freedom in the x -dimension (equivalently the y -dimension). The angular hp -elements (wedges) are colored according to the number of degrees of freedom in the θ -dimension.

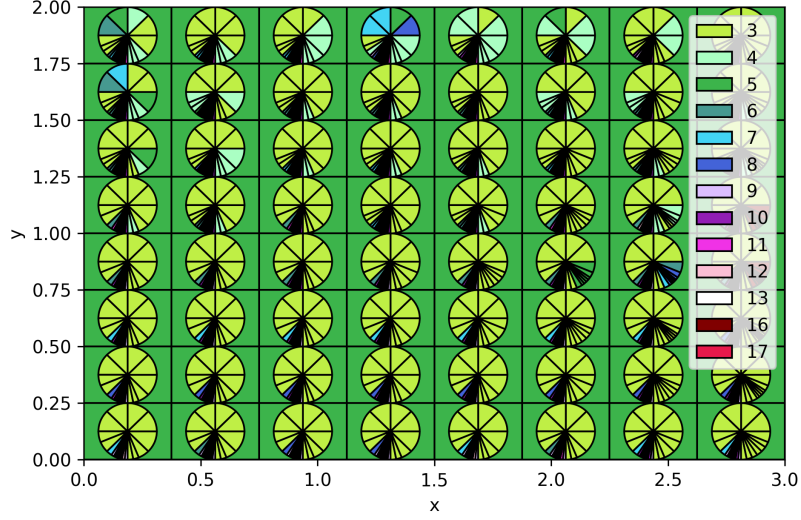


(a)

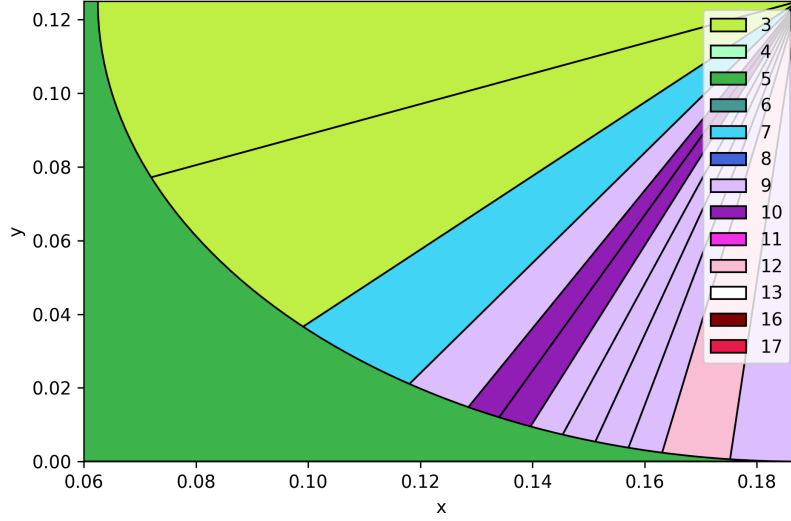


(b)

Figure SI-7: Visualizations of the final numerical solution for the adaptive angular h -refinement strategy (h -Adap. Ang.) for the *Manufactured Solution* test case. The angular (a) and horizontal (b) integral of $u_{hp}(x, y, \theta)$.

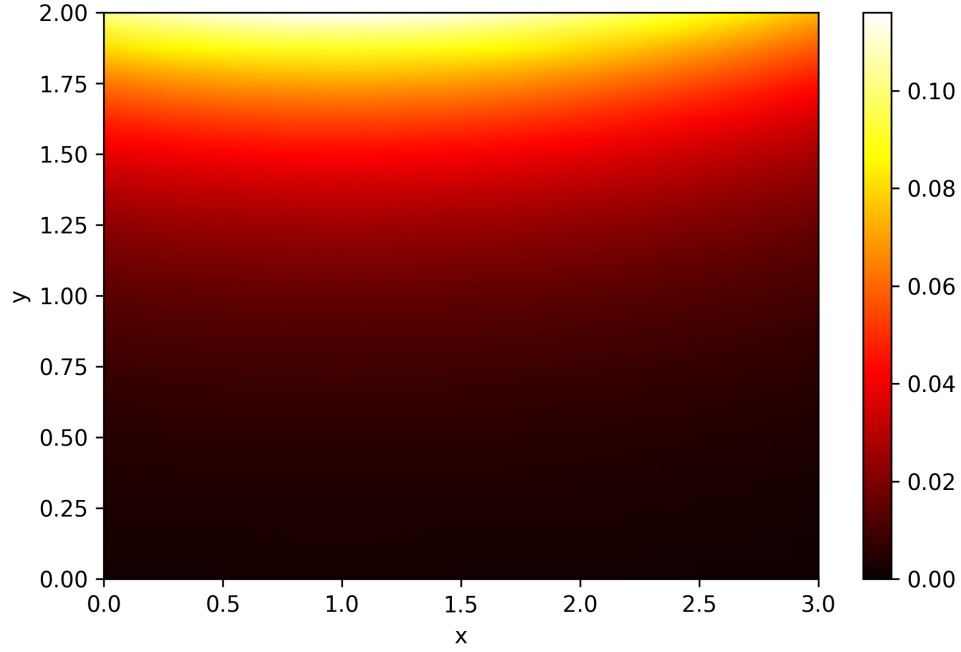


(a)

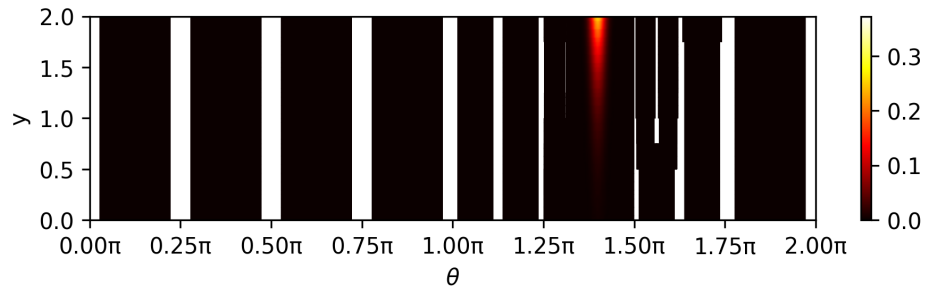


(b)

Figure SI-8: (a) The final spatial-angular hp -mesh for the adaptive angular hp -refinement strategy (hp -Adap. Ang.) for the *Manufactured Solution* test case. (b) A small section of the final spatial-angular hp -mesh highlighting the large number of h - and p -refinements in the direction of the radiation field propagation. The spatial hp -elements (rectangles) are colored according to the number of degrees of freedom in the x -dimension (equivalently the y -dimension). The angular hp -elements (wedges) are colored according to the number of degrees of freedom in the θ -dimension.



(a)



(b)

Figure SI-9: Visualizations of the final numerical solution for the adaptive angular hp -refinement strategy (hp -Adap. Ang.) for the *Manufactured Solution* test case. The angular (a) and horizontal (b) integral of $u_{hp}(x, y, \theta)$.

comparing it to an ‘analytic’ error indicator and a ‘high-resolution’ error indicator for the *Manufactured Solution* test case.

Specifically, the analytic error indicator for spatial-angular hp -element (C, K) is given by

$$\eta_{ant}^{(C, K)} := \sqrt{\frac{\int_{\mathcal{S}^K} \int_{\Omega^C} (u - u_{hp}) \, d\vec{x} \, d\vec{s}}{\int_{\mathcal{S}} \int_{\Omega} u^2 \, d\vec{x} \, d\vec{s}}}, \quad (8)$$

where u is the analytic solution and u_{hp} is the numeric solution.

The high-resolution error indicator for spatial-angular hp -element (C, K) follows the same form as Eqn. 8, replacing u by a ‘high-resolution’ numeric solution u_{hr} . This ‘high-resolution’ numeric solution is obtained from the numeric solution u_{hp} by refining the mesh on which u_{hp} was obtained. This ensures that u_{hr} is more refined than u_{hp} at each point in $\Omega \times \mathcal{S}$.

In this section, the high-resolution solution u_{hr} is obtained by p -refining the mesh of u_{hp} thrice in angle.

Figure SI-10 shows the convergence rate of the *Manufactured Solution* test case using each of the analytic (Figure SI-10a), high-resolution (Figure SI-10b), and angular-jump (Figure SI-10c) error indicators to determine adaptive refinement.

Further, Figure SI-11 shows the value of each error indicator on each spatial-angular hp -element on an example mesh. The different values on the color scale of these plots is largely unimportant. It is not the magnitude of the error indicator on each spatial-angular hp -element that is important, but rather the *distribution* of the error that determines with spatial-angular hp -elements are refined.

These three error indicators performed similarly for both adaptive angular h - and hp -refinement strategies - both in overall performance and in agreement of which spatial-angular hp -elements to refine. This suggests that the angular-jump error indicator is especially effective, as it is less expensive to calculate than the high-resolution error indicator

SI-6. Test case 2: Two strong scatterers - Numerical solutions and mesh visualizations

We provide visualizations for each refinement strategy for the *Two Strong Scatterers* test case below. These visualizations include the final numerical solution as well as sections of the final spatial-angular hp -mesh. The primary

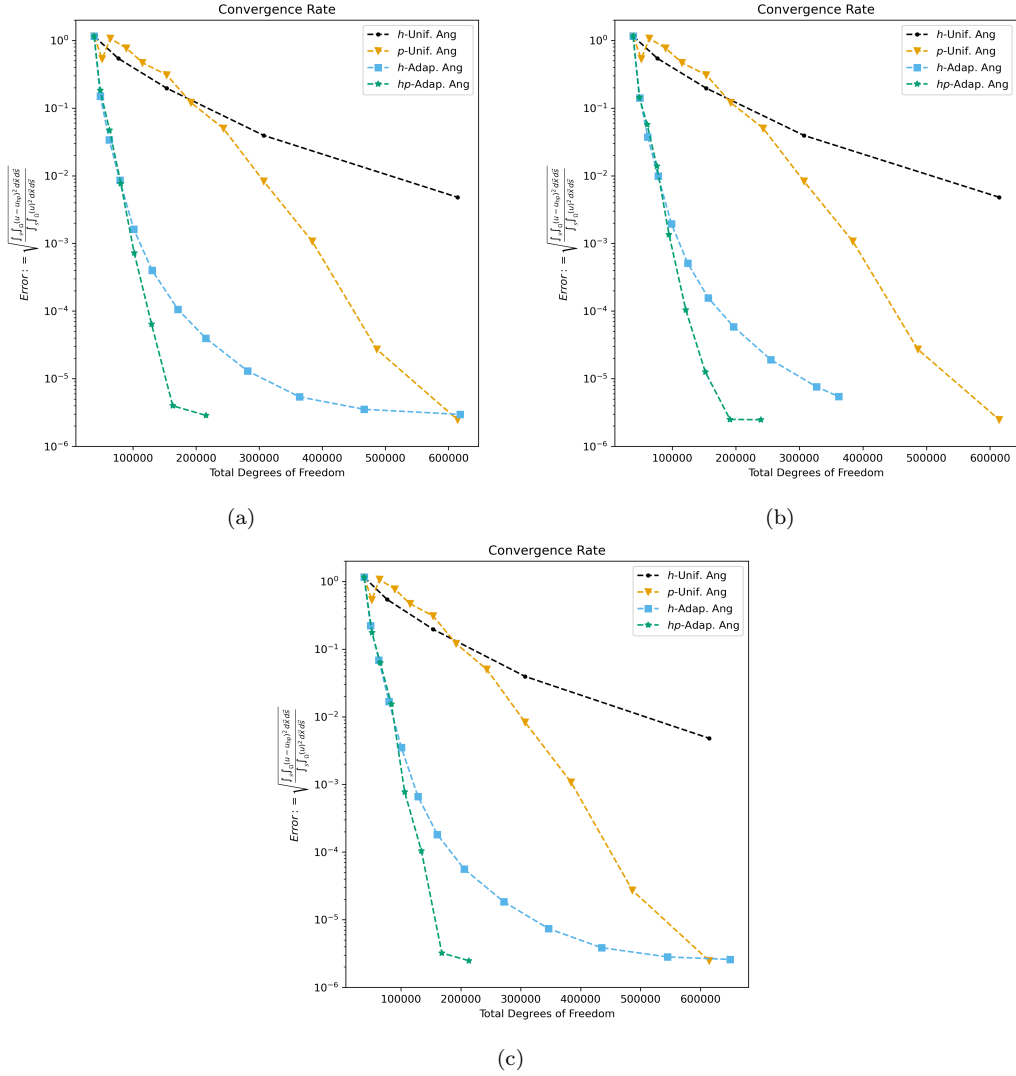


Figure SI-10: Convergence rates for each refinement strategy for the *Manufactured Solution* test case using the (a) analytic error indicator, (b) high-resolution error indicator, and (c) angular-jump error indicator to determine which spatial-angular *hp*-elements to angularly refine. Due to the computational expense of calculating the high-resolution error indicator, we were unable to use it for refining larger mesh sizes in (b).

