



Angular-spatial hp -adaptivity for radiative transfer with discontinuous Galerkin spectral element methods

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ABSTRACT

Radiative transfer is important for many science and engineering applications, and numerical simulations of radiative transfer can be challenging. For instance, the radiation field is seven-dimensional – three spatial, two angular, one wavelength, and one temporal – and often features steep gradients. Therefore, memory usage is a key issue. To reduce memory, some past work has investigated the use of adaptive mesh refinement (AMR), typically for either the spatial or angular coordinate, and typically for only h -adaptivity. Here, we propose the use of AMR for the spatial and angular coordinates together, and the use of h - and p -adaptivity together as hp -AMR for the potential for further memory savings. We implemented the proposed method for several test cases in two spatial and one angular dimension, with the discontinuous Galerkin spectral element method. These test cases featured highly anisotropic angular radiation, with or without steep spatial gradients. Our primary findings from these test cases were: (1) Angular hp -adaptivity can deliver the radiation solution with the same accuracy as, and with much less computational memory than, uniform angular h - or p -refinements, or angular h -adaptivity alone. This is most obvious when the incoming radiation is highly anisotropic, in which case the savings can be orders of magnitude. (2) Full spatial-angular hp -adaptivity is more efficient in solution representation, compared to solely spatial or solely angular hp -adaptivity. This is most evident when steep gradients are present in both the spatial and angular distribution. These results suggest that adaptive spatial-angular hp -refinement may perform well in large-scale seven-dimensional applications.

1. Introduction

The physical phenomenon of the propagation of electromagnetic radiation through a medium, called radiative transfer or radiation transport, is pervasive in nature and a good understanding of it is important for many human endeavors. Radiative transfer has a profound impact on, e.g., additive manufacturing and material science [1,2], astrophysics [2–7], climate and weather [5–15], combustion systems and turbulent reactive flows [5,16], neutron transport [2,5,17–19], medical imaging [2,5–7,19,20], remote sensing [2,6,21,22], video graphics [2], and high-temperature systems where radiation can account for the majority of the total energy and the dominant mode of energy transfer [4, 23–25].

In some applications, it is not necessary to calculate the full radiation field, e.g., when only a limited number of quantities are desired. A Monte Carlo or ray tracing method is generally preferred in such instances for its low memory requirements [13,26,27]. However, when

many quantities are desired or there are numerous sources of radiation, it is often more efficient to calculate the full radiation field [13,18,23, 26–29].

Several challenges arise when solving the radiative transfer equation (RTE) in full. The radiation field is a seven-dimensional quantity: three spatial, two angular, one wavelength, and one temporal. As a result, a naïve discretization can necessitate the use of many degrees of freedom, i.e., the ‘curse of dimensionality’. Further, the domain often features steep gradients or discontinuities in space, angle, and/or wavelength. The RTE may be dominated by the advection term in some subsets of the domain and by the scattering term in others [30–32]. Thus, the improper selection of matrix solver can lead to a slow convergence of the solution. It is well known that source iteration without use of a preconditioner can converge slowly in the strong scattering regime [27]. For many applications, such as climate simulations, the RTE must be

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solved alongside equations that govern momentum, energy transport, and chemical reactions [6,7,13–15,18,25–29,33].

Many of these difficulties could potentially be addressed by *hp*-adaptivity and the discontinuous Galerkin spectral element method (DGSEM). Several advances have been made in this direction and have been applied to the RTE. The RTE is closely related to the neutron transport equation and the Boltzmann transport equation (BTE). Oftentimes, numerical methods applied to these other equations may be applied to the RTE as well.

We provide a brief survey of such past advances here. A finite element method (FEM) in both space and angle for the RTE was utilized in [29], and an *a posteriori* error indicator in space was introduced to enable adaptive mesh refinement in future work. In [6], a discontinuous Galerkin in space, discrete ordinate method (DOM) in angle method for RTE was investigated. Yuan et al. developed positivity-preserving versions of this method for the RTE [34]. A discontinuous FEM (DFEM) in both space and angle for the RTE was studied in [35,36]. Similarly to [29], Hart et al. developed a residual-based spatial error estimator for a discontinuous Galerkin (DG) FEM [37]. In [38] a DGFEM method with a priori error analysis was applied to the space, angle, and energy domains. The idea of *h*-adaptivity in space was explored with a DGFEM in space, (DOM) in angle scheme in [39]. In [25,28,40], *hp*-adaptivity in space was investigated for the RTE using both a DGSEM in space and angle scheme as well as a DG method for simplified P_N approximations of the RTE. Angular adaptivity with Haar wavelets was studied in [26], using an unstructured spatial grid. In [41], angular *hp*-adaptivity was explored for a DFEM discretization in both space and angle. The work of Bennison introduced simultaneous spatial-angular *h*-adaptivity with a DGFEM in space and angle [42].

To the authors' knowledge, the work presented herein represents the first combination of the DGSEM in space and angle with *hp*-adaptivity in both space and angle for solving the RTE/BTE.

The complete *hp*-adaptive DGSEM has many beneficial properties. For instance, the DGSEM lends itself well to local numerical error estimators and inhomogeneous mesh refinement on arbitrary geometries, thereby increasing the efficiency of adaptive mesh refinement [18,29,37,39,42–44]. Further, DGSEM is highly scalable to high-performance computing systems, and to block-elimination matrix solution strategies [17,45–47]. Spectral element methods converge exponentially for smooth problems, meaning that fewer degrees of freedom (i.e., less memory) is required to achieve accuracy similar to typical finite difference or finite volume methods [28,47]. However, they do not converge exponentially for problems featuring steep gradients and can slow the convergence of iterative solvers, and so *h*-adaptivity is employed to isolate such local features and alleviate the slow convergence of iterative solvers [7,28]. It has been shown that the *hp*-adaptive DGSEM may converge exponentially for problems that include steep gradients [17,25,28].

Recently, a spatially-*hp*-adaptive DGSEM for radiative transfer was introduced in [28]. Through examples featuring smooth and steep gradients, as well as Rayleigh and Mie scattering, the method was shown to converge exponentially with memory and computational cost scaling linearly with the number of spatial degrees of freedom. However, there are other important effects aside from spatial effects, which are not addressed by spatial refinement [7,18,24,26,48]. Inhomogeneous anisotropic angular *h*-adaptivity was shown to ameliorate some such effects in [18,26]. In [26], Haar wavelets were used in the angular part of the domain, while constant, linear, Ω -linear, P_1 -like, and P_1 angular basis functions were investigated. For a review of works on overcoming angular resolution problems, we recommend that which was given in the introduction of [18].

The key innovation of the present work, which extends the work of [28], is the coordinated, simultaneous usage of inhomogeneous spatial and anisotropic angular *hp*-adaptivity for the RTE. The anisotropic angular resolution varies across space, allowing for further speedup and memory cost reduction. Such a flexible representation of the radiation

field would be useful in applications where both spatial and angular refinement may be required. For example, radiation can be highly peaked in angle in some medical imaging applications. Also, there are steep spatial gradients at the boundaries of atmospheric clouds, and the scattering of radiation within clouds smooths out its angular distribution. Here we demonstrate the advantages of spatial-angular *hp*-adaptivity across a range of two-dimensional radiative transfer problems featuring smooth and steep gradients in both space and angle.

Several complexities and challenges arise as a result of combining (1) angular, (2) spatial, and (3) *hp*-adaptivity all together. For instance, while this combination allows a more efficient representation of the numerical solution, it could potentially lead to a much more complicated pattern of inter-element communication. Thus, certain *hp*-spatial-angular mesh regularity criteria will be enforced to ensure the resulting numerical methods can be efficiently implemented. These considerations are one example of how spatial-angular adaptivity differs from spatial adaptivity alone or angular adaptivity alone. Also, in addition to the typical *h*-versus-*p* steering criterion which is needed for *hp*-adaptivity, we introduce an spatial-angular steering criterion to determine when and where a spatial or an angular refinement needs to be applied. In setting up these and other aspects of the methodology, the aim is to bring some amount of simplification to the setup while maintaining generality and flexibility.

Also note that there exist other reduction techniques that can be adopted either as alternative approaches or combined with the spatial-angular *hp*-adaptivity techniques developed here. Examples include P_N approximation [27,49], simplified P_N approximation [25,50], reduced bases/order methods [51,52], and sparse discrete ordinates [53]. While we do not consider these approaches here, they could potentially be considered in the future.

The remainder of the paper is organized as follows. In Section 2, we introduce the numerical methods including the choices of spectral element approximation spaces, the discontinuous Galerkin schemes, and the matrix solvers. Then, in Section 3, we discuss the strategy for angular *hp*-adaptivity, including angular error and regularity estimators, and spatial-angular steering criterion. Finally, in Section 4, we present the performance of the method across three numerical experiments. These experiments featured smooth and steep gradients in both space and angle, as well as various scattering phase functions.

The accompanying Supplemental Information contains detailed discussions and additional figures on the numerical method described here and these test cases, including the construction of the algebraic system, and mesh and numerical solution visualizations. Labels for material included in the Supplemental Information contains the prefix 'SI'.

2. Numerical methods

In this work, we focus on the time-independent, monochromatic radiative transfer equation, given by

$$\begin{aligned} \vec{s} \cdot \nabla_{\vec{x}} u(\vec{x}, \vec{s}) + \beta_e(\vec{x}) u(\vec{x}, \vec{s}) - \beta_e(\vec{x}) \tilde{\omega}(\vec{x}) \int_S \Phi(\vec{s}, \vec{s}') u(\vec{x}, \vec{s}') d\vec{s}' \\ = \beta_e(\vec{x}) (1 - \tilde{\omega}(\vec{x})) B(\vec{x}, \vec{s}) \end{aligned} \quad (1)$$

for $\vec{x} \in \Omega \subseteq \mathbb{R}^d$. The terms on the left-hand side of Eq. (1), from left to right, represent the effects of propagation, extinction, and scattering on the radiation field with intensity $u(\vec{x}, \vec{s})$. The vector $\vec{x}$ represents spatial position, while $\vec{s} \in S := \mathbb{S}^{d-1}$ denotes direction, where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$. The term on the right-hand side represents the effect of the black body radiation $B(\vec{x}, \vec{s})$ on the radiation field. The coefficients $\beta_e(\vec{x}) > 0$ and $1 \geq \tilde{\omega}(\vec{x}) \geq 0$ are the extinction parameter and single-scattering albedo, and $\Phi(\vec{s}, \vec{s}')$ is the scattering phase function. The scattering phase function satisfies

$$\int_S \Phi(\vec{s}, \vec{s}') d\vec{s}' = 1. \quad (2)$$

The values of $\tilde{\omega}(\vec{x})$ range from zero in a purely absorbing medium to one in a purely scattering medium [12].

