Supplemental Material for "Asymptotic Models for the Madden-Julian Oscillation and Tropical Geostrophic Balance"

Samuel N. Stechmann[1,2], Scott Hottovy [3] *

[1]Department of Mathematics, University of Wisconsin–Madison, Madison, Wisconsin

[2]Department of Atmospheric and Oceanic Sciences, University of Wisconsin–Madison, Madison, Wisconsin

[3]Department of Mathematics, United States Naval Academy, Annapolis, Maryland

Contents of this file

*Scott Hottovy, Department of Mathematics, United States Naval Academy, Mail Stop 9E Annapolis, MD, USA 21401, hottovy@usna.edu

Introduction

This Supporting Information describes details of the methods for deriving the asymptotic model. The text section is organized as follows:

Text S1.

The system (33) is changed to the characteristic variables, $r = \frac{u-\theta}{\sqrt{2}}$ and $l = \frac{u+\theta}{\sqrt{2}}$. Using these variables it is advantagouse to use the raising and lowering operators $L_\pm = \frac{1}{\sqrt{2}}(\partial_y \pm y)$ to denote derivatives in y. Thus system (33) is written as

$$-\frac{1}{\sqrt{2}}(L_+ - L_-)v - \frac{1}{\sqrt{2}}(\partial_x l - \partial_x r) = 0 \tag{1a}$$

$$L_+ r - L_- l = 0 \tag{1b}$$

$$\frac{\tilde{Q}_{low}}{\sqrt{2}}(\partial_x r + \partial_x l + (L_+ + L_-)v) = -\frac{1}{\tau_{low}} q_{low} \tag{1c}$$

$$\frac{\tilde{Q}_{mid}}{\sqrt{2}}(\partial_x r + \partial_x l + (L_+ + L_-)v) = -\frac{1}{\tau_{mid}} q_{mid} + b_{mid}\frac{1}{2}\left(L_+ + L_-\right)^2 q_{mid} \tag{1d}$$

$$\frac{1}{\sqrt{2}}\partial_t(l-r) - \frac{1}{\sqrt{2}}(1 - \tilde{Q}_{low} - \tilde{Q}_{mid})(\partial_x r + \partial_x l + (L_+ + L_-)v) = -\frac{1}{\sqrt{2}\tau_\theta}(l-r) + b_{low}\frac{1}{2}\left(L_+ + L_-\right)^2 q_{low} + b_{mid}\frac{1}{2}\left(L_+ + L_-\right)^2 q_{mid}. \tag{1e}$$

The variables are expanded with parabolic cylinder function, $\phi_m(y)$,

$$\begin{pmatrix} r \\ l \\ v \\ q_{low} \\ q_{mid} \end{pmatrix} = \sum_{m=0}^{\infty} \begin{pmatrix} r_m \\ l_m \\ v_m \\ q_{low,m} \\ q_{mid,m} \end{pmatrix} \phi_m. \tag{2}$$

The parabolic cylinder functions have the following property with the operators $L_\pm$,

$$L_+\phi_m = \sqrt{m}\phi_{m-1}, \quad L_-\phi_m = -\sqrt{m+1}\phi_{m+1}. \tag{3}$$

A truncation is applied to all variables $r, l, v, q_{mid}, q_{mid}$ at the ϕ_2 parabolic cylinder function. Balance equations are derived by setting the coefficients of ϕ_k equal to each other. The simplest equation to start with is (1b). It yields

$$L_+(r_0\phi_0 + r_1\phi_1 + r_2\phi_2) = L_-(l_0\phi_0 + l_1\phi_1 + l_2\phi_2). \tag{4}$$

Simplifying, this equation gives

$$r_1 = 0 \tag{5a}$$

$$\sqrt{2} r_2 = - l_0 \tag{5b}$$

$$-\sqrt{2} l_1 = 0 \tag{5c}$$

$$-\sqrt{3} l_2 = 0. \tag{5d}$$

for ϕ_0, ϕ_1, ϕ_2, and ϕ_3 respectively. Using the above existing relations, equation (1a) yields the system

$$r_2 = - \sqrt{2} r_0 \tag{6a}$$

$$v_1 = \frac{1}{\sqrt{2}} \partial_x r_2 \tag{6b}$$

$$v_0 = 0 \tag{6c}$$

$$-\sqrt{3} v_2 = 0 \tag{6d}$$

Note that all r, l, and v variables are zero or related to r_0 by,

$$r_1 = 0 \tag{7a}$$

$$r_2 = - \sqrt{2} r_0 \tag{7b}$$

$$v_0 = 0 \tag{7c}$$

$$v_1 = - \partial x r_0 \tag{7d}$$

$$v_2 = 0 \tag{7e}$$

$$l_0 = 2 r_0 \tag{7f}$$

$$l_1 = 0 \tag{7g}$$

$$l_2 = 0. \tag{7h}$$

Using these relations, equation (1c), yields the system

$$\tilde{Q}_{low}(2\partial_x r_0) = -\frac{\sqrt{2}}{\tau_{low}} q_{low,0} \quad (8a)$$

$$0 = -\frac{1}{\tau_{low}} q_{low,1} \quad (8b)$$

$$0 = -\frac{1}{\tau_{low}} q_{low,2} \quad (8c)$$

where $q_{low,1}$ is zero because $l_1 = r_1 = v_0 = v_2 = 0$ and $q_{low,2} = 0$ because of cancellation between r_2, l_2, and v_1.