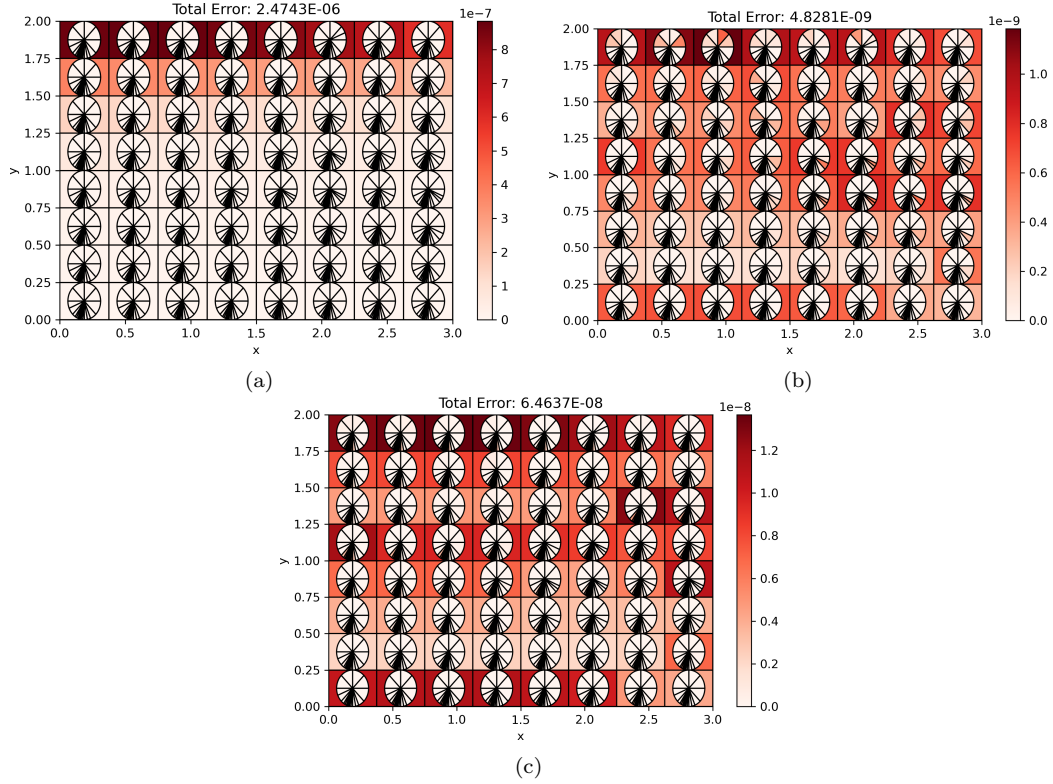


Figure SI-11: Error distribution on an example iteration for adaptive hp -refinement for the *Manufactured Solution* test case using the (a) analytic error indicator, (b) high-resolution error indicator, and (c) angular-jump error indicator to determine which spatial-angular hp -elements to angularly refine. Note that while the color scales differ, it is only the *distribution* of the error indicator that is important, as that determines which spatial-angular hp -elements are refined.

discussion of the results of the *Two Strong Scatterers* test case is given in Subsection 4.2.

SI-7. One weak scatterer test case

The *One Weak Scatterer* test case was designed to feature sharp gradients in angle with smooth gradients in space that were easy to resolve. We compared the following four angular refinement strategies:

1. uniform h -refinement in angle (h -Unif. Ang.);
2. uniform p -refinement in angle (p -Unif. Ang.);
3. spatially-inhomogeneous adaptive h -refinement in angle (h -Adap. Ang.), and;
4. spatially-inhomogeneous adaptive hp -refinement in angle (hp -Adap. Ang.).

We utilized an angular refinement tolerance of $\tau_{\vec{s}} = 0.9$. The spatial h -mesh for each strategy was a 16×16 grid of spatial h -elements, which was not refined in this test case. The associated angular h -mesh for each spatial h -element consisted of 8 elements. Each spatial-angular hp -element had three nodes in both the x - and y -dimensions, and four nodes in the θ dimension

SI-7.1. Boundary conditions, extinction and scattering coefficients, and scattering phase function

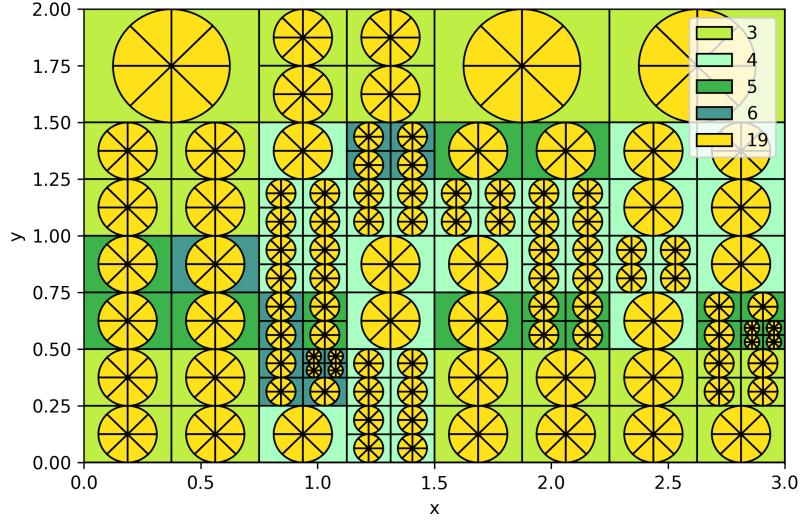
The *One Weak Scatterer* test case featured a circular optically-thin ‘object’ (Figure SI-18) and incoming radiation along the top and left domain boundaries that was sharply peaked in angle.

The extinction $\kappa(\vec{x})$ and scattering $\sigma(\vec{x})$ coefficients were given by

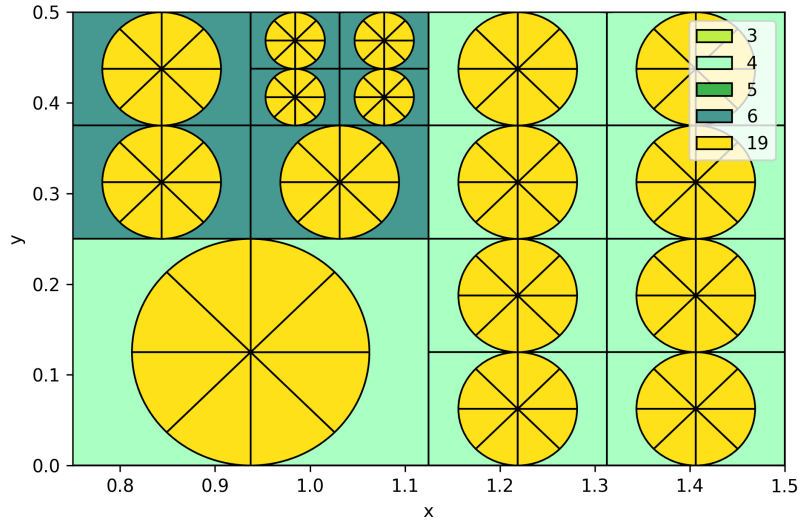
$$r(x, y) = \frac{L_y}{6} - \sqrt{\left(x - \frac{5L_x}{8}\right)^2 + \left(y - \frac{3L_y}{8}\right)^2}, \quad (9a)$$

$$\kappa(x, y) = \frac{5.5}{1 + e^{-15r(x, y)}}, \quad \sigma(x, y) = 0.7 \kappa(x, y). \quad (9b)$$

The length of the domain in the x - and y -dimensions were $L_x = 3$ and $L_y = 2$. The scattering phase function Φ was the Henyey–Greenstein scattering phase function (Eqn. 24) with $g = 0.7$. This value of g corresponds to anisotropic forward scattering.

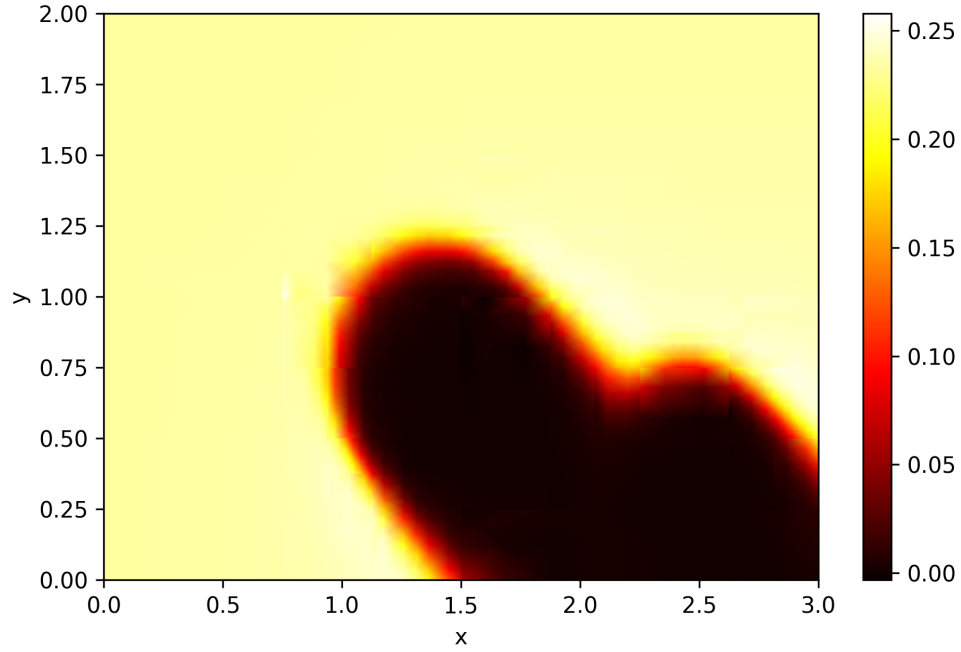


(a)

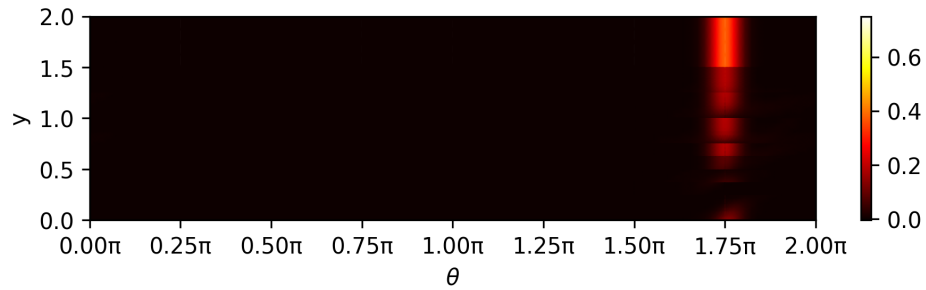


(b)

Figure SI-12: (a) The final spatial-angular hp -mesh for the adaptive spatial hp -refinement with angular uniform p -refinement strategy (hp -Adap. Spt., p -Unif. Ang.) for the *Two Strong Scatterers* test case. (b) A small section of the final spatial-angular hp -mesh highlighting the large number of p -refinements in angle to resolve the sharp angular distribution of the radiation field; and h - and p -refinements in space to resolve the steep spatial gradients of the radiation. The spatial hp -elements (rectangles) are colored according to the number of degrees of freedom in the x -dimension (equivalently the y -dimension). The angular hp -elements (wedges) are colored according to the number of degrees of freedom in the θ -dimension.

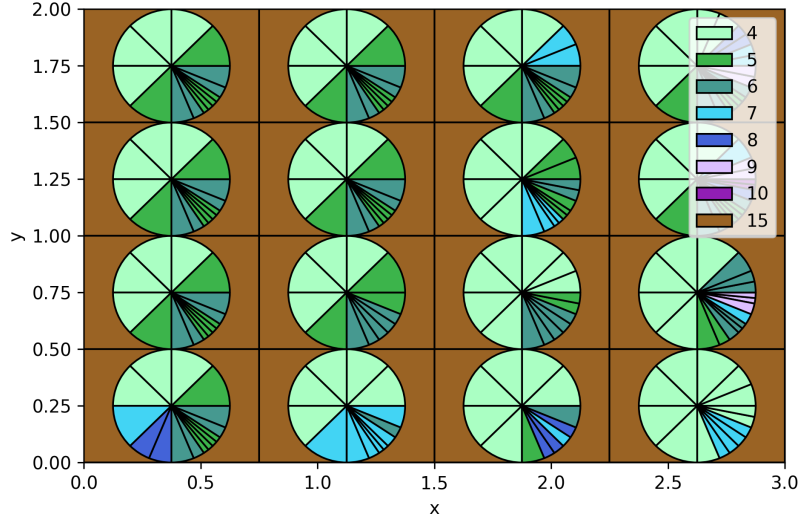


(a)

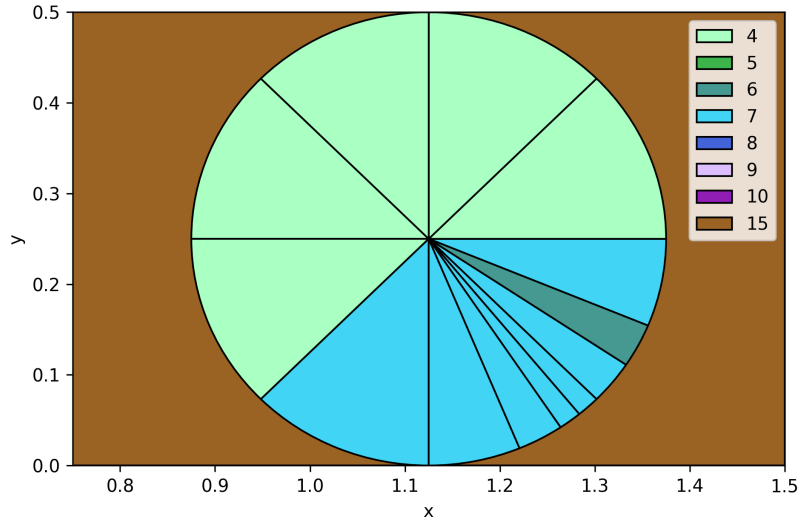


(b)

Figure SI-13: Visualizations of the final numerical solution for the adaptive spatial hp -refinement with angular uniform p -refinement strategy (hp -Adap. Spt., p -Unif. Ang.) for the *Two Strong Scatterers* test case. The angular (a) and horizontal (b) integral of $u_{hp}(x, y, \theta)$.