We may re-write Eq. (1) in simpler notation as

$$\vec{s} \cdot \nabla_{\vec{x}} u(\vec{x}, \vec{s}) + \kappa(\vec{x}) u(\vec{x}, \vec{s}) - \sigma(\vec{x}) \int_{\mathcal{S}} \Phi(\vec{s}, \vec{s}') u(\vec{x}, \vec{s}') d\vec{s}' = f(\vec{x}, \vec{s}), \quad (3)$$

where we require $\kappa(\vec{x}) > \sigma(\vec{x}) \geq 0$ and $f(\vec{x}, \vec{s}) \geq 0$.

In order for Eq. (3) to possess a unique solution, we must additionally impose boundary conditions on $u(\vec{x}, \vec{s})$. Throughout this work, we consider the inflow Dirichlet boundary condition

$$u(\vec{x}, \vec{s}) = u^{in,bd}(\vec{x}, \vec{s}), \quad \text{for } \vec{x} \in \partial\Omega, \vec{s} \cdot \hat{n}_{\partial\Omega} \leq 0, \quad (4)$$

where $\partial\Omega$ denotes the boundary of the spatial part of the domain Ω , and $\hat{n}_{\partial\Omega}$ its outward unit normal vector.

2.1. Domain discretization

In this subsection, we discuss the construction of the computational mesh used to discretize the radiative transfer equation (RTE; Eq. (3)).

We focus solely on the two-dimensional (2-D) steady monochromatic radiative transfer problem in a rectangular spatial domain in this work. Specifically, the 2-D steady monochromatic RTE we consider has two spatial dimensions ($\Omega \subseteq \mathbb{R}^2$) and one angular dimension $\mathcal{S} = \mathbb{S}^1$. This follows the problem considered in [28], which our work extends.

This version of the 2-D steady monochromatic RTE problem is valuable as it allows for faster simulations for testing purposes that are still computationally intense. Further, the discretized domain may be visualized using three-dimensional visualization tools, which enables easier understanding of the discretization and debugging of the software implementation. The implementation in higher dimensions is left as future work. The methods used here have a natural extension to higher dimensions owing to the discontinuous Galerkin spectral element method framework.

In short, the mesh is '2 + 1'-dimensional, i.e., a two-dimensional spatial mesh is constructed, and each spatial element has an associated angular mesh.

The spatial domain is the rectangle $\Omega = [0, L_x] \times [0, L_y]$ and $\mathcal{S} = \mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$. In this setting, we have $\vec{x} = (x, y)$ and $\vec{s} = (\cos(\theta), \sin(\theta))$ with $x \in [0, L_x]$, $y \in [0, L_y]$, and $\theta \in [0, 2\pi)$.

We construct a mesh to discretize the domain by first meshing the spatial domain Ω by a collection of rectangles. We require that this collection of rectangles is 1-irregular. That is, each rectangle has at most two neighbors through each face [28,54].

We refer to each of these rectangles as a *spatial h-element*, and the domain of a spatial h-element C is denoted by Ω^C . The collection $\mathcal{C}$ of all spatial h-elements is referred to as the *spatial h-mesh*. With the spatial h-refinement strategy used here (Section 3.2), a spatial h-mesh is 1-irregular if neighboring spatial h-elements differ by no more than one level of spatial h-refinement. We show an example spatial h-mesh in Fig. 1(a).

Each spatial h-element C has an associated *angular h-mesh*, which is the meshing of the angular domain $\mathcal{S}$ by a collection of arcs $\mathcal{K}^C$. For each angular h-mesh $\mathcal{K}^C$, we adopt a criterion similar to the 1-irregularity criterion. Specifically, the ratio of the length of neighboring arcs is no greater than two.

We refer to each of these arcs as an *angular h-element*, and the domain of an angular h-element K is denoted by $\mathcal{S}^K$. With the angular h-refinement strategy used here (Section 3.2), an angular h-mesh is 1-irregular if neighboring angular h-elements differ by no more than one level of angular h-refinement.

The space formed by the product of a spatial h-element C and an angular h-element $K \in \mathcal{K}^C$ is a *spatial-angular h-element*. We denote a spatial-angular h-element with the ordered pair (C, K) . The domain of spatial-angular h-element (C, K) is the product space $\Omega^C \times \mathcal{S}^K$.

The collection of all spatial-angular h-elements forms the *spatial-angular h-mesh* $\mathcal{M}_h$. Here, we use h to refer to elements/meshes without mention of their spectral basis (see Section 2.2). Later, we use hp to refer to such elements/meshes equipped with a spectral basis. Similarly to the spatial h-mesh and each angular h-mesh, we require that the spatial-angular h-mesh obeys the three-dimensional 1-irregularity condition. That is, each spatial-angular element has at most four neighbors through each face [28,54]. With the spatial and angular h-refinement strategies used here (Section 3.2), a spatial-angular h-mesh is 1-irregular if neighboring spatial-angular h-elements differ by no more than one level of spatial and/or angular h-refinement. We provide two different visualizations of the same spatial-angular h-mesh in Figs. 1(b), 1(c).

2.2. Spectral bases

In this subsection, we discuss the spectral bases used in the discretization of the radiative transfer equation (RTE; Eq. (3)).

To each spatial h-element (Section 2.1), we will associate a spectral basis of polynomials. The spatial h-element together with its associated spectral basis is a *spatial hp-element*. We likewise associate a spectral basis of polynomials to each angular h-element, which together we refer to as an *angular hp-element*.

The tensor product of the spectral basis of a given spatial hp-element C and an angular hp-element $K \in \mathcal{K}^C$ forms a spectral basis for the spatial-angular h-element (C, K) . This spectral basis, together with the spatial-angular h-element (C, K) , forms an *spatial-angular hp-element*.

For simplicity of notation, we henceforth refer to a spatial-angular hp-element by its spatial-angular h-element (C, K) .

In greater detail, the spectral basis of a spatial-angular hp-element is a tensor product of two spatial spectral bases (one in each of the x - and y -dimensions) and an angular spectral basis. The spatial spectral bases are constructed from the Legendre–Gauss–Lobatto quadrature, and the angular spectral bases are constructed from the Legendre–Gauss quadrature. The values for these quadratures, among others, are given in [44,55].

The basis functions in the spectral bases are the Lagrange polynomials that interpolate the quadrature nodes. Note that each basis function has a value of 1 at exactly one quadrature node and zero at all other nodes. We denote the spatial and angular spectral bases of spatial-angular hp-element (C, K) by

$$\left\{ \left(x_i^C, w_i^{C,x}, \phi_i^C(x) \right) \right\}_{i=0}^{n_x^C-1}, \quad (5a)$$

$$\left\{ \left(y_j^C, w_j^{C,y}, \psi_j^C(y) \right) \right\}_{j=0}^{n_y^C-1}, \quad (5b)$$

$$\left\{ \left(\theta_a^K, w_a^{K,\theta}, \xi_a^K(\theta) \right) \right\}_{a=0}^{n_\theta^K-1}. \quad (5c)$$

Here, n_x^C , n_y^C , and n_θ^K are the number of quadrature nodes, weights, and basis functions for (C, K) in the x - and y -spatial dimensions and the θ -angular dimension. These are also the number of x -, y -, and θ -degrees of freedom for (C, K) . The quadrature nodes in each dimension are given by x_i^C , y_j^C , and θ_a^K , respectively, and $w_i^{C,x}$, $w_j^{C,y}$, and $w_a^{K,\theta}$ are the quadrature weights. The basis functions ϕ_i^C , ψ_j^C , and ξ_a^K are Lagrange polynomials interpolating the quadrature nodes, and are visualized in Section SI-1.

The collection of all spatial-angular hp-elements forms the *spatial-angular hp-mesh* $\mathcal{M}_{hp}$. Here, p represents the polynomial degree of the mesh, defined as the maximal degree of x -, y -, and θ -basis functions across all spatial-angular hp-elements.

The structure of the spatial-angular hp-mesh leads naturally to a piecewise spectral representation of the radiation field $u(\vec{x}, \vec{s}) = u(x, y, \theta)$. We denote the spectral representation of the radiation field across the entire domain as $u_{hp}(x, y, \theta)$.