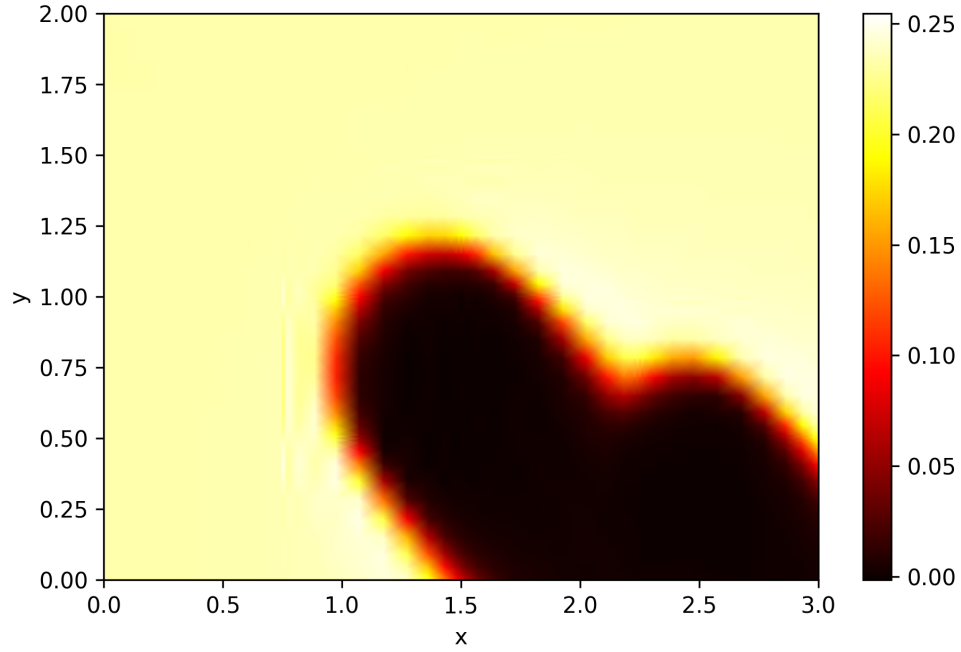


(a)

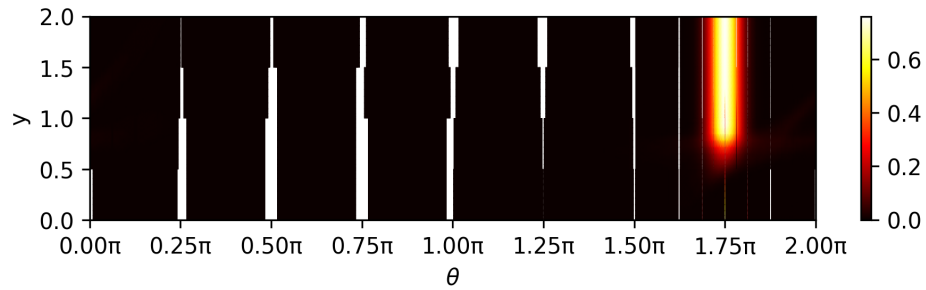


(b)

Figure SI-14: (a) The final spatial-angular hp -mesh for the adaptive angular hp -refinement with spatial uniform p -refinement strategy (hp -Adap. Ang., p -Unif. Spt.) for the *Two Strong Scatterers* test case. (b) A small section of the final spatial-angular hp -mesh highlighting the large number of p -refinements in space to resolve the steep spatial gradients of the radiation; and h - and p -refinements in angle to resolve the sharp angular distribution of the radiation field. The spatial hp -elements (rectangles) are colored according to the number of degrees of freedom in the x -dimension (equivalently the y -dimension). The angular hp -elements (wedges) are colored according to the number of degrees of freedom in the θ -dimension.

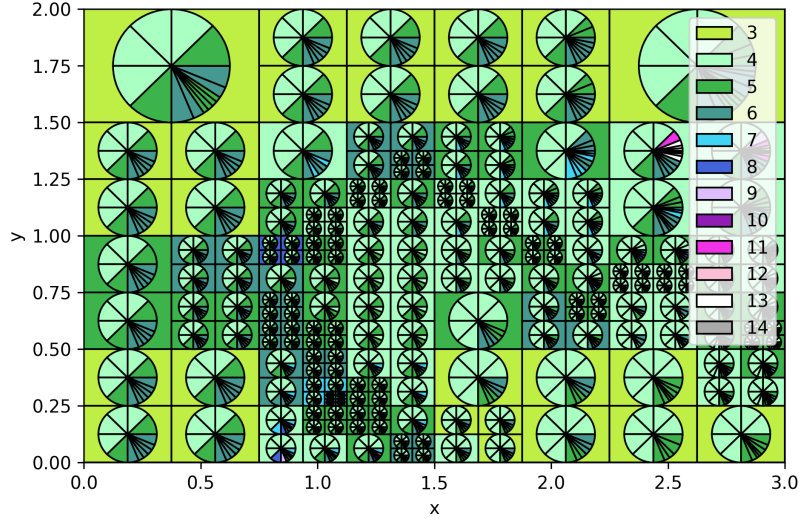


(a)

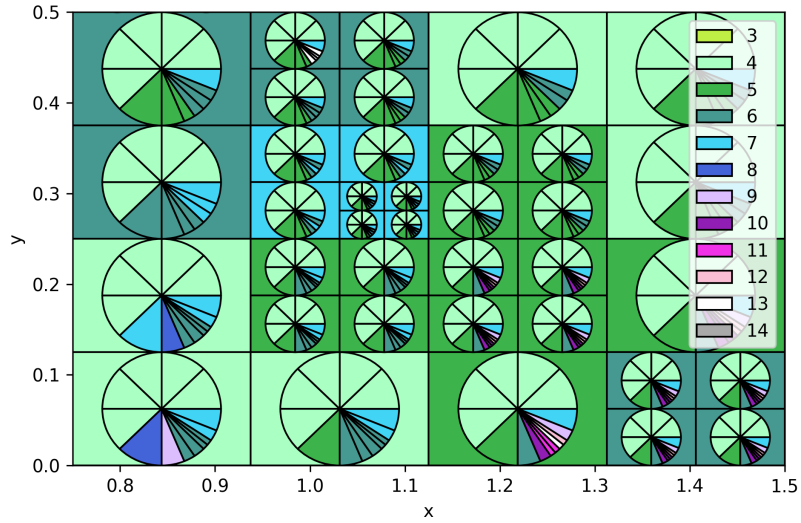


(b)

Figure SI-15: Visualizations of the final numerical solution for the adaptive angular hp -refinement with spatial uniform p -refinement strategy (hp -Adap. Ang., p -Unif. Spt.) for the *Two Strong Scatterers* test case. The angular (a) and horizontal (b) integral of $u_{hp}(x, y, \theta)$.

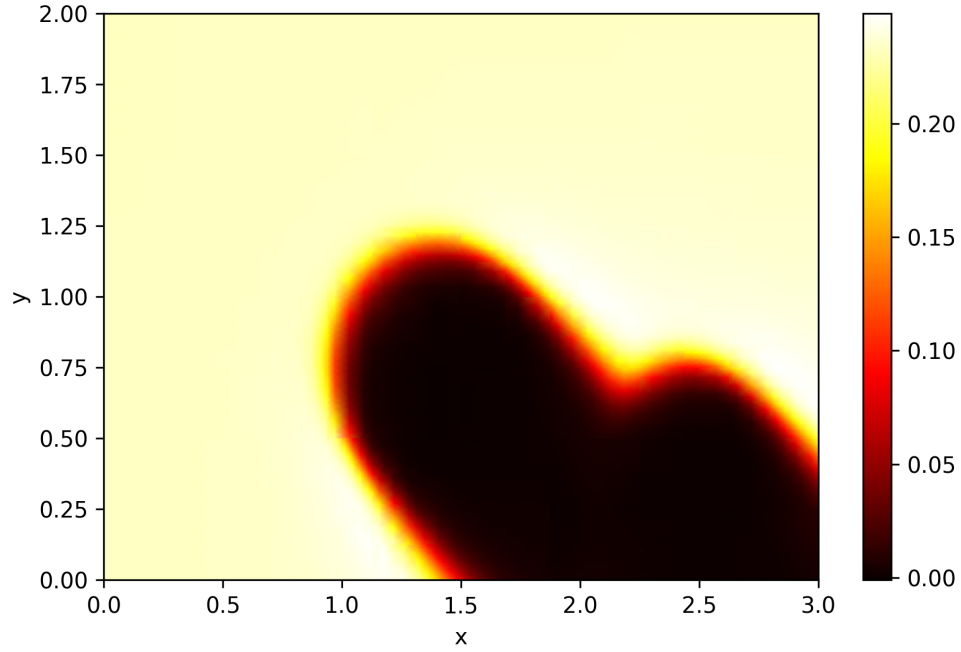


(a)

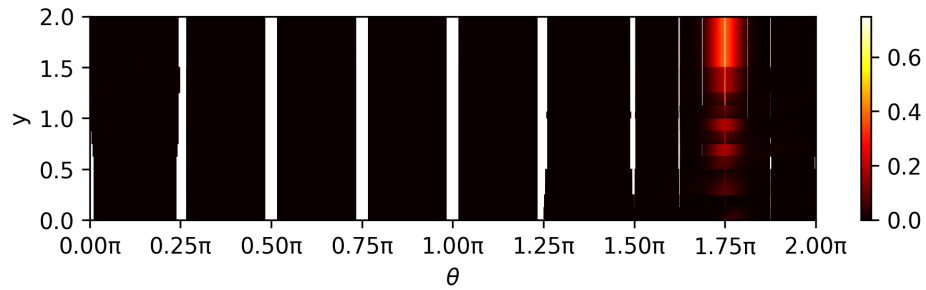


(b)

Figure SI-16: [Reproduction of Figure 9] (a) The final spatial-angular hp -mesh for the adaptive spatial hp -refinement with angular adaptive hp -refinement strategy (hp -Adap. Spt., hp -Adap. Ang.) for the *Two Strong Scatterers* test case. (b) A small section of the final spatial-angular hp -mesh highlighting the large number of h - and p -refinements in angle to resolve the sharp angular distribution and steep spatial gradients of the radiation field. The spatial hp -elements (rectangles) are colored according to the number of degrees of freedom in the x -dimension (equivalently the y -dimension). The angular hp -elements (wedges) are colored according to the number of degrees of freedom in the θ -dimension.



(a)



(b)

Figure SI-17: Visualizations of the final numerical solution for the adaptive spatial hp -refinement with angular adaptive p -refinement strategy (hp -Adap. $Spt.$, p -Adap. $Ang.$) for the *Two Strong Scatterers* test case. The angular (a) and horizontal (b) integral of $u_{hp}(x, y, \theta)$.

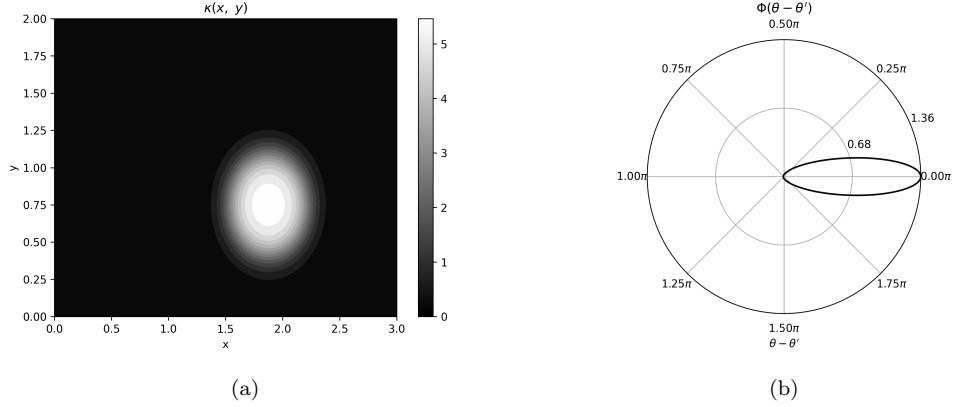


Figure SI-18: The extinction coefficient (a) and scattering phase function (b) for Test Case 2: *One Weak Scatterer*. Note that the scattering phase function $\Phi(\theta, \theta')$ is plotted as a function of $\theta - \theta'$.

The incoming radiation was given by

$$u^{in,bd}(x, y, \theta) = \begin{cases} e^{-\left(\frac{192}{2\pi} \left(\theta - \frac{7\pi}{4}\right)\right)^2}, & y = y_{top} \text{ or } x = x_{left}, \\ 0, & \text{otherwise,} \end{cases} \quad (10)$$

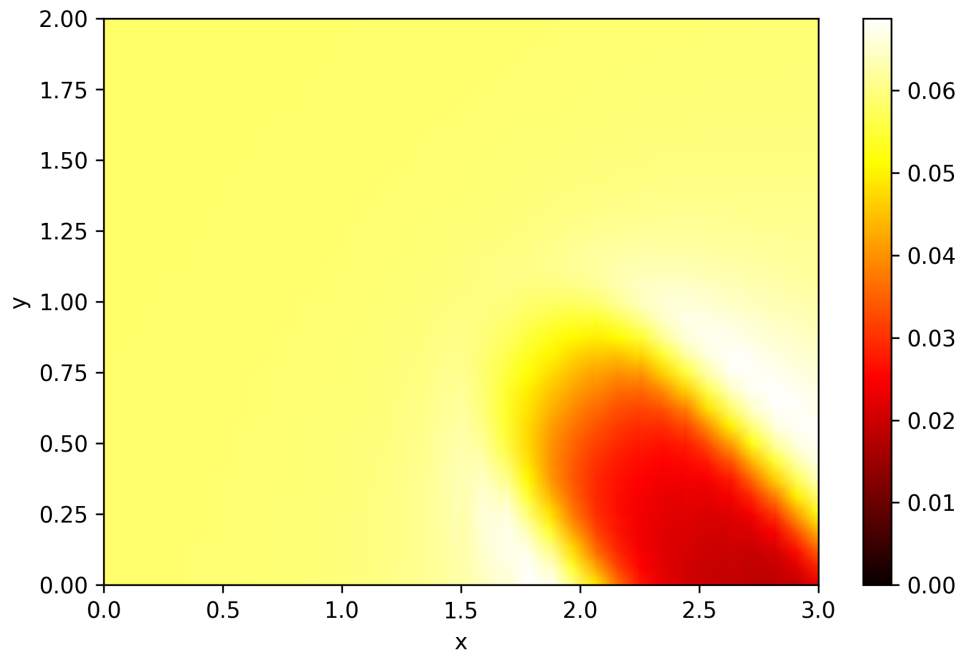
where $y_{top} = L_y = 2$ was the value of y at the top of the domain and $x_{left} = 0$ was the value of x at the left of the domain.

Figure SI-19 shows two visualizations of a high-resolution numerical solution. Note the sharp angular distribution of the radiation field at all vertical levels of the domain (and indeed throughout the entire domain). Such a narrow function is difficult to resolve spectrally.

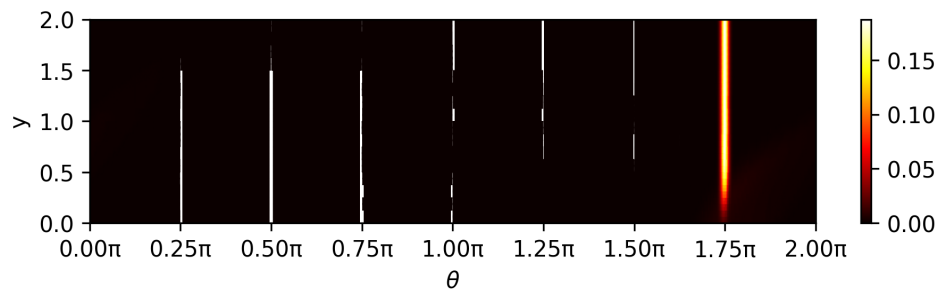
SI-7.2. Angular refinement behavior and convergence results

The final numerical solutions for the *h-Unif. Ang.* and *p-Unif. Ang.* refinement strategies are shown in Figures SI-20 and SI-21. As expected, each uniform refinement strategy successfully resolved the narrow angular distribution of the radiation field.

The *h-Adap. Ang.* refinement strategy successfully identified and resolved the narrow angular distribution of the radiation field about $\frac{7\pi}{4}$. This was apparent by the distribution of smaller angular *hp*-elements about this angle. Further, we observed that there were fewer such elements in the shadow of the ‘object’. The final numerical solution for this refinement strategy is shown in Figure SI-23, and the final spatial-angular *hp*-mesh is shown in Figure SI-22.

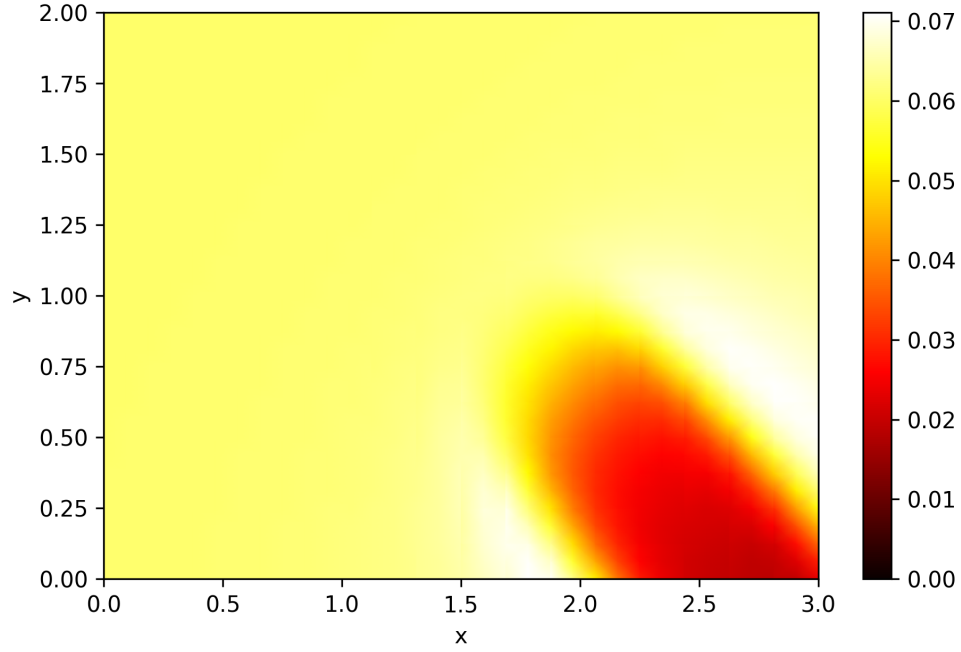


(a)

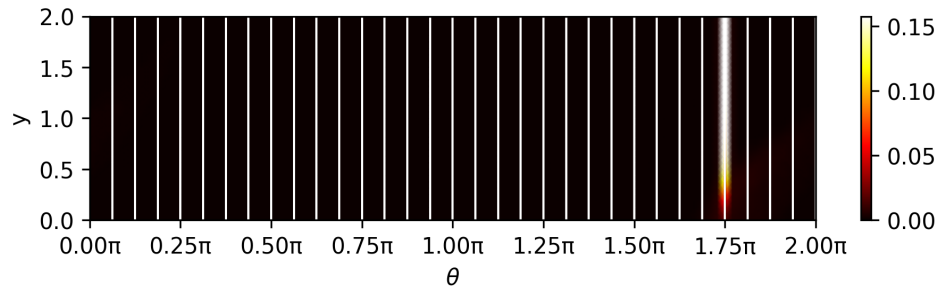


(b)

Figure SI-19: Visualizations of a high-resolution numerical solution for Test Case 2: *One Weak Scatterer*; the angular (a) and horizontal (b) integral of $u_{hr}(x, y, \theta)$.

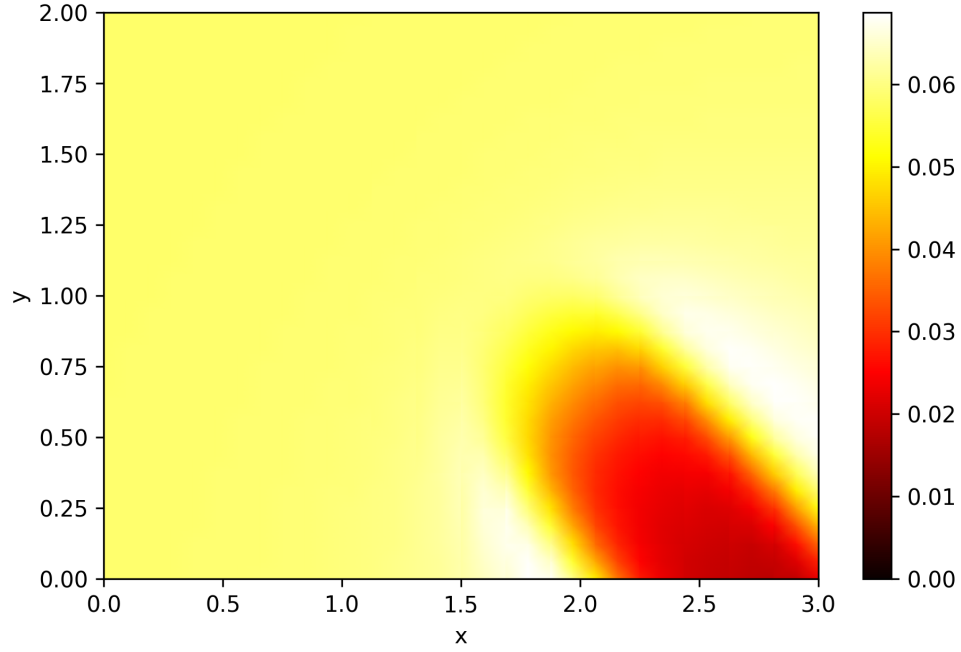


(a)

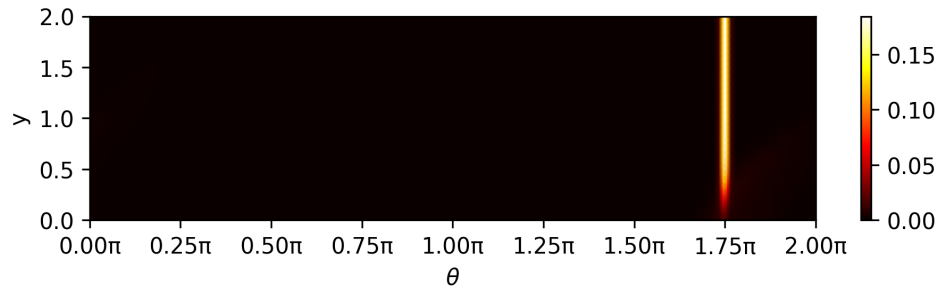


(b)

Figure SI-20: Visualizations of the final numerical solution for the uniform angular h -refinement strategy (h -Unif. Ang.) for the *One Weak Scatterer* test case. The angular (a) and horizontal (b) integral of $u_{hp}(x, y, \theta)$.

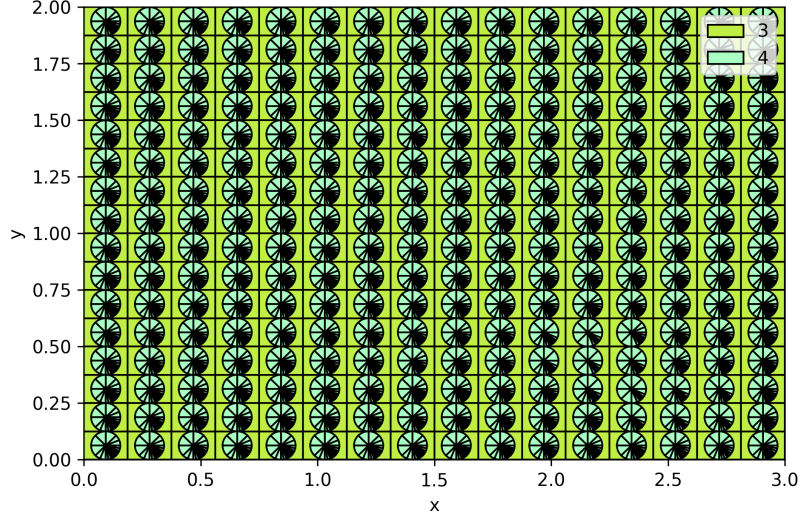


(a)

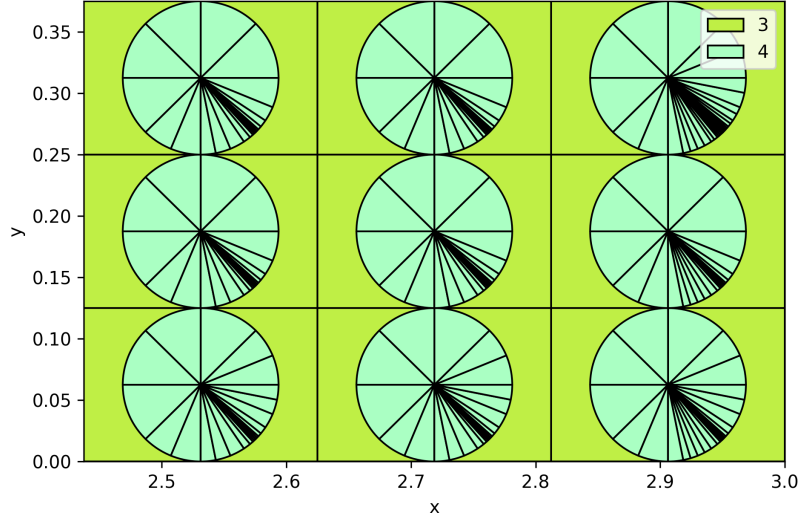


(b)

Figure SI-21: Visualizations of the final numerical solution for the uniform angular p -refinement strategy (p -Unif. Ang.) for the *One Weak Scatterer* test case. The angular (a) and horizontal (b) integral of $u_{hp}(x, y, \theta)$.

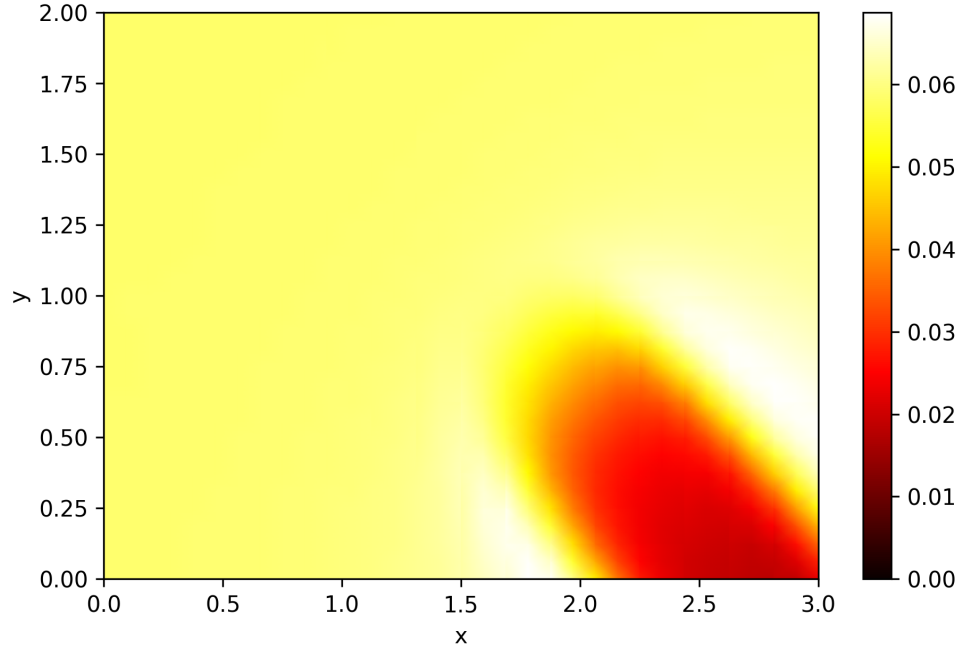


(a)

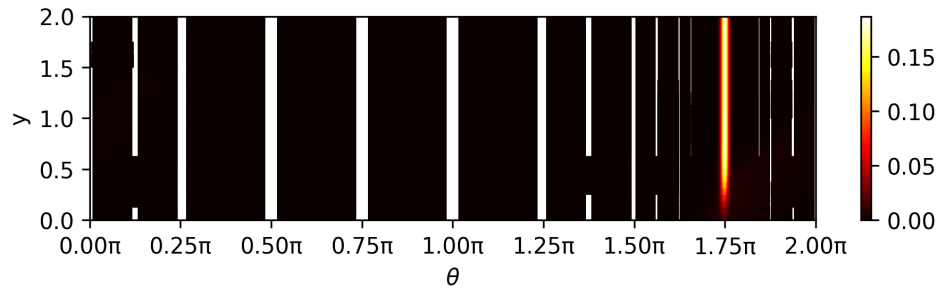


(b)

Figure SI-22: (a) The final spatial-angular hp -mesh for the adaptive angular h -refinement strategy (h -Adap. Ang.) for the *One Weak Scatterer* test case. (b) A small section of the final spatial-angular hp -mesh highlighting the large number of h -refinements in the direction of the radiation field propagation. The spatial hp -elements (rectangles) are colored according to the number of degrees of freedom in the x -dimension (equivalently the y -dimension). The angular hp -elements (wedges) are colored according to the number of degrees of freedom in the θ -dimension.