We denote the spectral representation of the radiation field in a single spatial-angular hp-element (C, K) by $u_{hp}^{(C,K)}(x, y, \theta)$. We may

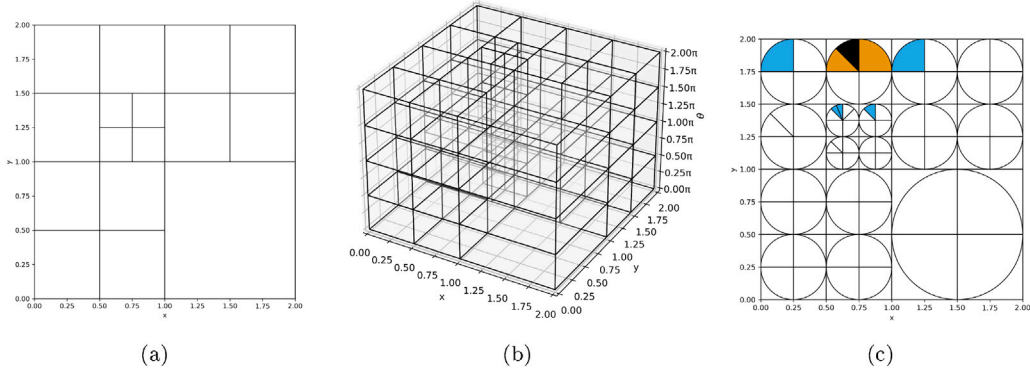


Fig. 1. Visualizations of a spatial-angular h -mesh. (a) A 1-irregular meshing of the spatial domain Ω by a collection of rectangles, forming a spatial h -mesh. (b) A three-dimensional rendering of a spatial-angular h -mesh as a box, where the vertical axis represents the angular dimension. (c) An alternate visualization of a spatial-angular h -mesh, in which the angular h -mesh for each spatial h -element is shown as a disk. The neighbors of the black spatial-angular h -element within the angular h -mesh are shown in orange, while its neighbors in spatially-adjacent angular h -meshes are shown in blue.

write $u_{hp}^{(C,K)}(x, y, \theta)$ as the sum

$$u_{hp}^{(C,K)}(x, y, \theta) = \sum_{i=0}^{n_x^C-1} \sum_{j=0}^{n_y^C-1} \sum_{a=0}^{n_\theta^K-1} u_{ija}^{C,K} \phi_i^C(x) \psi_j^C(y) \xi_a^K(\theta). \quad (6)$$

2.3. Discontinuous Galerkin

In this section, we briefly introduce the weak formulation of the radiative transfer equation (RTE; Eq. (3)). Further details of the construction of the matrices for the numerical solver are given in Sections SI-10, SI-11, SI-12, and SI-13.

In deriving the weak formulation, we utilize approximation spaces that are naturally suggested by the structure of the spatial-angular hp -mesh (Section SI-1).

We define the approximation space for spatial-angular hp -element (C, K) as the space spanned by the spectral basis of (C, K) , i.e.,

$$\mathcal{V}_{hp}^{(C,K)} := \text{span} \left\{ \begin{cases} \phi_i^C(x) \psi_j^C(y) \xi_a^K(\theta) & i = 0, \dots, n_x^C - 1, \\ & j = 0, \dots, n_y^C - 1, \\ & a = 0, \dots, n_\theta^K - 1 \end{cases} \right\}. \quad (7)$$

The global approximation space is therefore

$$\mathcal{V}_{hp} := \prod_{(C,K) \in \mathcal{M}_{hp}} \mathcal{V}_{hp}^{(C,K)}. \quad (8)$$

We give the discontinuous Galerkin (DG) method for Eq. (3) based on the DG spectral element approximation space $\mathcal{V}_{hp}$ defined in Eq. (8). Multiplying Eq. (3) by a test function $v(x, y, \theta) \in \mathcal{V}_{hp}$ and integrating on spatial-angular hp -element (C, K) , we obtain

$$\begin{aligned} & \int_{\Omega^C} \int_{S^K} \vec{s} \cdot \nabla_{\vec{x}} u(\vec{x}, \vec{s}) v(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \\ & + \int_{\Omega^C} \int_{S^K} \kappa(\vec{x}) u(\vec{x}, \vec{s}) v(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \\ & - \int_{\Omega^C} \int_{S^K} \sigma(\vec{x}) \left[\int_S \Phi(\vec{s}, \vec{s}') u(\vec{x}, \vec{s}') d\vec{s}' \right] v(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \\ & = \int_{\Omega^C} \int_{S^K} f(\vec{x}, \vec{s}) v(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \end{aligned} \quad (9)$$

We may perform an integration-by-parts of the first term to obtain the following variational form: find $u_{hp} \in \mathcal{V}_{hp}$ such that

$$a(u_{hp}, v_{hp}) = (f, v_{hp}), \quad \text{for all } v_{hp} \in \mathcal{V}_{hp} \quad (10)$$

where

$$\begin{aligned} a(u, v) := & \sum_{(C,K) \in \mathcal{M}_{hp}} \left[\int_{\partial\Omega^C} \int_{S^K} (\vec{s} \cdot \hat{n}_{\partial\Omega^C}) \hat{u}(\vec{x}, \vec{s}) v(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \right. \\ & \left. - \int_{\Omega^C} \int_{S^K} \vec{s} \cdot \nabla_{\vec{x}} v(\vec{x}, \vec{s}) u(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \right] \end{aligned}$$

$$+ \int_{\Omega^C} \int_{S^K} \kappa(\vec{x}) u(\vec{x}, \vec{s}) v(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \quad (11a)$$

$$- \int_{\Omega^C} \int_{S^K} \sigma(\vec{x}) \left(\int_S \Phi(\vec{s}, \vec{s}') u(\vec{x}, \vec{s}') d\vec{s}' \right) v(\vec{x}, \vec{s}) d\vec{s} d\vec{x},$$

$$(f, v) := \sum_{(C,K) \in \mathcal{M}_{hp}} \left[\int_{\Omega^C} \int_{S^K} f(\vec{x}, \vec{s}) v(\vec{x}, \vec{s}) d\vec{s} d\vec{x} \right], \quad (11b)$$

where $\hat{n}_{\partial\Omega^C}$ is the outward unit normal of spatial h -element C along its

To complete the definition, we adopt the upwinding form of the numerical trace of u along the boundary $\partial\Omega^C$ of each spatial h -element C

$$\hat{u}(\vec{x}, \vec{s}) = \begin{cases} u^{in}(\vec{x}, \vec{s}), & \vec{x} \in \partial\Omega^C, \vec{s} \cdot \hat{n}_{\partial\Omega^C} < 0, \\ u(\vec{x}, \vec{s}), & \text{otherwise,} \end{cases} \quad (12)$$

where $u^{out}(\vec{x}, \vec{s})$ and $u^{in}(\vec{x}, \vec{s})$ are the outflow and inflow restriction of $u(\vec{x}, \vec{s})$ on $\partial\Omega^C$. The value of $u^{in}(\vec{x}, \vec{s})$ comes from either the neighboring spatial h -elements of C or the inflow boundary data $u^{in,bd}(\vec{x}, \vec{s})$.

3. hp -adaptivity

In this section, we describe the method utilized for spatial-angular hp -refinement of the spatial-angular hp -mesh that can be spatially inhomogeneous and angularly anisotropic. The essential pieces to formulate an hp -adaptive mesh refinement method, based on some key ideas from [56], are as follows:

1. An estimate of the error of the numerical solution for each part of the spatial-angular hp -mesh, which we refer to as an *error indicator*;
2. A refinement strategy which utilizes the error indicator to indicate which parts of the spatial-angular hp -mesh need to be refined;
3. A spatial-angular steering criterion to determine whether to perform the spatial refinements or the angular refinements, and;
4. An hp -steering criterion to determine whether to perform h - or p -refinement.

Each of the parts of the hp -adaptive mesh refinement method are described in Sections 3.1, 3.2, 3.3, and 3.4, respectively.

3.1. Error indicators

In this subsection, we describe the spatial and angular error indicators used for the spatial and angular hp -refinement. These are the *spatial*- and *angular-jump error indicators*, which are based on the jumps of the numerical solution u_{hp} across interfaces between neighboring spatial-angular hp -elements. As the mesh is refined, these jumps

approach zero at a similar rate of convergence as the numerical solution [28,39]. These jumps are inexpensive and straightforward to compute, as they rely solely on the numerical solution that has already been calculated. It is possible to use more advanced, and more expensive, error estimators, and such options are explored in a general context in [29,43].

The *spatial-jump error indicator* is defined for a collection of spatial-angular elements $(C, K)_{K \in \mathcal{K}^C}$. The spatial-jump error indicator is defined as the root-mean-square of the sum of the jump in the numerical solution between spatial-angular hp -elements that are neighboring in space

$$\eta_{\vec{x}}^C := \sqrt{\frac{1}{|\partial\Omega^C| |S|} \int_{\partial\Omega^C} \int_S (\|u_{hp}(\vec{x}, \vec{s})\|_{\vec{x}})^2 d\vec{s} d\vec{x}}. \quad (13)$$

The spatial jump operator $\|\cdot\|_{\vec{x}}$ is the difference in the numerical solution at the interface between spatial-angular hp -elements that are neighboring in space. Further, $|\partial\Omega^C|$ represents the surface area (in three spatial dimensions) or perimeter (in two spatial dimensions) of C , and $|S|$ represents the surface area (in two angular dimensions) or perimeter (in one angular dimension) of $S = \mathbb{S}^{d-1}$, the unit sphere in $\mathbb{R}^d$.

The *angular-jump error indicator* is the root-mean-square of the jump between spatial-angular hp -elements that are neighboring in angle

$$\eta_{\vec{s}}^{(C,K)} := \sqrt{\frac{1}{|\partial S^K| |\Omega^C|} \int_{\partial S^K} \int_{\Omega^C} (\|u_{hp}(\vec{x}, \vec{s})\|_{\vec{s}})^2 d\vec{x} d\vec{s}}. \quad (14)$$

The angular jump operator $\|\cdot\|_{\vec{s}}$ is the difference in the numerical solution at the interface between spatial-angular hp -elements that are neighboring in angle. Further, $|\partial S^K|$ represents the side length (in two angular dimensions) of S^K embedded on the unit sphere in $\mathbb{R}^d$, and $|\Omega^C|$ represents the volume (in three spatial dimensions) or area (in two spatial dimensions) of C . We use $|\partial S^K| = 2$ in one angular dimension, as ∂S^K is two discrete points.

3.2. Refinement strategy

In this subsection, we give an overview of the refinement strategy, including how spatial and angular refinements are performed.