(a)



(b)

Figure SI-23: Visualizations of the final numerical solution for the adaptive angular h -refinement strategy (h -Adap. Ang.) for the *One Weak Scatterer* test case. The angular (a) and horizontal (b) integral of $u_{hp}(x, y, \theta)$.

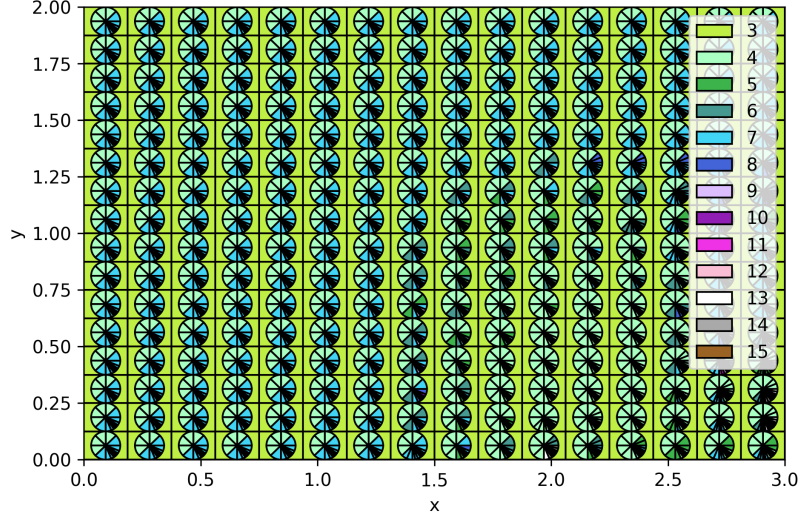
The *hp-Adap. Ang.* refinement strategy also successfully identified and resolved the narrow angular distribution of the radiation field about $\frac{7\pi}{4}$. This was evident by the large number of small angular *hp*-elements with many degrees of freedom in angle about this angle. Similar to the other adaptive refinement strategy for this test case, observed that angular *hp*-elements in the shadow of the scatterer were refined less frequently than those outside of the shadow. More of those refinements in the shadow of the scatterer were *p*-refinements. This reflected the fact that the angular distribution of the radiation there would be wider with a smaller gradient due to the scattering of the radiation. The final numerical solution for this strategy is shown in Figure SI-25, and the final spatial-angular *hp*-mesh is shown in Figure SI-24.

The convergence results for each of these strategies is shown in Figure SI-26. Error was measured against a high-resolution solution u_{hr} . The high-resolution solution u_{hr} was calculated by *p*-refining the low-resolution *hp*-mesh thrice in angle. We compared this ‘high-resolution error’ for each strategy as a function of total degrees of freedom in the angular-spatial *hp*-mesh and as a function of the number of non-zero entries in the matrix that arose from the discretization of Eqn. 10.

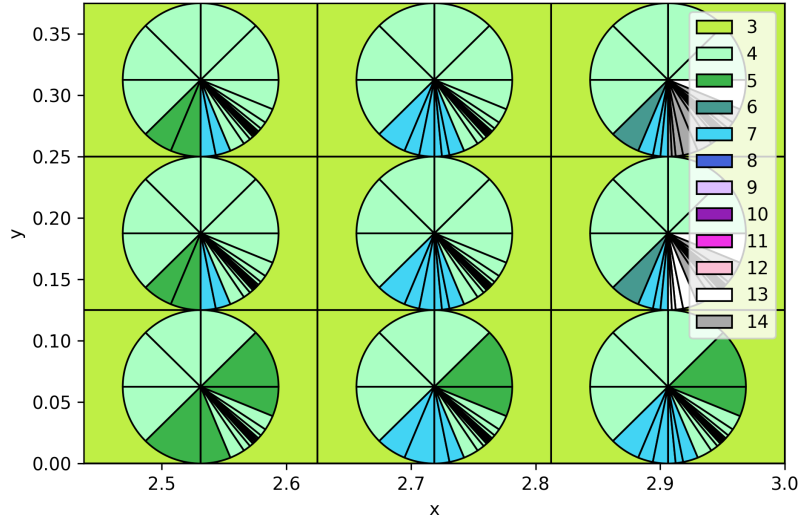
We observed that the *h-Unif. Ang.* refinement strategy performed the worst of these three strategies, while the *hp-Adap. Ang* strategy had the best performance for mid- to large-mesh sizes. The two uniform refinement strategy used many degrees of freedom in angles where there was no radiation present, contributing to their computational inefficiency. Both adaptive strategies focused the available degrees of freedom on the angle that the radiation propagated in. The *hp-Adap. Ang* strategy further capitalized on the spectral convergence of *p*-refinements to more efficiently resolve the angular distribution of radiation. The *h-Adap. Ang* strategy was ten times more accurate than *p-Unif. Ang.* for a mesh with approximately 250 thousand degrees of freedom. Further, *hp-Adap. Ang* was ten times more accurate than *h-Adap. Ang* for a mesh at this size.

SI-8. Further details on the discontinuous Galerkin spectral element method: basis functions, and pull-back and push-forward maps

When discussing the discontinuous Galerkin spectral element method (DGSEM), it is useful to introduce a one-dimensional reference element $\bar{E}$ with domain $[-1, 1]$, with associated quadrature nodes, weights, and spectral

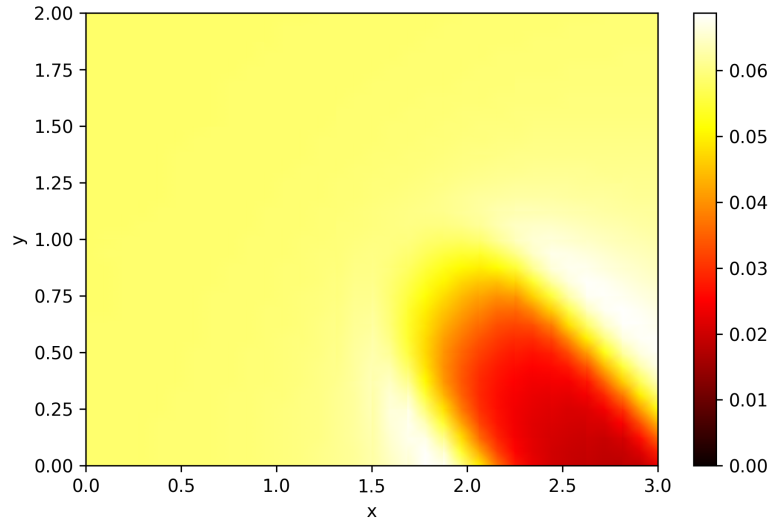


(a)

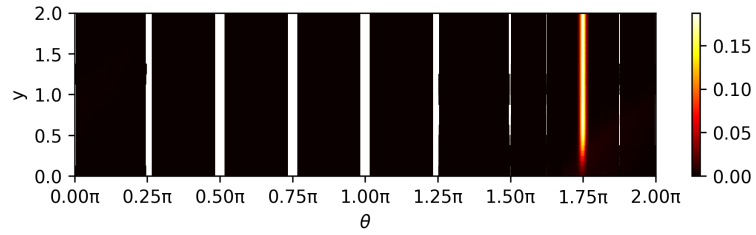


(b)

Figure SI-24: (a) The final spatial-angular hp -mesh for the adaptive angular hp -refinement strategy (hp -Adap. Ang.) for the *One Weak Scatterer* test case. (b) A small section of the final spatial-angular hp -mesh highlighting the large number of h - and p -refinements in the direction of the radiation field propagation. The spatial hp -elements (rectangles) are colored according to the number of degrees of freedom in the x -dimension (equivalently the y -dimension). The angular hp -elements (wedges) are colored according to the number of degrees of freedom in the θ -dimension.



(a)



(b)

Figure SI-25: Visualizations of the final numerical solution for the adaptive angular hp -refinement strategy (hp -Adap. Ang.) for the *One Weak Scatterer* test case. The angular (a) and horizontal (b) integral of $u_{hp}(x, y, \theta)$.

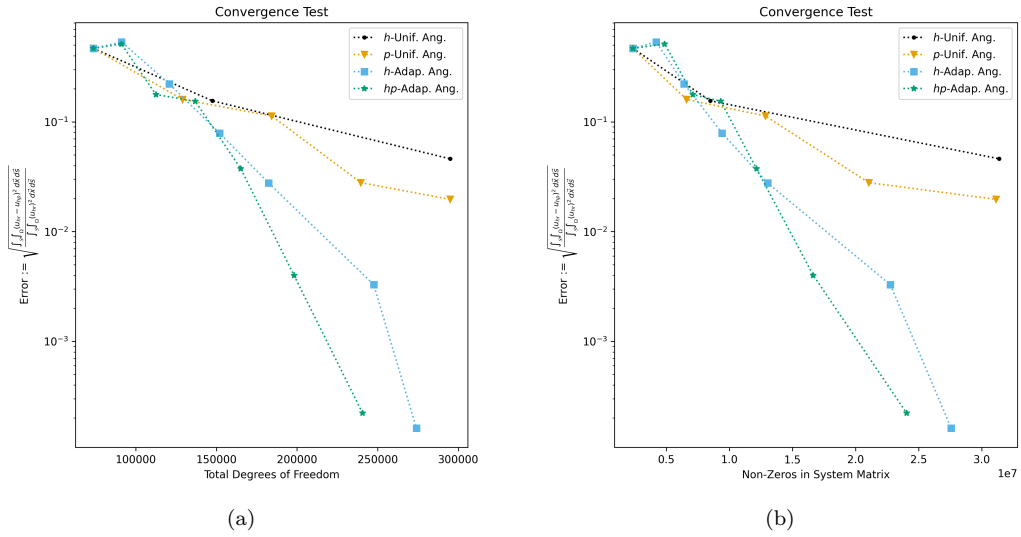


Figure SI-26: The convergence test of the uniform angular h -refinement (h -Unif. Ang.), uniform angular p -refinement (p -Unif. Ang.), adaptive angular h -refinement (h -Adap. Ang.), and adaptive angular hp -refinement (hp -Adap. Ang.) refinement strategies for Test Case 2: *One Weak Scatterer*. Note that since each strategy utilizes the same initial mesh, the left-most point is shared amongst all four data sets. The hp -Adap. Ang strategy far outperformed the other three. For larger mesh sizes, hp -Adap. Ang was an order of magnitude more accurate than h -Adap. Ang, and two orders of magnitude more accurate than the uniform refinement strategies.

basis functions given by

$$\left\{ \bar{z}_l^{\bar{E}}, w_l^{\bar{E}, \bar{z}}, \bar{\zeta}_l^{\bar{E}}(\bar{z}) \right\}_{l=0}^{n_{\bar{z}}^{\bar{E}}-1}. \quad (11)$$

The spectral basis functions $\bar{\zeta}_l^{\bar{E}}(\bar{z})$ are defined such that they take a value of zero when $\bar{z}$ is outside the domain of $\bar{E}$. Further, $\bar{\zeta}_l^{\bar{E}}(\bar{z}_m^{\bar{E}}) = 1$ if $l = m$ and is otherwise zero. The superscript indicates the range of valid values of the indices in the subscript. In this case $l, m = 0, \dots, n_{\bar{z}}^{\bar{E}} - 1$.

We may map the domain of the reference element $\bar{E}$ to any one-dimensional element E via the *push-forward map* $\mathcal{T}^{E,z} : [-1, 1] \rightarrow [z_0^E, z_f^E]$ defined as

$$z = \mathcal{T}^{E,z}(\bar{z}) = \frac{\Delta z^E}{2} (\bar{z} + 1) - z_0^E. \quad (12)$$

where $\{z_m^E\}_{m=0}^{n_z^E-1}$ are the quadrature nodes of E . We use the subscript f to denote the final valid index, in this case $z_f^E = z_{n_z^E-1}^E$, and $\Delta z^E = z_f^E - z_0^E$.

The inverse of the push-forward map, called the *pull-back map* $(\mathcal{T}^{E,z})^{-1} : [z_0^E, z_f^E] \rightarrow [-1, 1]$, maps the domain of E back to $\bar{E}$

$$\bar{z} = (\mathcal{T}^{E,z})^{-1}(z) = \frac{2}{\Delta z^E} (z - z_0^E) - 1. \quad (13)$$

The spectral basis functions of E may thus be written as the composition

$$\zeta_m^E(z) = \bar{\zeta}_m^{\bar{E}} \circ (\mathcal{T}^{E,z})^{-1}(z). \quad (14)$$

For simplicity, we write E in place of $\bar{E}$ in such contexts, to denote that m indexes the quadrature nodes and weights, and spectral basis functions, of element E .