Given spatial $0 \leq \tau_{\vec{x}} \leq 1$ and angular $0 \leq \tau_{\vec{s}} \leq 1$ refinement tolerances, a stopping criterion, and an initial spatial-angular hp -mesh $\mathcal{M}_{hp}^{(0)}$, our refinement algorithm can be summarized in the following steps: Setting $l = 0$ to begin,

1. Calculate the numeric solution $u_{hp}(\vec{x}, \vec{s})$ on the spatial-angular hp -mesh $\mathcal{M}_{hp}^{(l)}$.
2. Calculate the spatial error indicator $\eta_{\vec{x}}^C$ on each spatial hp -element and angular error indicator $\eta_{\vec{s}}^{(C,K)}$ on each spatial-angular hp -element.
3. Check if the stopping criterion has been met, e.g., if the maximal spatial error indicator is below a pre-determined threshold.
4. Calculate the spatial and angular refinement thresholds, given respectively by

$$T_{\vec{x}} = \tau_{\vec{x}} \max_C \left(\eta_{\vec{x}}^C \right), \quad T_{\vec{s}} = \tau_{\vec{s}} \max_{(C,K)} \left(\eta_{\vec{s}}^{(C,K)} \right). \quad (15)$$

Mark for refinement each spatial-angular hp -element whose spatial error indicator exceeds the spatial refinement threshold, and likewise for spatial-angular hp -elements whose angular error exceeds the angular error indicator refinement threshold.

5. Use the spatial-angular steering criterion (Section 3.3) to determine whether to refine all spatial-angular hp -elements marked for spatial refinement or all spatial-angular hp -elements marked for angular refinement.
6. For each of the marked elements, use the hp -steering criteria (Section 3.4) to determine whether h - or p -refinement should be

performed. Perform all refinements to obtain the next spatial-angular hp -mesh $\mathcal{M}_{hp}^{(l+1)}$.

7. Increment l to $l + 1$, and return to step 1.

To perform an h -refinement, we divide spatial-angular hp -elements into similar spatial-angular hp -elements of a smaller size. In particular,

- A spatial h -refinement is performed on the collection of spatial-angular hp -elements $(C, K)_{K \in \mathcal{K}^C}$ simultaneously. This design was chosen to for ease of implementation of both the mesh and the calculation of the integral in Eq. (3). For simplicity, we refer to this as performing a spatial h -refinement on spatial hp -element C . Each spatial-angular hp -element $(C, K)_{K \in \mathcal{K}^C}$ is replaced by four spatial-angular hp -elements with the same angular domain and same spectral basis as (C, K) . The side lengths of the new spatial-angular hp -elements in the spatial domain are half of that of (C, K) .
- An angular h -refinement is performed on a spatial-angular hp -element (C, K) by replacing it with two spatial-angular hp -elements with the same spatial domain and spectral basis as (C, K) . The angular extent of each of the new spatial-angular hp -elements is half of that of (C, K) . For example, if the angular extent of the parent spatial-angular hp -element (C, K) is $[0, \pi/4]$, then the angular extent of each child spatial-angular hp -element is $[0, \pi/8]$ and $[\pi/8, \pi/4]$, respectively.

If performed naïvely, h -refinements may violate the 1-irregularity of the spatial-angular hp -mesh. While this violation may result in a more efficient representation of the numerical solution, it also increases the complexity of inter-element communication. We adopt the 1-irregularity criterion here as a balance point in the tradeoff of representational capacity and inter-element communication, similarly to [28]. Thus, when performing an h -refinement, we recursively check if neighboring elements need to be h -refined to maintain the 1-irregularity criterion. We show a visualization of a spatial h -refinement of a spatial-angular hp -element in Figure SI-2 and an angular h -refinement in Fig. 2.

To perform a p -refinement, we simply add one degree of freedom in each spatial or each angular dimension. In particular,

- A spatial p -refinement is performed on the collection of spatial-angular hp -elements $(C, K)_{K \in \mathcal{K}^C}$ simultaneously. For simplicity, we refer to this as performing a spatial p -refinement on spatial hp -element C . We add one to the number of nodes in the x - and y -dimensions in of each spatial-angular hp -element (C, K) .
- An angular p -refinement is performed on a spatial-angular hp -element (C, K) adding one to its number of nodes in the θ -dimension.

There is a similar trade-off in p -refinement as there is in h -refinement regarding the efficiency of solution representation and inter-element communication complexity. While allowing arbitrary p -refinement gives more flexibility in solution representation, it also leads to more complicated and expensive inter-element communication. Suppose the maximal number of degrees of freedom in each spatial dimension of a spatial-angular hp -element in a 1-irregular spatial mesh is n_{max} . Then the total number of inter-element communication matrices is $\mathcal{O}(n_{max}^3)$. For large n_{max} , the storage (or the repeated computation) of all possible communication matrices becomes prohibitively expensive.

Similarly to [28], we adopt the even-degree criterion for the spatial-angular hp -mesh. That is, if a spatial-angular hp -element (C, K) has multiple spatial-angular hp -elements neighboring through a face of (C, K) , then each of those neighboring elements must have the same number of spatial degrees of freedom and the same number of angular degrees of freedom. We visualize a p -refinement of a spatial element in Figure SI-3 and of an angular element in Fig. 3.

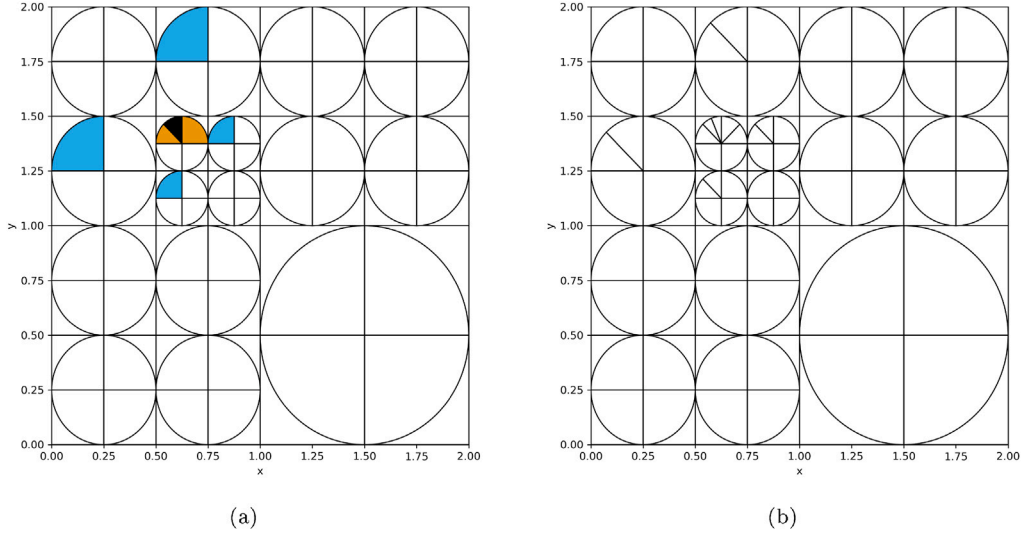


Fig. 2. Demonstration of an angular h -refinement of a spatial-angular hp -element (black). (a) We first check if spatial-angular hp -elements that neighbor in angle (orange) or space (blue) must be h -refined in angle to ensure that the angular 1-irregularity criterion is preserved, and (b) the resultant spatial-angular hp -mesh after an angular h -refinement has been performed.

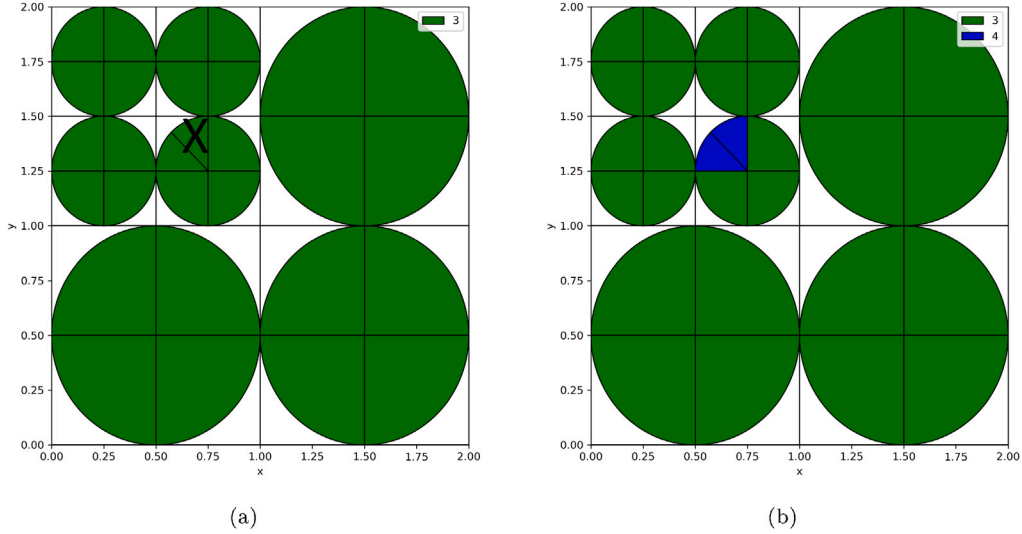


Fig. 3. Demonstration of an angular p -refinement of a spatial-angular hp -element (black 'X'). (a) Before the angular p -refinement, and (b) the resultant spatial-angular hp -mesh after the angular p -refinement has been performed. The color of each spatial-angular hp -element shows its number of angular degrees of freedom.

3.3. Spatial-angular steering criterion

In this section, we present the criterion used to determine whether to perform the angular or spatial refinements determined via the error indicators (Section 3.1).

We first calculate the average spatial and angular error indicators across all spatial-angular hp -elements marked for refinement, which we denote $\bar{\eta}_x$ and $\bar{\eta}_\theta$. We refer to these as the *spatial* and *angular error attempted to resolve*, or *spatial EA2R* and *angular EA2R*.

We then make a random selection between spatial or angular refinement, with probability weighted by their error indicators. We chose to utilize stochasticity in the spatial-angular steering criterion to increase the robustness of the algorithm. We observe in our numerical tests that introducing stochasticity can reduce the likelihood that the refinement gets stuck in a local minimum of reducing the error. Specifically, we perform the spatial refinements across all marked spatial-angular

hp -elements with probability

$$p_{\bar{x}} = \frac{\bar{\eta}_x}{\bar{\eta}_x + \bar{\eta}_\theta}. \quad (16)$$

If we do not perform the spatial refinements, then we perform the angular refinements.

3.4. hp -steering criteria

In this section, we present the criteria used to determine where to h - or p -refine a spatial-angular hp -element marked for refinement via the error indicator (Section 3.1). Each are based on a regularity estimator of an integral of the radiation field $u_{hp}(x, y, \theta)$ in the element.

For hp -steering for spatial refinement, we use the regularity estimator of [28], as derived from results from [56]. We review the spatial regularity estimator in Section SI-3.

We utilize a similar regularity estimator for the hp -steering criterion for angular refinement. Without loss of generality, consider a function $g^{\bar{K}}(\bar{\theta})$ defined on the reference angular hp -element $\bar{K}$. The

reference angular hp -element has domain $S^{\bar{K}} = [-1, 1]$, with associated Legendre–Gauss quadrature nodes and weights, and basis functions denoted

$$\left\{ \left(\bar{\theta}_a^{\bar{K}}, w_a^{\bar{K},\theta}, \bar{\xi}_a^{\bar{K}}(\bar{\theta}) \right) \right\}_{a=0}^{n_{\bar{\theta}}^{\bar{K}}-1}. \quad (17)$$

The Legendre coefficients of $g^{\bar{K}}(\bar{\theta})$ are given by

$$\gamma_r^{\bar{K}} \approx \begin{cases} \frac{2r+1}{2} \sum_{a=0}^{n_{\bar{\theta}}^{\bar{K}}-1} g_a^{\bar{K}} \left(w_a^{\bar{K},\theta} \mathcal{L}_r \left(\bar{\theta}_a^{\bar{K}} \right) \right), & 0 \leq r \leq n_{\bar{\theta}}^{\bar{K}} - 1, \\ 0, & \text{otherwise.} \end{cases} \quad (18a)$$

$$l_p^{\theta} := \frac{\log \left(\frac{2n_{\bar{\theta}}^{\bar{K}}-1}{2|\gamma_j^{\bar{K}}|^2} \right)}{2 \log(n_{\bar{\theta}}^{\bar{K}}-1)}. \quad (18b)$$

Then, if $l_p^{\theta} - \frac{1}{2} \geq n_{\bar{\theta}}^{\bar{K}}$, we perform p -refinement; otherwise, we perform h -refinement. We apply this hp -steering criterion when performing angular refinement. Specifically, we evaluate the regularity of the spatial integral

$$g(\theta) = \int_{\Omega^C} u_{hp}(\vec{x}, \vec{s}) d\vec{x} \quad (19)$$

on each spatial-angular hp -element (C, K) .

4. Numerical experiments

Here we present the results of several numerical experiments. These experiments were designed to demonstrate the efficacy of angular hp -adaptive mesh refinement that is inhomogeneous in space and anisotropic in angle. The designs span a range of scenarios:

1. *Test Case 1: Manufactured Solution* - The *Manufactured Solution* test case arose from a radiation field $u(\vec{x}, \vec{s})$ that was given analytically. The radiation field featured smooth spatial gradients and a sharp peak in angle.

We selected the scattering phase function to be Rayleigh scattering (Eq. (21)), and selected the extinction coefficient $\kappa(\vec{x})$ and scattering coefficient $\sigma(\vec{x})$ to have smooth spatial gradients. The forcing function $f(\vec{x}, \vec{s})$ was calculated such that Eq. (3) would be satisfied.

We compared different angular refinement strategies as a baseline to support the use of angular hp -adaptivity for radiation transfer calculations.

2. *Test Case 2: Two Strong Scatterers* - The *Two Strong Scatterers* test case was designed to be the most complex test case. The system contained two optically-thick circular ‘objects’ to scatter the radiation. The incoming radiation was spatially homogeneous across the left and top boundaries of the domain, and was sharply peaked in angle. Hence, the radiation field would have sharp gradients in both space and angle. Radiation scattered by one ‘object’ would further interact with the second ‘object’. The test case was designed to emulate common systems in nature, such as multiple layers of optically thick clouds in the sky. We investigated the simultaneous use of adaptive mesh refinement in both space and angle to resolve the sharp gradients in angle and space. The goal of this test was to investigate the efficacy of combining hp -adaptivity in both space and angle.

We give further details on the construction on the algebraic system in the Supplemental Information (Sections SI-8, SI-9, SI-10, SI-11, SI-12, and SI-13). We solve the algebraic system using preconditioned iterative matrix solves, as implemented in `petsc4py` [57,58]. For the *Manufactured Solution* test case, we utilized the `gmres` solver with the

`bjacobi` preconditioner. For the *Two Strong Scatterers* test case, we utilized the `fgmres` solver with the `kaczmarz` preconditioner.

In the Supplemental Information, we include an additional test case: *One Weak Scatterer*. The *One Weak Scatterer* test case featured a known incoming radiation field and a single optically-thin circular ‘object’ to scatter the radiation, and incoming radiation similar to the *Two Strong Scatterer* test case.

This test case was more complex than the *Manufactured Solution* test case, as it featured more spatial inhomogeneities in the radiation field. For the *One Weak Scatterer* test case, we utilized the `fgmres` solver with the `kaczmarz` preconditioner. Discussion of the *One Weak Scatterer* test case is reserved for Section SI-7.

4.1. Test case 1: Manufactured solution

The *Manufactured Solution* test case was designed to investigate the efficacy of spatially-inhomogeneous angularly-anisotropic hp -adaptivity to calculate radiation fields with steep angular gradients.

The test case used a known radiation field that was separable in x , y , and θ with a narrow angular distribution and smooth spatial gradients. We compared the following four angular refinement strategies:

1. uniform h -refinement in angle (*h-Unif. Ang.*);
2. uniform p -refinement in angle (*p-Unif. Ang.*);
3. spatially-inhomogeneous adaptive h -refinement in angle (*h-Adap. Ang.*, e.g., Fig. 1(c)), and;
4. spatially-inhomogeneous adaptive hp -refinement in angle (*hp-Adap. Ang.*).

We utilized an angular refinement tolerance of $\tau_s = 0.8$. The spatial h -mesh for each strategy was an 8×8 grid of spatial h -elements, which was not refined in this test case. The associated angular h -mesh for each spatial h -element consisted of 8 elements. Each spatial-angular hp -element had five nodes in both the x - and y -dimensions (Q_4 element), and three nodes in the θ -dimension (P_2 element).

4.1.1. Analytic solution, extinction and scattering coefficients, and scattering phase function

The *Manufactured Solution* test case featured a smooth extinction coefficient with a gentle vertical and horizontal gradient (Fig. 4) with weak scattering.

The extinction $\kappa(\vec{x})$ and scattering $\sigma(\vec{x})$ coefficients were given by

$$\kappa(x, y) = e^{-\left(\frac{1}{L_x} \left(x - \frac{L_x}{2}\right)\right)^2} e^{-\frac{y}{L_y}}, \quad \sigma(x, y) = 0.1 \kappa(x, y). \quad (20a)$$

The length of the domain in the x - and y -dimensions were $L_x = 3$ and $L_y = 2$, respectively. The scattering phase function Φ was the Rayleigh scattering phase function

$$\Phi_R(\theta, \theta') = \frac{1}{3\pi} \left(1 + (\cos(\theta - \theta'))^2 \right). \quad (21)$$

The true radiation field was given by

$$u(x, y, \theta) = e^{-\left(\frac{1}{L_x} \left(x - \frac{L_x}{2}\right)\right)^2} e^{-4\frac{L_y-y}{L_y}} e^{\left(\frac{96}{2\pi} \left(\theta - \frac{7\pi}{5}\right)\right)^2}. \quad (22)$$

This was used for the inflow boundary conditions for the numerical solutions.

The forcing function $f(x, y, \theta)$ was calculated by substituting the extinction coefficient $\kappa(x, y)$, the scattering coefficient $\sigma(x, y)$, the Rayleigh scattering phase function $\Phi_R(\theta, \theta')$, and the manufactured solution $u(x, y, \theta)$ into the radiative transfer equation (Eq. (3)). All integrals for calculating the forcing function were estimated using the high-order numerical quadrature function `quad` of `SciPy` [59].

Fig. 5 shows two visualizations of the true radiation field.

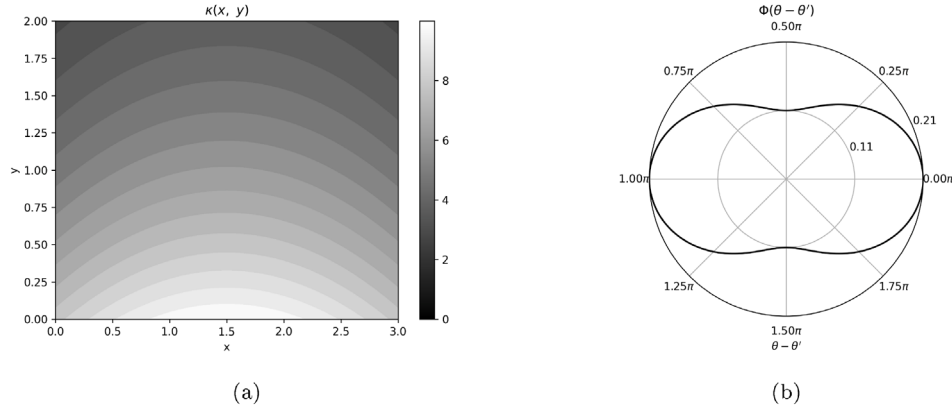


Fig. 4. The extinction coefficient (a) and scattering phase function (b) for Test Case 1: *Manufactured Solution*. Note that the scattering phase function $\Phi(\theta, \theta')$ is plotted as a function of $\theta - \theta'$.