Hence, for a function $f(z)$ with sufficient regularity defined on $[z_0^E, z_f^E]$, we may utilize the quadrature rule for the reference element $\bar{E}$ to calculate

¹In cases where the quadrature nodes do not include the endpoints of the domain of E , then δz^E is the width of the domain of E , not the difference between the zeroth and final quadrature nodes of E .

the following integrals:

$$\begin{aligned}
\int_{z_0^E}^{z_f^E} f(z) dz &= \frac{\Delta z^E}{2} \int_{-1}^1 f(\mathcal{T}^{E,z}(\bar{z})) d\bar{z} \\
&\approx \frac{\Delta z^E}{2} \sum_{i=0}^{n_z^E-1} w_m^{E,z} f(\mathcal{T}^{E,z}(\bar{z}_i^E)) \\
&\approx \frac{\Delta z^E}{2} \sum_{i=0}^{n_z^E-1} w_m^{E,z} f(z_m^E),
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
\int_{z_0^E}^{z_f^E} f(z) \zeta_m^E(z) dz &= \frac{\Delta z^E}{2} \int_{-1}^1 f(\mathcal{T}^{E,z}(\bar{z})) \bar{\zeta}_n^E(\bar{z}) d\bar{z} \\
&\approx \frac{\Delta z^E}{2} \sum_{m=0}^{n_z^E-1} w_m^{E,z} f(\mathcal{T}^{E,z}(\bar{z}_m^E)) \bar{\zeta}_n^E(\bar{z}_m^E) \\
&\approx \frac{\Delta z^E}{2} w_n^{E,z} f(z_n^E).
\end{aligned} \tag{16}$$

The last line arises from the usage of the Lagrange polynomials interpolating the quadrature nodes, i.e., $\bar{\zeta}_n^E(\bar{z}_m^E) = \delta_{mn}^{EE}$.

We may write the L_2 -projection $f_h^E(z)$ of $f(z)$ onto the spectral basis for angular-spatial element E as

$$f_h^E(z) = \sum_{i=0}^{n_z^E-1} f(z_m^E) \zeta_m^E(z). \tag{17}$$

In future calculations, it will be desirable to integrate the product of $f_h^E(z)$ and spectral basis functions for another element E' whose domain intersects that of E ,

$$\int_{z_0^E}^{z_f^E} f_h(z) \zeta_n^{E'}(z) dz = \int_{z_0^{E'}}^{z_f^{E'}} f_h(z) \zeta_n^{E'}(z) dz. \tag{18}$$

The equality of the left- and right-hand sides of this equation comes from the fact that the spectral basis functions of an element are only non-zero on

the domain of the element. To avoid discontinuities in basis functions at the boundary of the interval on which they are supported, we utilize the side of this equation that corresponds to the element with the smaller domain; i.e., if $[z_0^E, z_f^E] \subseteq [z_0^{E'}, z_f^{E'}]$, then we evaluate the left-hand side using numerical quadrature, and otherwise we evaluate the right-hand side with numerical quadrature.

When determining how many points to use for the numerical quadrature to evaluate in integral in Eqn. 18, we note that the Legendre–Gauss–Lobatto and Legendre–Gauss quadrature rules with N nodes and weights are exact for polynomials with degree up to $2N - 1$ [2]. Hence, the exact evaluation of Eqn. 18 requires at least $\frac{1}{2} (n_z^E + n_z^{E'} + 1)$ quadrature points and weights. For simplicity and sparsity of the resultant matrix, however, we instead utilize $N = \max \{n_z^E, n_z^{E'}\}$ quadrature points and weights.

Thus, the evaluation of the integral in Eqn. 18 is given by the following: let $A \in \{E, E'\}$ with the larger number of quadrature points and weights, and $B \in \{E, E'\}$ be the element with the smaller domain. If the domains of E and E' are equal, we choose $A = B$. We obtain

$$\begin{aligned} \int_{z_0^E}^{z_f^E} f_h(z) \zeta_n^{E'}(z) dz &= \int_{z_0^B}^{z_f^B} f_h(z) \zeta_n^{E'}(z) dz \\ &= \frac{\Delta z^B}{2} \sum_{i=0}^{n_z^E-1} \left[\int_{-1}^1 f(z_m^E) \zeta_m^E(\mathcal{T}^{B,z}(\bar{z})) \zeta_n^{E'}(\mathcal{T}^{B,z}(\bar{z})) d\bar{z} \right] \\ &\approx \frac{\Delta z^B}{2} \sum_{i=0}^{n_z^E-1} \left[\sum_{i'=0}^{n_z^A-1} w_{i'}^{A,z} f(z_m^E) \zeta_m^E(\mathcal{T}^{B,z}(\bar{z}_{i'}^A)) \zeta_n^{E'}(\mathcal{T}^{B,z}(\bar{z}_{i'}^A)) \right]. \quad (19) \end{aligned}$$

When A and B are specified, the expression in the final line of the above equation may reduce significantly. For example, let E' have a larger number of quadrature points and weights and a smaller domain than E . The

expression reduces to

$$\begin{aligned}
& \frac{\Delta z^{E'}}{2} \sum_{i=0}^{n_z^E-1} \left[\sum_{i'=0}^{n_z^{E'}-1} w_{i'}^{E',z} f(z_m^E) \zeta_m^E(\mathcal{T}^{E',z}(\bar{z}_{i'}^{E'})) \zeta_n^{E'}(\mathcal{T}^{E',z}(\bar{z}_{i'}^{E'})) \right] \\
&= \frac{\Delta z^{E'}}{2} \sum_{i=0}^{n_z^E-1} \left[\sum_{i'=0}^{n_z^{E'}-1} w_{i'}^{E',z} f(z_m^E) \zeta_m^E(\mathcal{T}^{E',z}(\bar{z}_{i'}^{E'})) \bar{\zeta}_n^{E'}(\bar{z}_{i'}^{E'}) \right] \quad (20) \\
&= \frac{\Delta z^{E'}}{2} \sum_{i=0}^{n_z^E-1} w_r^{E',z} f(z_m^E) \zeta_m^E(\mathcal{T}^{E',z}(\bar{z}_r^{E'})).
\end{aligned}$$

SI-9. Further details on the numerical solution of the variational form: Global matrix construction

When writing discrete forms of the terms in Eqn. 10, we will encounter expressions of the form

$$\sum_{i=0}^{n_x^C-1} \sum_{j=0}^{n_y^C-1} \sum_{a=0}^{n_\theta^K-1} u_{ija}^K g_{ijapqr}^{KK'} \quad (21)$$

where i, j, a index the quadrature points and weights in spatial element C and element $K \in \mathcal{K}^C$, and p, q, r index the quadrature points and weights in spatial element C' and element $K' \in \mathcal{K}^{C'}$, where we may have $C = C'$ and/or $K = K'$. We may write such an expression as a matrix multiplication $\mathbf{M}^{KK'} \vec{u}^K$ by mapping the triples (i, j, a) and (p, q, r) to single indices α and β , respectively, via

$$\alpha(i, j, a) = n_x^K n_y^C a + n_x^C j + i, \quad (22a)$$

$$\beta(p, q, r) = n_x^{C'} n_y^{C'} r + n_x^{C'} q + p. \quad (22b)$$

The matrix $\mathbf{M}^{KK'}$ is the *local-angular-spatial element interaction matrix*, and, using all such local-angular-spatial element interaction matrices, we may assemble them into a *local-spatial element interaction matrix*

$$\mathbf{M}^{CC'} = \begin{bmatrix} \mathbf{M}^{K_0 K'_0} & \mathbf{M}^{K_0 K'_1} & \dots \\ \mathbf{M}^{K_1 K'_0} & \mathbf{M}^{K_1 K'_1} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \quad (23)$$

where $\mathcal{K}^C = \{K_0, K_1, \dots\}$ and $\mathcal{K}^{C'} = \{K'_0, K'_1, \dots\}$. The local-spatial element interaction matrices are then assembled into the *global matrix* similarly

$$\mathbf{M}^{\mathcal{G}} = \begin{bmatrix} \mathbf{M}^{C_0 C'_0} & \mathbf{M}^{C_0 C'_1} & \dots \\ \mathbf{M}^{C_1 C'_0} & \mathbf{M}^{C_1 C'_1} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}. \quad (24)$$

SI-10. Further details on the numerical solution of the variational form: The extinction matrix

The *extinction matrix* arises from the discretization of the third term of Eqn. 11a, which corresponds to the extinction of radiation. Using the spectral representation for $u(\vec{x}, \vec{s})$ and the general basis function $\phi_p^C(x) \psi_y^C(y) \xi_r^K(\theta)$ of $\mathcal{V}_h^K$ for $v(\vec{x}, \vec{s})$, we obtain

$$\begin{aligned} & \int_{\Omega^C} \int_{\mathcal{S}^K} \kappa(\vec{x}) u(\vec{x}, \vec{s}) v(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \\ & \approx \sum_{i=0}^{n_x^C-1} \sum_{j=0}^{n_y^C-1} \sum_{a=0}^{n_\theta^K-1} u_{ija}^K \left(\int_{y_0^C}^{y_f^C} \left[\int_{x_0^C}^{x_f^C} \kappa(x, y) \phi_i^C(x) \phi_p^C(x) dx \right] \psi_j^C(y) \psi_q^C(y) dy \right) \\ & \quad \cdot \left(\int_{\theta_0^K}^{\theta_f^K} \xi_a^K(\theta) \xi_r^K(\theta) d\theta \right), \end{aligned} \quad (25)$$

where the spatial domain of spatial element C is $\Omega^C = [x_0^C, x_f^C] \times [y_0^C, y_f^C]$, the angular domain of angular-spatial element K is $\mathcal{S}^K = [\theta_0^K, \theta_f^K]$, and n_x^C , n_y^C , and n_θ^K are the number of quadrature points in x and y for spatial element C and in θ for angular-spatial element K , respectively.

Using the quadrature rule for spatial element C , we are able to discretize the spatial integrals as

$$\begin{aligned} & \int_{y_0^C}^{y_f^C} \left[\int_{x_0^C}^{x_f^C} \kappa(x, y) \phi_i^C(x) \phi_p^C(x) dx \right] \psi_j^C(y) \psi_q^C(y) dy \\ & \approx \frac{\Delta x^C \Delta y^C}{4} w_i^{C,x} w_j^{C,y} \kappa(x_i^C, y_j^C) \delta_{ip}^{CC} \delta_{jq}^{CC}, \end{aligned} \quad (26)$$

where $\Delta x^C = x_f^C - x_0^C$, $\Delta y^C = y_f^C - y_0^C$, $w_i^{C,x}$ and $w_j^{C,y}$ are the quadrature weights in the x - and y -dimensions for spatial element C , and δ is the Kronecker delta. We utilize a superscript C to denote that the range of valid

values for their associated index subscript, e.g., the valid values for the index i are $0, \dots, n_x^C - 1$.

Using the quadrature rule for angular-spatial element K , we are able to discretize the angular integral as

$$\int_{\theta_0^K}^{\theta_f^K} \xi_a^K(\theta) \xi_r^K(\theta) d\theta \approx \frac{\Delta\theta^K}{2} w_a^{K,\theta} \delta_{ar}^{KK}, \quad (27)$$

where $\Delta\theta^K = \theta_f^K - \theta_0^K$, and $w_a^{K,\theta}$ are the quadrature weights in the θ -dimension for angular-spatial element K .

Together, these yield the discretized form of the extinction term in Eqn. 10

$$\begin{aligned} & \int_{\Omega^C} \int_{\mathcal{S}^K} \kappa(\vec{x}) u(\vec{x}, \vec{s}) v(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \\ & \approx \sum_{i=0}^{n_x^C-1} \sum_{j=0}^{n_y^C-1} \sum_{a=0}^{n_\theta^K-1} u_{ija}^K \left(\frac{\Delta x^C \Delta y^C \Delta\theta^K}{8} w_i^{C,x} w_j^{C,y} w_a^{K,\theta} \kappa(x_i^C, y_j^C) \delta_{ip}^{CC} \delta_{jq}^{CC} \delta_{ar}^{KK} \right). \end{aligned} \quad (28)$$

We show a local-angular-spatial element, a local-spatial element, and the global extinction matrix in Figure SI-27, constructed using the methodology discussed in Appendix SI-9.

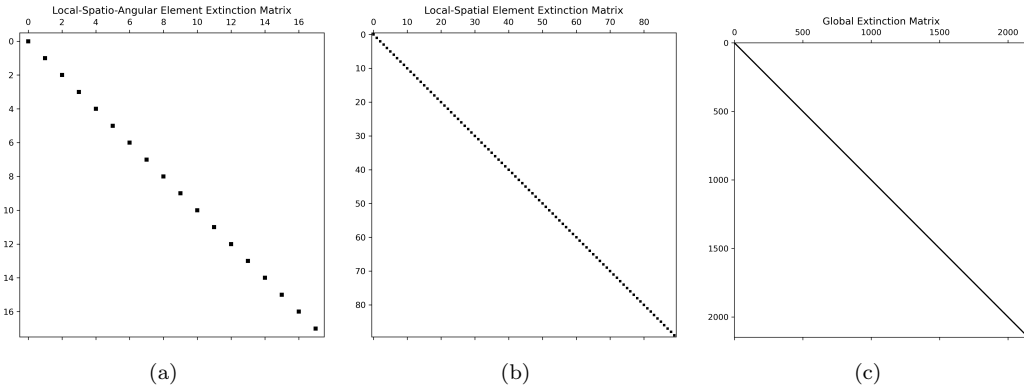


Figure SI-27: (a) A local-angular-spatial element, (b) a local-spatial element, and (c) the global extinction matrix for a simple radiative transfer problem with a mesh that has been randomly h -refined in space and angle.