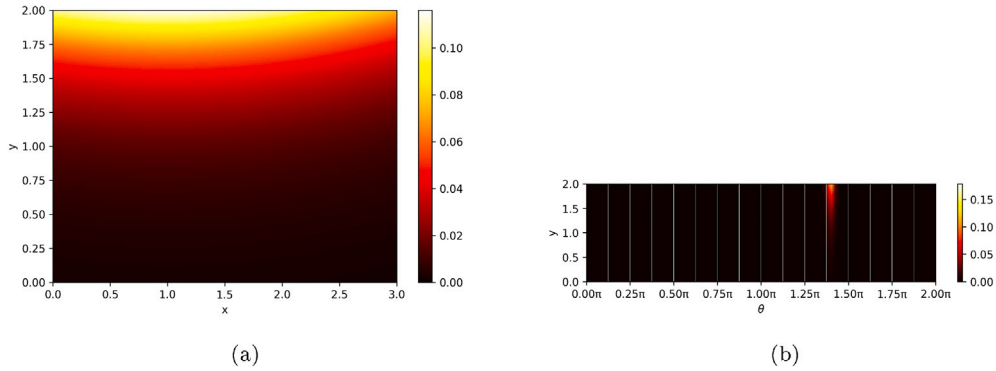


Fig. 5. Visualizations of the true solution for Test Case 1: *Manufactured Solution*; the angular (a) and horizontal (b) integrals of $u(x, y, \theta)$. The vertical white lines arise from the choice of quadrature nodes in angle, which do not include the interval endpoints for each angular hp -element.

4.1.2. Angular refinement behavior and convergence results

The final numerical solution for the uniform angular refinement strategies are shown in Figures SI-4 and SI-5. As expected, each uniform refinement strategy successfully converged toward the true solution (Fig. 6).

The *h-Adap. Ang.* strategy successfully identified and resolved the narrow angular distribution of the radiation field about $\frac{7\pi}{5}$. This was apparent by the final angular hp -meshes (Figure SI-6) having many small angular hp -elements near this angle across the spatial domain, especially toward the bottom. The final numerical solution for Strategy 3 is shown in Figure SI-7.

The *hp-Adap. Ang.* strategy also successfully identified and resolved the narrow angular distribution of the radiation field about $\frac{7\pi}{5}$. This was apparent by the final angular hp -meshes (Figure SI-8) having many small angular hp -elements with high polynomial degree near this angle, especially toward the bottom of the spatial domain. Many of these angular refinements were p -refinements. This reflected the fact that the angular distribution of the radiation field was flatter toward the bottom of the spatial domain; see Fig. 5(b). The final numerical solution for Strategy 4 is shown in Figure SI-9.

Error was measured against the analytic solution u . We compare this error for each strategy as a function of total degrees of freedom in the spatial-angular hp -mesh, as well as a function of the number of non-zero entries in the matrix that arise from the discretization of Eq. (10). We use the number of non-zeros in the system matrix as a proxy for computational expense and memory cost, as the cost of constructing and solving the system matrix depends on algorithm implementation details and computer architecture, among other factors. The number of non-zeros in the system matrix is independent of these factors. The

convergence results for each of these refinement strategies is shown in Fig. 6.

The *h-Unif. Ang.* refinement strategy was the least efficient of these four strategies. The use of this strategy required the largest number of degrees of freedom, generated the most dense system matrices, and was the least accurate. Oppositely, the *hp-Adap. Ang.* strategy had the best performance for every mesh size. Using this strategy gave an error on the order of 10^{-6} using only roughly 150,000 degrees of freedom and generated a system matrix with about 2×10^7 non-zero entries. The second best-performing strategy, *h-Adap. Ang.* required over three times as many degrees of freedom, and generated a system matrix with eight times as many non-zero entries, to achieve this level of accuracy.

4.2. Test case 2: Two strong scatterers

The *Two Strong Scatterers* test case was designed to feature sharp gradients in both space and angle. We compared the following three adaptive refinement strategies:

1. adaptive hp -refinement in space with uniform p -refinement in angle (*hp-Adap. Spt.*, *p-Unif. Ang.*);
2. spatially-inhomogeneous adaptive hp -refinement in angle with uniform p -refinement in space (*hp-Adap. Ang.*, *p-Unif. Spt.*, e.g., Fig. 1(c)), and;
3. spatially-inhomogeneous adaptive hp -refinement in angle with adaptive hp -refinement in space (*hp-Adap. Ang.*, *hp-Adap. Spt.*).

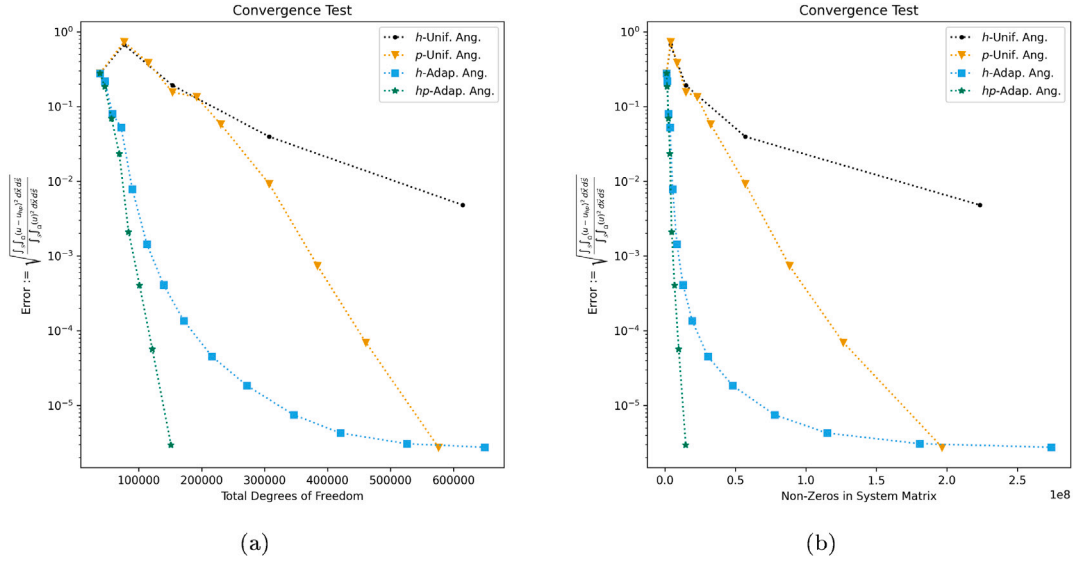


Fig. 6. The convergence test of uniform angular h -refinement (h -Unif. Ang.), uniform angular p -refinement (p -Unif. Ang.), adaptive angular h -refinement (h -Adap. Ang.), and adaptive angular hp -refinement (hp -Adap. Ang.), for Test Case 1: *Manufactured Solution*. Note that since each strategy utilized the same initial mesh, the left-most point is shared amongst all four data sets. The hp -Adap. Ang. far outperformed the other refinement strategies. The use of the strategy achieved high accuracy with meshes with fewer degrees of freedom and whose system matrices had fewer non-zero entries.

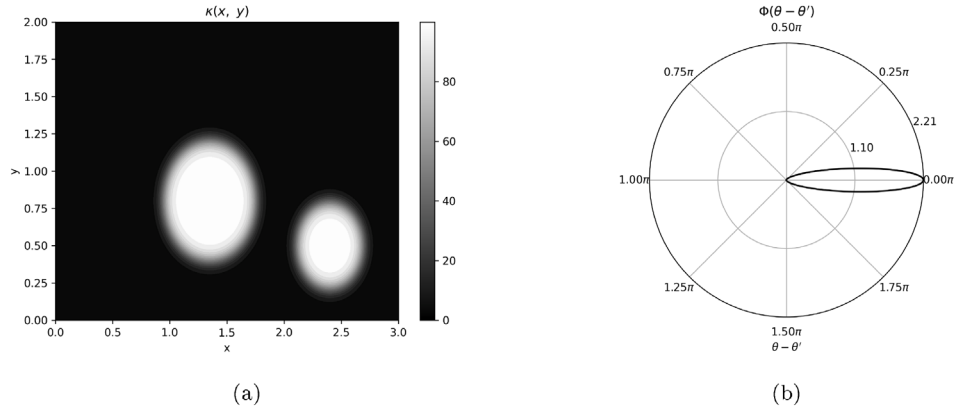


Fig. 7. The extinction coefficient (a) and scattering phase function (b) for Test Case 3: *Two Strong Scatterers*. Note that the scattering phase function $\Phi(\theta, \theta')$ is plotted as a function of $\theta - \theta'$.

Each of these adaptive refinement strategies utilized the spatial-angular steering criterion described in Section 3.3 to determine whether to refine in angle or space at each iteration.

We utilized spatial and angular refinement tolerances of $\tau_{\vec{x}} = \tau_{\vec{s}} = 0.8$. The initial spatial h -mesh for each strategy was a 4×4 grid of spatial h -elements. The associated angular h -mesh for each spatial h -element consisted of 8 elements. Each spatial-angular hp -element had three nodes in both the x - and y -dimensions, and four nodes in the θ dimension.

4.2.1. Boundary conditions, extinction and scattering coefficients, and scattering phase function

The *Two Strong Scatterers* test case featured two circular optically-thick ‘objects’ (Fig. 7) and incoming radiation along the top and left domain boundaries that was sharply peaked in angle.

The extinction $\kappa(\vec{x})$ and scattering $\sigma(\vec{x})$ coefficients were given by

$$r_1(x, y) = \frac{L_y}{5} - \sqrt{\left(x - \frac{9L_x}{20}\right)^2 + \left(y - \frac{2L_y}{5}\right)^2}, \quad (23a)$$

$$\kappa_1(x, y) = \frac{100}{1 + e^{-30r_1(x, y)}}, \quad (23b)$$

$$r_2(x, y) = \frac{L_y}{7} - \sqrt{\left(x - \frac{4L_x}{5}\right)^2 + \left(y - \frac{L_y}{4}\right)^2}, \quad (23c)$$

$$\kappa_2(x, y) = \frac{100}{1 + e^{-30r_2(x, y)}} \quad (23d)$$

$$\kappa(x, y) = \kappa_1(x, y) + \kappa_2(x, y), \quad \sigma(x, y) = 0.7\kappa(x, y). \quad (23e)$$

The length of the domain in the x - and y -dimensions were $L_x = 3$ and $L_y = 2$. The scattering phase function Φ was the Henyey–Greenstein scattering phase function

$$\Phi_{HG}(\theta, \theta'; g) = \frac{1}{C(g)} \left(\frac{1 - g^2}{(1 + g^2 - 2g \cos(\theta - \theta'))^{\frac{3}{2}}} \right) \quad (24)$$

with $g = 0.8$. This value of g corresponds to anisotropic forward scattering.

The incoming radiation was given by

$$u^{in,bd}(x, y, \theta) = \begin{cases} e^{-\left(\frac{48}{2\pi}(\theta - \frac{7\pi}{4})\right)^2}, & y = y_{top} \text{ or } x = x_{left}, \\ 0, & \text{otherwise,} \end{cases} \quad (25)$$

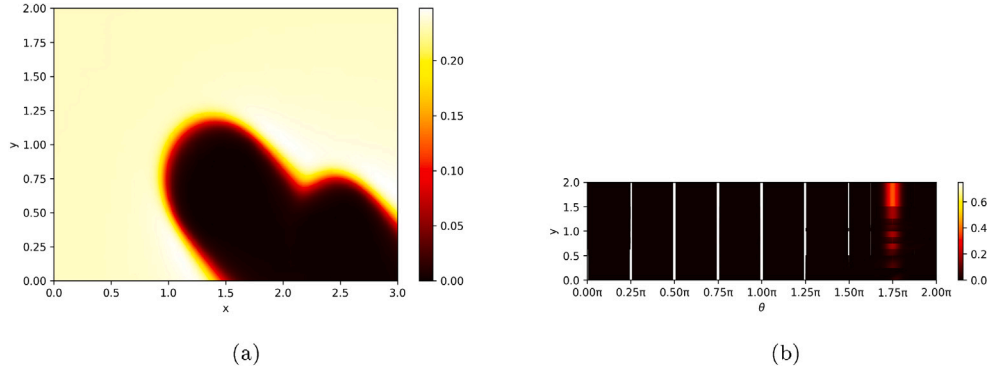


Fig. 8. Visualizations of a high-resolution numerical solution for Test Case 3: Two Strong Scatterers; the angular (a) and horizontal (b) integral of $u_{hr}(x, y, \theta)$.

where $y_{top} = L_y = 2$ was the value of y at the top of the domain and $x_{left} = 0$ was the value of x at the left of the domain.

Fig. 8 shows two visualizations of a high-resolution numerical solution. Note the sharp spatial gradient of the radiation field near each of the two ‘objects’ in the domain. Some radiation scattered by the center-most object was also absorbed and scattered by the object toward the lower-right corner. This is akin to, e.g., multiple layers of clouds in the atmosphere.

4.2.2. Spatial-angular refinement behavior and convergence results

The *hp-Adap. Spt., p-Unif. Ang.* refinement strategy primarily performed angular refinements until the angular error attempted to resolve (EA2R; Section 3.3) approached the spatial EA2R (approximately at 8 thousand degrees of freedom). This strategy then regularly alternated between angular and spatial refinements. The angular EA2R remained between 6×10^{-2} and 10^{-3} for meshes with more than 8 thousand degrees of freedom. The spatial EA2R continued to decrease almost monotonically throughout the entire test, regardless if angular or spatial refinements were performed.

The final spatial and angular *hp*-meshes for the *hp-Adap. Spt., p-Unif. Ang.* strategy are shown in Figure SI-12, and the final numerical solution is shown in Figure SI-13. We observed that the *hp*-adaptive spatial refinements successfully identified regions of the domain with large spatial gradients. This was evident by the distribution of smaller spatial *hp*-elements with high polynomial degree. This strategy performed many uniform angular *p*-refinements to resolve the sharp angular gradients of the radiation field present toward the top and left of the domain. This was despite the fact that the angular distribution of the radiation in the shadows of the ‘objects’ was fairly flat.

The *hp-Adap. Ang., p-Unif. Spt.* refinement strategy behaved similarly for smaller mesh sizes. The angular EA2R was close to the spatial EA2R at larger mesh sizes (approximately at 14 thousand degrees of freedom). This strategy overall favored angular refinements over spatial refinements. The angular EA2R remained between 9×10^{-3} and 10^{-3} for meshes larger than 55 thousand degrees of freedom. The spatial EA2R continued to decrease almost monotonically throughout the entire test, regardless if angular or spatial refinements were performed.

The final spatial and angular *hp*-meshes for the *hp-Adap. Ang., p-Unif. Spt.* refinement strategy are shown in Figure SI-14, and the final numerical solution is shown in Figure SI-15. We observed that the *hp*-adaptive angular refinements successfully identified the sharp angular distribution in radiation at $\frac{7\pi}{4}$. This was evident by the distribution of smaller angular *hp*-elements with high polynomial degree at this angle. We also observed several angular refinements along the lower- and rightmost-spatial elements in the 0π and $\frac{5\pi}{4}$ directions. These seemingly spurious angular refinements arose from the known jump discontinuity along the spatial boundary between the inflow and outflow radiation. The size of this jump is dependent on the parameters of

the problem, e.g. extinction and scattering coefficients, and is present in linear half-space equations [60]. This strategy called for a high polynomial degree in space in each spatial *hp*-element to resolve the sharp spatial gradients of the radiation field. This was despite the fact that the radiation field had very small spatial gradients throughout most of the spatial domain.

The *hp-Adap. Spt., hp-Adap. Ang.* refinement strategy behaved similarly to the *hp-Adap. Ang., p-Unif. Spt.* refinement strategy for meshes smaller than 24 thousand degrees of freedom. After that, the angular EA2R consistently hovered between 4×10^{-2} and 8×10^{-3} . The spatial EA2R decreased slowly, but monotonically, regardless of if spatial or angular refinements were performed.

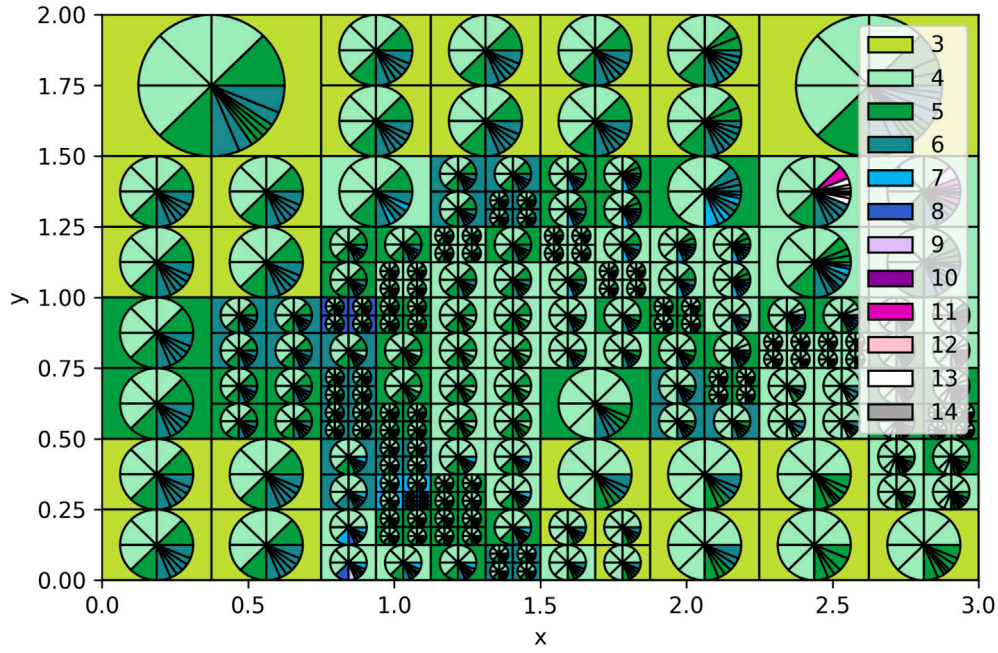
The final spatial and angular *hp*-meshes for the *hp-Adap. Spt., hp-Adap. Ang.* refinement strategy are shown in Fig. 9 (reproduced in Figure SI-16), and the final numerical solution is shown in Figure SI-17. This strategy successfully identified the sharp gradients in both space and angle. This was evident by the many angular *hp*-elements about the $\frac{7\pi}{4}$ direction and the abundance of spatial *hp*-elements surround the two circular optically-thick ‘objects’. A distinct advantage of this strategy was the flexibility for different spatial *hp*-elements to have different angular *hp*-meshes. Specifically, if two spatial *hp*-elements arose from a single spatial *hp*-element, they may later have different angular *hp*-meshes. Neither of the other two strategies could achieve this flexibility.

Fig. 9(b) demonstrates this advantage in spatial elements at different locations along the edge of the shadows cast by the two optically-thick ‘objects’. The angular meshes outside the shadows use a greater number of degrees of freedom than within the shadow.