SI-11. Further details on the numerical solution of the variational form: The scattering matrix

The *scattering matrix* arises from the discretization of the fourth term of Eqn. 11a, which corresponds to the scattering of radiation. Before discretizing this integral, we rewrite the scattering term as

$$\sum_{K' \in C} \int_{\Omega^C} \int_{S^K} \sigma(\vec{x}) \left(\int_{S^{K'}} \Phi(\vec{s}, \vec{s}') u(\vec{x}, \vec{s}') d\vec{s}' \right) v(\vec{x}, \vec{s}) d\vec{s} d\vec{x}. \quad (29)$$

Each term of the summation corresponds to a local-angular-spatial element scattering matrix, and we therefore discretize each term in a general form. Using the spectral representation for $u(\vec{x}, \vec{s})$ and the general basis function $\phi_p^C(x) \psi_y^C(y) \xi_r^K(\theta)$ of $\mathcal{V}_h^K$ for $v(\vec{x}, \vec{s})$, we obtain

$$\begin{aligned} & \int_{\Omega^C} \int_{S^K} \sigma(\vec{x}) \left(\int_{S^{K'}} \Phi(\vec{s}, \vec{s}') u(\vec{x}, \vec{s}') d\vec{s}' \right) v(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \\ & \approx \sum_{i=0}^{n_x^C-1} \sum_{j=0}^{n_y^C-1} \sum_{a=0}^{n_\theta^{K'}-1} u_{i,j,a}^{K'} \left(\int_{y_0^C}^{y_f^C} \left[\int_{x_0^C}^{x_f^C} \sigma(x, y) \phi_i^C(x) \phi_p^C(x) dx \right] \psi_j^C(y) \psi_q^C(y) dy \right) \\ & \quad \cdot \left(\int_{\theta_0^K}^{\theta_f^K} \left[\int_{\theta_0^{K'}}^{\theta_f^{K'}} \Phi(\theta, \theta') \xi_a^{K'}(\theta) d\theta' \right] \xi_r^K(\theta) d\theta \right), \end{aligned} \quad (30)$$

The discretization of the spatial integral here is similar to that of the spatial integral for the extinction term (Appendix SI-10),

$$\begin{aligned} & \int_{y_0^C}^{y_f^C} \left[\int_{x_0^C}^{x_f^C} \sigma(x, y) \phi_i^C(x) \phi_p^C(x) dx \right] \psi_j^C(y) \psi_q^C(y) dy \\ & \approx \frac{\Delta x^C \Delta y^C}{4} w_i^{C,x} w_j^{C,y} \sigma(x_i^C, y_j^C) \delta_{ip}^{CC} \delta_{jq}^{CC}. \end{aligned} \quad (31)$$

The θ' -integral is discretized using the quadrature rule for the angular part of element K'

$$\int_{\theta_0^{K'}}^{\theta_f^{K'}} \Phi(\theta, \theta') \xi_a^{K'}(\theta) d\theta' \approx \frac{\Delta \theta^{K'}}{2} w_a^{K',\theta} \Phi(\theta, \theta_a^{K'}), \quad (32)$$

which leads to the final discretization of the angular integrals

$$\int_{\theta_0^K}^{\theta_f^K} \left[\frac{\Delta_{\theta}^{K'}}{2} w_a^{K',\theta} \Phi(\theta, \theta_a^{K'}) \right] \xi_r^K(\theta) d\theta \approx \frac{\Delta_{\theta}^K \Delta_{\theta}^{K'}}{4} w_a^{K',\theta} w_r^{K,\theta} \Phi(\theta_r^K, \theta_a^{K'}). \quad (33)$$

Together with the discretization of the spatial integral, this yields the discretized form of the extinction term in Eqn. 10

$$\begin{aligned} & \int_{\Omega^C} \int_{\mathcal{S}^K} \sigma(\vec{x}) \left(\int_{\mathcal{S}^{K'}} \Phi(\vec{s}, \vec{s}') u(\vec{x}, \vec{s}') d\vec{s}' \right) v(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \\ & \approx \sum_{i=0}^{n_x^C-1} \sum_{j=0}^{n_y^C-1} \sum_{a=0}^{n_{\theta}^{K'}-1} u_{ija}^{K'} \left(\frac{\Delta x^C \Delta y^C \Delta_{\theta}^K \Delta_{\theta}^{K'}}{16} w_i^{C,x} w_j^{C,y} w_a^{K',\theta} w_r^{K,\theta} \right. \\ & \quad \left. \sigma(x_i^C, y_j^C) \Phi(\theta_r^K, \theta_a^{K'}) \delta_{ip}^{CC} \delta_{jq}^{CC} \right), \end{aligned} \quad (34)$$

We show a local-angular-spatial element, a local-spatial element, and the global scattering matrix in Figure SI-28, constructed using the methodology discussed in Appendix SI-9.

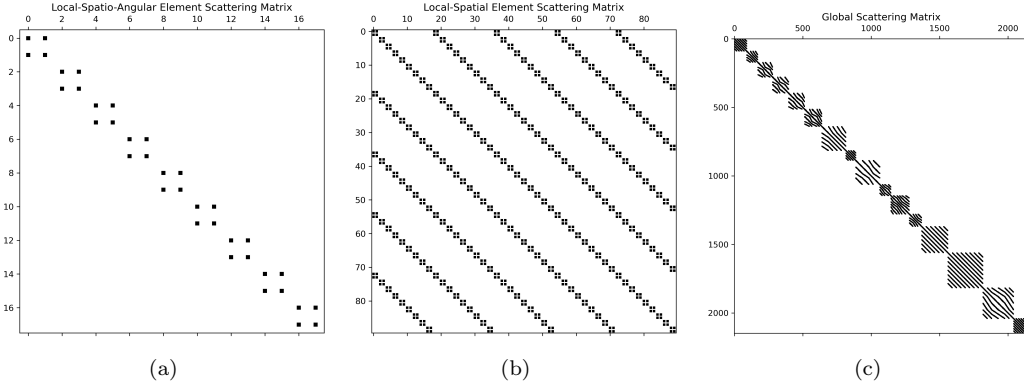


Figure SI-28: (a) A local-angular-spatial element, (b) a local-spatial element, and (c) the global scattering matrix for a simple radiative transfer problem with a mesh that has been randomly h -refined in space and angle.

SI-12. Further details on the numerical solution of the variational form: The interior propagation matrix

The *interior propagation matrix* arises from the discretization of the second term of Eqn. 11a, which arises from performing integration-by-parts on

the term that corresponds to the propagation of radiation. Using the spectral representation for $u(\vec{x}, \vec{s})$ and the general basis function $\phi_p^C(x) \psi_y^C(y) \xi_r^K(\theta)$ of $\mathcal{V}_h^K$ for $v(\vec{x}, \vec{s})$, we obtain

$$\begin{aligned} & \int_{\Omega^C} \int_{\mathcal{S}^K} \vec{s} \cdot \nabla_{\vec{x}} v(\vec{x}, \vec{s}) u(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \\ & \approx \sum_{i=0}^{n_x^C-1} \sum_{j=0}^{n_y^C-1} \sum_{a=0}^{n_\theta^K-1} u_{ija}^K \\ & \quad \cdot \left[\left(\int_{x_0^C}^{x_f^C} \phi_i^C(x) \dot{\phi}_p^C(x) dx \right) \left(\int_{y_0^C}^{y_f^C} \psi_j^C(y) \psi_q^C(y) dy \right) \left(\int_{\theta_0^K}^{\theta_f^K} \cos(\theta) \xi_a^K(\theta) \xi_r^K(\theta) d\theta \right) \right. \\ & \quad \left. + \left(\int_{x_0^C}^{x_f^C} \phi_i^C(x) \phi_p^C(x) dx \right) \left(\int_{y_0^C}^{y_f^C} \psi_j^C(y) \dot{\psi}_q^C(y) dy \right) \left(\int_{\theta_0^K}^{\theta_f^K} \sin(\theta) \xi_a^K(\theta) \xi_r^K(\theta) d\theta \right) \right], \end{aligned} \quad (35)$$

where the overdot diacritic denotes the derivative.

The discretization of each of these integrals utilizing the quadrature rules for spatial element C and angular-spatial element K follow similarly to integrals calculated in Appendices SI-8 and SI-10. We list them below

$$\int_{x_0^C}^{x_f^C} \phi_i^C(x) \dot{\phi}_p^C(x) dx \approx \frac{\Delta x^C}{2} w_i^{C,x} \dot{\phi}_p^C(x_i^C), \quad (36a)$$

$$\int_{y_0^C}^{y_f^C} \psi_j^C(y) \psi_q^C(y) dy \approx \frac{\Delta y^C}{2} w_j^{C,y} \delta_{jq}^{CC}, \quad (36b)$$

$$\int_{\theta_0^K}^{\theta_f^K} \cos(\theta) \xi_a^K(\theta) \xi_r^K(\theta) d\theta \approx \frac{\Delta \theta^K}{2} w_a^{K,\theta} \cos(\theta_a^K) \delta_{ar}^{KK}, \quad (36c)$$

$$\int_{x_0^C}^{x_f^C} \phi_i^C(x) \phi_p^C(x) dx \approx \frac{\Delta x^C}{2} w_i^{C,x} \delta_{ip}^{CC}, \quad (36d)$$

$$\int_{y_0^C}^{y_f^C} \psi_j^C(y) \dot{\psi}_q^C(y) dy \approx \frac{\Delta y^C}{2} w_j^{C,y} \dot{\psi}_q^C(y_j^C), \quad (36e)$$

$$\int_{\theta_0^K}^{\theta_f^K} \sin(\theta) \xi_a^K(\theta) \xi_r^K(\theta) d\theta \approx \frac{\Delta \theta^K}{2} w_a^{K,\theta} \sin(\theta_a^K) \delta_{ar}^{KK}. \quad (36f)$$

Combining these, we obtain the discretized form of the interior propagation

term in Eqn. 11a

$$\begin{aligned}
& \int_{\Omega^C} \int_{\mathcal{S}^K} \vec{s} \cdot \nabla_{\vec{x}} v(\vec{x}, \vec{s}) u(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \\
& \approx \sum_{i=0}^{n_x^C-1} \sum_{j=0}^{n_y^C-1} \sum_{a=0}^{n_\theta^K-1} u_{ija}^K \left(\frac{\Delta x^C \Delta y^C \Delta \theta^K}{8} w_i^{C,x} w_j^{C,y} w_a^{K,\theta} \right. \\
& \quad \cdot \left. \left[\cos(\theta_a^K) \dot{\phi}_p^C(x_i^C) \delta_{jq}^{CC} + \sin(\theta_a^K) \dot{\psi}_q^C(y_j^C) \delta_{ip}^{CC} \right] \delta_{ar}^{KK} \right).
\end{aligned} \tag{37}$$

We show a local-angular-spatial element, a local-spatial element, and the global interior propagation matrix in Figure SI-29, constructed using the methodology discussed in Section SI-9.

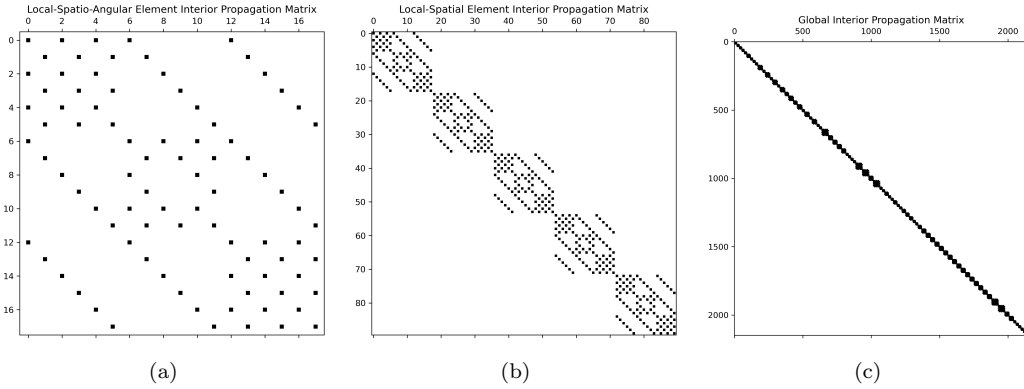


Figure SI-29: (a) A local-angular-spatial element, (b) a local-spatial element, and (c) the global interior propagation matrix for a simple radiative transfer problem with a mesh that has been randomly h -refined in space and angle.

SI-13. Further details on the numerical solution of the variational form: The boundary propagation matrix

The *boundary propagation matrix* arises from the discretization of the second term of Eqn. 11a, which arises from performing integration-by-parts on the term that corresponds to the propagation of radiation. The discretization of this term is the most complicated of any term in Eqn. 11a, as we must account for the upwinding form for the numerical trace of u along the boundary $\partial\Omega^C$ of each spatial element C , given by Eqn. 12. To discretize this term,

we utilize $u(\vec{x}, \vec{s})$ restricted to angular-spatial element K in spatial element C or neighboring element K' in neighboring spatial element C' according to

$$u(\vec{x}, \vec{s}) = \begin{cases} u^{K'}(\vec{x}, \vec{s}), & \text{if } \vec{x} \in \partial\Omega^C \not\subseteq \partial\Omega, \text{ and } \vec{s} \cdot \hat{n}_{\partial\Omega^C} < 0, \\ u^K(\vec{x}, \vec{s}), & \text{otherwise.} \end{cases} \quad (38)$$

The complication of the condition of this piecewise expression is reduced by the structure of the mesh. In particular, the θ -domain of each element in the mesh is a subset of one of the following intervals

$$\mathcal{S}_0 = \left[0, \frac{\pi}{2}\right], \quad \mathcal{S}_1 = \left[\frac{\pi}{2}, \pi\right], \quad \mathcal{S}_2 = \left[\pi, \frac{3\pi}{2}\right], \quad \mathcal{S}_3 = \left[\frac{3\pi}{2}, 2\pi\right]. \quad (39)$$

The rectangular spatial boundary $\partial\Omega^C$ of spatial element C may be separated into four faces $\mathcal{F}_F^C$, indexed by $F = 0, \dots, 3$ starting with the right face and proceeding counterclockwise. We may then express the condition of the piecewise expression in terms of the angular domain of the angular-spatial element K and the facial index F . This is summarized in Table 1.