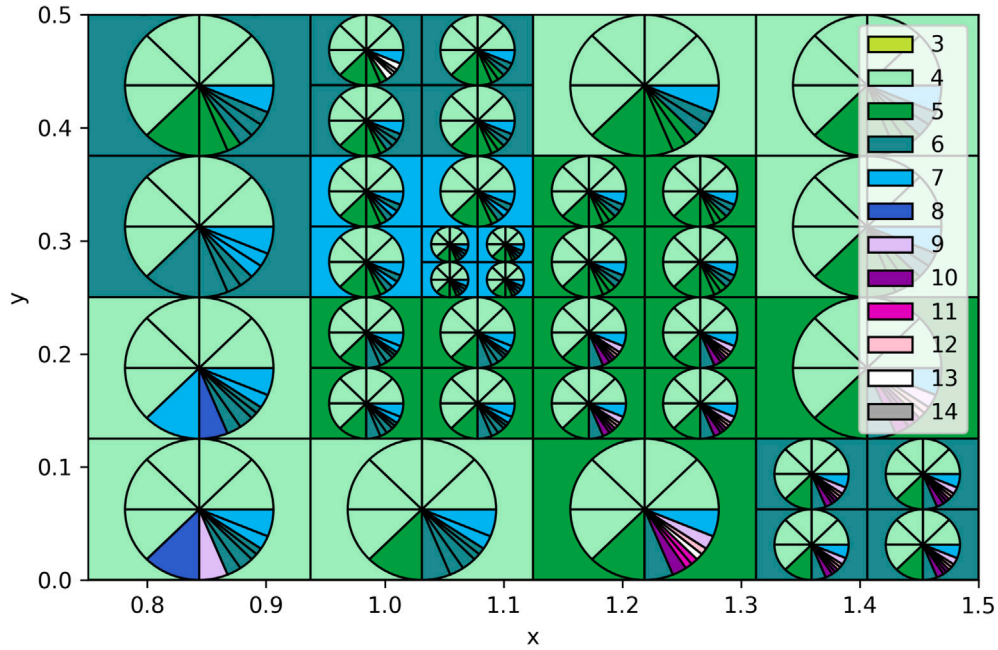
The convergence results for each of these strategies is shown in Fig. 10. Error was measured against high-resolution solutions u_{hr} . For each iteration of the mesh throughout the experiment, an associated high-resolution solution was calculated by *p*-refining the low-resolution *hp*-mesh twice in both angle and space. This ensures that the associated high-resolution solution is indeed higher resolution everywhere than the computed numerical solution. The high-resolution solution uses notably more degrees of freedom throughout the test. For example, in the *hp-Adap. Spt., hp-Adap. Ang.* refinement strategy, the initial and final meshes had 4608 and 248,160 degrees of freedom, respectively, while their high-resolution counterparts had 19,200 and over one million degrees of freedom.

We compare this ‘high-resolution error’ for each strategy as a function of total degrees of freedom in the spatial-angular *hp*-mesh and as a function of the number of non-zero entries in the matrix that arose from the discretization of Eq. (10).

We observed that the *hp-Adap. Spt., p-Unif. Ang.* strategy performed the least efficient of these three strategies. The *hp-Adap. Spt., hp-Adap. Ang.* refinement strategy performed the best of these strategies, especially for meshes with more than approximately 75 thousand degrees



(a)



(b)

Fig. 9. (a) The final spatial-angular hp -mesh for the adaptive spatial hp -refinement with angular adaptive hp -refinement strategy (hp -Adap. Spt., hp -Adap. Ang.) for the *Two Strong Scatterers* test case. (b) A small section of the final spatial-angular hp -mesh highlighting the large number of h - and p -refinements in angle to resolve the sharp angular distribution and steep spatial gradients of the radiation field. The spatial hp -elements (rectangles) are colored according to the number of degrees of freedom in the x -dimension (equivalently the y -dimension). The angular hp -elements (wedges) are colored according to the number of degrees of freedom in the θ -dimension.

of freedom. The two strategies with hp -adaptivity in angle were able to more efficiently resolve the sharp angular distribution with fewer degrees of freedom.

In addition to generating the largest meshes, the system matrices for the hp -Adap. Spt., p -Unif. Ang. strategy were denser than those of

the other two strategies. While both of the angularly-adaptive strategies were able to leverage the spatial inhomogeneity of the adaptive angular refinement, only that which was also spatially-adaptive was able to capitalize on this advantage with an inhomogeneous spatial mesh. In particular, the hp -Adap. Spt., hp -Adap. Ang. strategy performed spatial

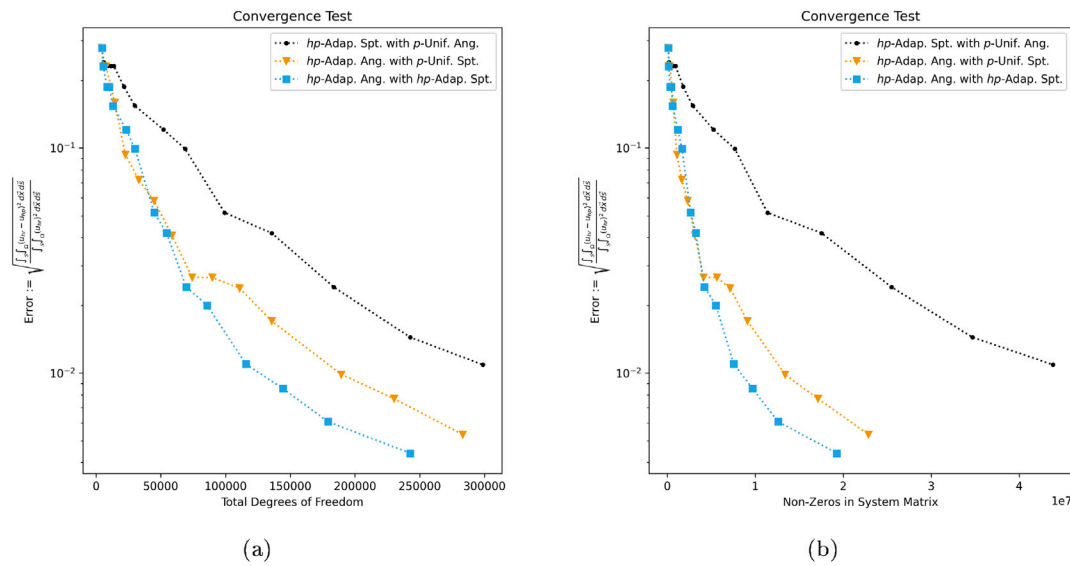


Fig. 10. The convergence test of adaptive spatial hp -refinement with uniform angular p -refinement (hp -Adap. Spt., p -Unif. Ang.), adaptive angular hp -refinement with uniform spatial p -refinement (hp -Adap. Ang., p -Unif. Spt.), and adaptive angular hp -refinement with adaptive spatial hp -refinement (hp -Adap. Spt., hp -Adap. Ang.). for Test Case 3: *Two Strong Scatterers*. Note that since each strategy utilized the same initial mesh, the left-most point is shared amongst all four data sets.

h -refinements in the regions of greater radiative intensity surrounding the scatterers, and each of these additional spatial elements had varying associated angular meshes.

5. Conclusion

In this work, we derived a discontinuous Galerkin spectral element method for the radiative transfer equation (Section 2) featuring spatially-varying hp -adaptive mesh refinement in the angular coordinate (Section 3). We demonstrated this refinement strategy across three numerical experiments featuring highly-anisotropic inflowing radiation (Sections 4 and SI-7). The last test case also featured steep spatial gradients, and we combined angular hp -adaptivity with the spatial hp -adaptive strategy introduced in [28].

The investigations here showed that spatially-varying hp -adaptivity in the angular coordinate can improve the accuracy of the radiation field while utilizing less memory. This improvement varied across different mesh sizes. In Test Case 1: *Manufactured Solution* (Section 4.1), the adaptive angular hp -refinement strategy outperformed uniform refinement strategies by as much as 5 orders of magnitude (Fig. 6). In Test Case 2: *Single Weak Scatterer* (Section SI-7), the adaptive angular hp -refinement strategy outperformed uniform refinement strategies by almost three orders of magnitude for similar mesh sizes. The adaptive angular hp -refinement strategy performed best when combined with the adaptive spatial hp -refinement strategy in Test Case 3: *Two Strong Scatterers* (Section 4.2), outperforming the second-best refinement strategy investigated by half an order of magnitude for larger mesh sizes.

Many possible future directions could potentially be explored. For instance, when calculating the matrix arising from scattering integral in Eq. (3) (see Section SI-11), we utilized the quadrature rule suggested by the angular mesh associated with each spatial hp -element C . These angular meshes were refined to best represent the radiation field $u_{hp}(\vec{x}, \vec{s})$, which may not be suitable to resolve the integral operator. It would be interesting to create a secondary angular mesh for each spatial h -element that is adaptively refined to best represent the integral operator.

Further, it would be insightful to extend the methodology to fully three-dimensional steady monochromatic radiative transfer in large domains by leveraging high performance computing machines with

mixed CPU–GPU architectures. In these large-scale cases, the full advantages of the parallelizability of discontinuous Galerkin spectral element method may be leveraged; including the simultaneous construction of the local spatial element matrices (see Sections SI-10, SI-11, SI-12, and SI-13) and highly-scalable iterative methods, such as domain decomposition, diffusion synthetic acceleration, block-sweeping, the multi-grid method, and the preconditioned Krylov subspace method, which was explored in [28].

For non-manufactured test cases with scattering on or near the spatial domain boundary, we observed that prescribed Dirichlet inflow boundary conditions that were highly peaked in angle were discontinuous with the numerically obtained solution at the outflow boundary. This is a known phenomenon, and is explored and successfully numerically resolved in a wide range of linear half-space kinetic equations in [60]. The size of and how far into the domain this discontinuity permeates is dependent on the parameters of the problem. This phenomenon complicates the choice of error indicator in angle. Since the jump error indicator is dependent on the underlying assumption of continuity of the true solution, it may be inefficient or ineffective for some problems. However, since this discontinuity is well-characterized, one would be able to adjust the angular jump error indicator to ignore this discontinuity. It would be of great value to explore this modification, as well as other angular error indicators, in future work.

It would also be interesting to investigate adaptive mesh coarsening, especially for spatial-angular hp -adaptive mesh refinement/coarsening. As the algorithm refines the solution in space, for example, it may reveal that some previous angular refinements were inefficient, and vice versa. We would expect this feature to greatly improve the performance of spatial-angular hp -adaptivity over the other spatial-angular refinement strategies investigated in the *Two Strong Scatterers* test case.

CRedit authorship contribution statement

Jason L. Torchinsky: Conceptualization, Investigation, Methodology, Software, Validation, Visualization, Writing – original draft, Writing – review & editing, Funding acquisition. **Shukai Du:** Conceptualization, Investigation, Resources, Software, Supervision, Visualization, Writing – original draft, Writing – review & editing. **Samuel N. Stechmann:** Conceptualization, Funding acquisition, Methodology, Project administration, Supervision, Writing – original draft, Writing – review & editing.

Declaration of competing interest

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.jqsrt.2025.109687>.

Data availability

The source code used for this work is archived at <https://zenodo.org/records/14885262>.

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