Table 1: Selection of $u^{K'}(\vec{x}, \vec{s})$ versus $u^K(\vec{x}, \vec{s})$

If $\mathcal{F}_F^C \subseteq \partial\Omega$,	use $u^K(\vec{x}, \vec{s})$.	Otherwise, ...
If $\overline{\mathcal{S}^K} \subseteq \overline{\mathcal{S}^0}$,	use $u^{K'}(\vec{x}, \vec{s})$ for	$\overline{\mathcal{S}^K} \not\subseteq \overline{\mathcal{S}^0}$, and $u^K(\vec{x}, \vec{s})$ otherwise.
$\mathcal{S}^K \subseteq \mathcal{S}^0$		$F = 2, 3$
$\mathcal{S}^K \subseteq \mathcal{S}^1$		$F = 3, 0$
$\mathcal{S}^K \subseteq \mathcal{S}^2$		$F = 0, 1$
$\mathcal{S}^K \subseteq \mathcal{S}^3$		$F = 1, 2$

With this in mind, we rewrite the boundary integral as the sum of integrals over each face of spatial element C

$$\begin{aligned} & \int_{\partial\Omega^C} \int_{\mathcal{S}^K} (\vec{s} \cdot \hat{n}_{\partial\Omega^C}) u(\vec{x}, \vec{s}) v(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \\ &= \sum_{F=0}^3 \int_{\mathcal{F}_F^C} \int_{\theta_0^K}^{\theta_f^K} \omega(\theta; F) u(\vec{x}, \theta) v(\vec{x}, \theta) d\theta d\vec{x}. \end{aligned} \quad (40)$$

where

$$\vec{s} \cdot \hat{n}_{\partial\Omega^C} = \omega(\theta; F) = \cos\left(\theta - F \frac{\pi}{2}\right). \quad (41)$$

Upon substitution by spectral basis functions, the integrand is separable into the product of functions of x , y , and θ . Thus, the computation of the boundary propagation integral may be thought of as three components: the x -component, the y -component, and the θ -component. We denote these components by $X_{ip}^{DC}(F)$, $Y_{jq}^{DC}(F)$, and $\Theta_{ar}^{JK}(F)$. Here $D \in \{C, C'\}$ and $J \in \mathcal{K}^D$ are used for full generality of the derivation of the discretization. The two cases of $D = C$ or $D = C'$ are specified later.

Using the spectral representation for $u(\vec{x}, \vec{s})$ and the general basis function $\phi_p^C(x) \psi_y^C(y) \xi_r^K(\theta)$ of $\mathcal{V}_h^K$ for $v(\vec{x}, \vec{s})$, we obtain

$$\begin{aligned} & \int_{\mathcal{F}_F^C} \int_{\theta_0^K}^{\theta_f^K} \omega(\theta; F) u^J(\vec{x}, \theta) v(\vec{x}, \theta) d\theta d\vec{x} \\ & \approx \sum_{i=0}^{n_x^D-1} \sum_{j=0}^{n_y^D-1} \sum_{a=0}^{n_\theta^J-1} u_{ija}^J X_{ip}^{DC}(F) Y_{jq}^{DC}(F) \Theta_{ar}^{JK}(F), \end{aligned} \quad (42)$$

For $F = 0$ or 2 , $X_{ip}^{DC}(F)$ corresponds to an evaluation of u^D and v at $x = x_f^C$ or x_0^C , respectively. For $F = 1$ or 3 , $X_{ip}^{DC}(F)$ corresponds to the following integral

$$\int_{x_0^C}^{x_f^C} \phi_i^D(x) \phi_p^C(x) dx. \quad (43)$$

This integral is of the form of that in Eqn. 19, where basis functions from (possibly) different elements K, K' in different spatial elements C, C' , respectively, are in the same integrand. Hence, different cases arise when, e.g., $[x_0^C, x_f^C] \subseteq [x_0^{C'}, x_f^{C'}]$ or $n_x^C \leq n_x^{C'}$. We focus on this integral, as a similar situation arises for the y - and θ -components $Y_{jq}^{DC}(F)$, and $\Theta_{ar}^{JK}(F)$.

As in the calculation of Eqn. 19, let $A \in \{C, D\}$ be the spatial element with the greater number of quadrature points and weights in the x -dimension, and $B \in \{C, D\}$ be the spatial element with the smaller domain in the x -dimension. If the domains of C and D are equal in the x -dimension, let $A = B$. We obtain

$$\int_{x_0^C}^{x_f^C} \phi_i^D(x) \phi_p^C(x) dx \approx \frac{\Delta x^B}{2} \sum_{i'=0}^{n_x^A-1} w_{i'}^{A,x} \phi_i^D(\mathcal{T}^{B,x}(\bar{x}_{i'}^A)) \phi_p^C(\mathcal{T}^{B,x}(\bar{x}_{i'}^A)). \quad (44)$$

The summation on the right-hand side sometimes reduces significantly when A and B are explicitly given. Denoting this integral by $\mathcal{E}_{ip}^{DC,x}$, we

calculate

$$\mathcal{E}_{ip}^{DC,x} \approx \begin{cases} \frac{\Delta x^C}{2} w_p^{C,x} \phi_i^D (\mathcal{T}^{C,x} (\bar{x}_p^C)), & \text{if } A = B = C, \\ \frac{\Delta x^D}{2} \sum_{i'=0}^{n_x^C-1} w_{i'}^{C,x} \bar{\phi}_i^D (\bar{x}_{i'}^C) \phi_p^C (\mathcal{T}^{D,x} (\bar{x}_{i'}^C)), & \text{if } A = C, B = D, \\ \frac{\Delta x^C}{2} \sum_{i'=0}^{n_x^D-1} w_{i'}^{D,x} \phi_i^D (\mathcal{T}^{C,x} (\bar{x}_{i'}^D)) \bar{\phi}_p^C (\bar{x}_{i'}^D), & \text{if } A = D, B = C, \\ \frac{\Delta x^D}{2} w_i^{D,x} \phi_p^C (\mathcal{T}^{D,x} (\bar{x}_i^D)), & \text{if } A = B = D. \end{cases} \quad (45)$$

This is further reduced in the case that $D = C$

$$\mathcal{E}_{ip}^{CC,x} \approx \frac{\Delta x^C}{2} w_i^{C,x} \delta_{ip}^{CC}. \quad (46)$$

This leads to the definitions

$$X_{ip}^{CC} (F) := \begin{cases} \delta_{if}^{CC} \delta_{pf}^{CC}, & \text{if } F = 0, \\ \delta_{i0}^{CC} \delta_{p0}^{CC}, & \text{if } F = 2, \\ \frac{\Delta x^C}{2} w_i^{C,x} \delta_{ip}^{CC}, & \text{if } F = 1, 3, \end{cases} \quad (47a)$$

$$X_{ip}^{C'C} (F) := \begin{cases} \delta_{i0}^{C'C'} \delta_{pf}^{CC}, & \text{if } F = 0, \\ \delta_{if}^{C'C'} \delta_{p0}^{CC}, & \text{if } F = 2, \\ \mathcal{E}_{ip}^{C'C,x}, & \text{if } F = 1, 3. \end{cases} \quad (47b)$$

The definition of the y -, and θ -components, $Y_{jq}^{DC} (F)$ and $\Theta_{ar}^{JK} (F)$, respectively, follow similarly

$$Y_{jq}^{CC} (F) := \begin{cases} \delta_{jf}^{CC} \delta_{qf}^{CC}, & \text{if } F = 1, \\ \delta_{j0}^{CC} \delta_{q0}^{CC}, & \text{if } F = 3, \\ \frac{\Delta y^C}{2} w_j^{C,y} \delta_{jq}^{CC}, & \text{if } F = 0, 2, \end{cases} \quad (48a)$$

$$Y_{jq}^{C'C} (F) := \begin{cases} \delta_{j0}^{C'C'} \delta_{qf}^{CC}, & \text{if } F = 1, \\ \delta_{jf}^{C'C'} \delta_{q0}^{CC}, & \text{if } F = 3, \\ \mathcal{E}_{jq}^{C'C,y}, & \text{if } F = 0, 2, \end{cases} \quad (48b)$$

where

$$\mathcal{E}_{jq}^{C'C,y} \approx \begin{cases} \frac{\Delta y^C}{2} w_q^{C,y} \psi_j^{C'} (\mathcal{T}^{C,y} (\bar{y}_q^C)), & A = B = C, \\ \frac{\Delta y^{C'}}{2} \sum_{j'=0}^{n_y^C-1} w_{j'}^{C,y} \bar{\psi}_j^{C'} (\bar{y}_{j'}^C) \psi_q^C (\mathcal{T}^{C',y} (\bar{y}_{j'}^C)), & A = C, B = C', \\ \frac{\Delta y^C}{2} \sum_{j'=0}^{n_y^{C'}-1} w_{j'}^{C',y} \psi_j^{C'} (\mathcal{T}^{C,y} (\bar{y}_{j'}^{C'})) \bar{\psi}_q^C (\bar{y}_{j'}^{C'}), & A = C', B = C, \\ \frac{\Delta y^{C'}}{2} w_j^{C',y} \psi_q^C (\mathcal{T}^{C',y} (\bar{y}_j^{C'})), & A = B = C', \end{cases} \quad (49)$$

and

$$\Theta_{ar}^{KK} (F) := \text{frac} \Delta \theta^K 2 w_a^{K,\theta} \Theta (\theta_a^K; F) \delta_{ar}^{KK} \quad (50a)$$

$$\Theta_{ar}^{K'K} (F) := \mathcal{E}_{ar}^{K'K,\theta} (F) \quad (50b)$$

where

$$\mathcal{E}_{ar}^{K'K,\theta} (F) := \begin{cases} \frac{\Delta \theta^K}{2} w_r^{K,\theta} \omega (\theta_r^K; F) \xi_a^{K'} (\mathcal{T}^{K,\theta} (\bar{\theta}_r^K)), & A=B=K, \\ \frac{\Delta \theta^{K'}}{2} \sum_{a'=0}^{n_\theta^K-1} w_{a'}^{K,\theta} \omega (\mathcal{T}^{K',\theta} (\bar{\theta}_{a'}^K); F) \bar{\xi}_a^{K'} (\bar{\theta}_{a'}^K) \xi_r^K (\mathcal{T}^{K',\theta} (\bar{\theta}_{a'}^K)), & A=K, B=K', \\ \frac{\Delta \theta^K}{2} \sum_{a'=0}^{n_\theta^{K'}-1} w_{a'}^{K',\theta} \omega (\mathcal{T}^{K,\theta} (\bar{\theta}_{a'}^{K'}); F) \xi_a^{K'} (\mathcal{T}^{K,\theta} (\bar{\theta}_{a'}^{K'})) \bar{\xi}_r^K (\bar{\theta}_{a'}^{K'}), & A=K', B=K, \\ \frac{\Delta \theta^{K'}}{2} w_a^{K',\theta} \omega (\theta_a^{K'}; F) \xi_r^K (\mathcal{T}^{K',\theta} (\bar{\theta}_a^{K'})), & A=B=K'. \end{cases} \quad (51)$$

In Eqn. 49, $A \in \{C, C'\}$ is the spatial element with the greater number of quadrature points and weights in the y -dimension, and $B \in \{C, C'\}$ be the spatial element with the smaller domain in the y -dimension, where $A = B$ if the domains of C and C' in the y -dimension are equal. Likewise, in Eqn. 51, $A \in \{K, K'\}$ is the element with the greater number of quadrature points and weights in the θ -dimension, and $B \in \{K, K'\}$ be the cell with the smaller domain in the θ -dimension, where $A = B$ if the domains of K and K' in the θ -dimension are equal.

With these, the discretization of Eqn. 40 may be written as

$$\int_{\partial \Omega^C} \int_{\mathcal{S}^K} (\bar{s} \cdot \hat{n}_{\partial \Omega^C}) u(\bar{x}, \bar{s}) v(\bar{x}, \bar{s}) d\bar{s} d\bar{x} \approx \sum_{F=0}^3 \sum_{i=0}^{n_x^D-1} \sum_{j=0}^{n_y^D-1} \sum_{a=0}^{n_\theta^J-1} u_{ija}^J X_{ip}^{DC} (F) Y_{jq}^{DC} (F) \Theta_{ar}^{JK} (F) \quad (52)$$

We show a local-angular-spatial element, a local-spatial element, and the global boundary propagation matrix in Figure SI-30, constructed using the methodology discussed in Section SI-9.

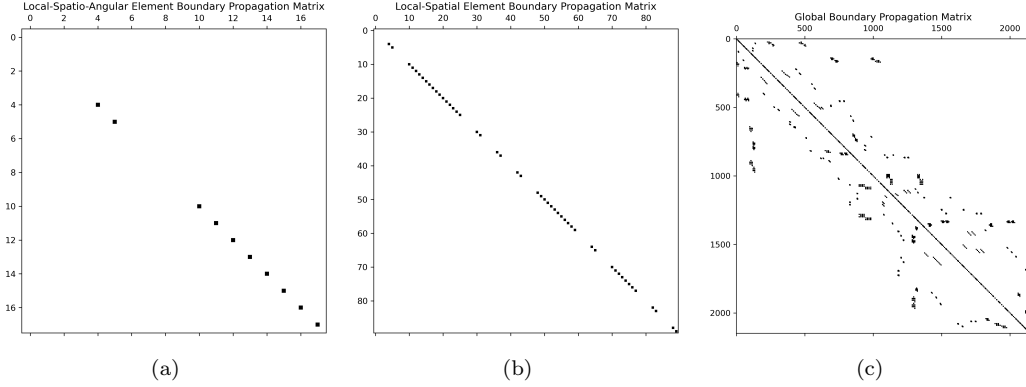


Figure SI-30: (a) A local-angular-spatial element, (b) a local-spatial element, and (c) the global boundary propagation matrix for a simple radiative transfer problem with a mesh that has been randomly h -refined in space and angle.

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