Equation (1d) yields the system

$$q_{mid,2} = (\frac{\sqrt{2}}{\tau_{mid}} + \frac{5}{2} b_{mid})^{-1} \frac{\sqrt{2}}{2} b_{mid} q_{mid,0} \quad (9a)$$

$$q_{mid,0} = 2\tilde{Q}_{mid} \left[-\frac{\sqrt{2}}{\tau_{mid}} - \frac{\sqrt{2}}{2} b_{mid} + \frac{b_{mid}}{2} \left(\frac{\sqrt{2}}{\tau_{mid}} + \frac{5}{2} b_{mid} \right)^{-1} \frac{\sqrt{2}}{2} b_{mid} \right]^{-1} \partial_x r_0. \quad (9b)$$

This system is solved for one of the variables r_0. Using this system, the resulting equation is

$$\frac{1}{\sqrt{2}} \partial_t(r_0) - \frac{1}{\sqrt{2}}(1 - \tilde{Q}_{low} - \tilde{Q}_{mid})(2\partial_x r_0) = -\frac{1}{\sqrt{2}\tau_\theta}(r_0) + b_{low} \frac{\tau_{low}\tilde{Q}_{low}}{\sqrt{2}} \partial_x r_0 \quad (10)$$
$$+ \frac{b_{mid}}{2}(-2\tilde{Q}_{mid} \left[-\frac{\sqrt{2}}{\tau_{mid}} - \frac{\sqrt{2}}{2} b_{mid} + \sqrt{2} \left(\frac{\sqrt{2}}{\tau_{mid}} + \frac{5}{2} b_{mid} \right)^{-1} \frac{\sqrt{2}}{2} b_{mid} \right]^{-1} \partial_x r_0$$
$$+ \sqrt{2}(\frac{\sqrt{2}}{\tau_{mid}} + \frac{5}{2} b_{mid})^{-1} \frac{\sqrt{2}}{2} b_{mid} 2\tilde{Q}_{mid}$$
$$\cdot \left[-\frac{\sqrt{2}}{\tau_{mid}} - \frac{\sqrt{2}}{2} b_{mid} + \sqrt{2} \left(\frac{\sqrt{2}}{\tau_{mid}} + \frac{5}{2} b_{mid} \right)^{-1} \frac{\sqrt{2}}{2} b_{mid} \right]^{-1} \partial_x r_0.$$

If zonal diffusion is retained, the evolution equation (33d) is now changed to

$$\frac{\partial \theta}{\partial t} - (1 - \tilde{Q}_{low} - \tilde{Q}_{mid}) \nabla \cdot \mathbf{u} = -\frac{1}{\tau_\theta} \theta + b_{low} \partial_y^2 q_{low} + b_{mid} \partial_y^2 q_{mid} + b_{mid} \partial_x^2 q_{mid}. \quad (11)$$

Through a similar analysis as above, this leads to a change in the asymptotic equation

$$\frac{1}{\sqrt{2}}\partial_t(r_0) - \frac{1}{\sqrt{2}}(1-\tilde{Q}_{low}-\tilde{Q}_{mid})(2\partial_x r_0) = -\frac{1}{\sqrt{2}\tau_\theta}(r_0) + b_{low}\frac{\tau_{low}\tilde{Q}_{low}}{\sqrt{2}}\partial_x r_0 \tag{12}$$
$$+\frac{b_{mid}}{2}(-2\tilde{Q}_{mid}\left[-\frac{\sqrt{2}}{\tau_{mid}} - \frac{\sqrt{2}}{2}b_{mid} + \sqrt{2}\left(\frac{\sqrt{2}}{\tau_{mid}} + \frac{5}{2}b_{mid}\right)^{-1}\frac{\sqrt{2}}{2}b_{mid}\right]^{-1}\partial_x r_0$$
$$+\sqrt{2}(\frac{\sqrt{2}}{\tau_{mid}} + \frac{5}{2}b_{mid})^{-1}\frac{\sqrt{2}}{2}b_{mid}2\tilde{Q}_{mid}$$
$$\cdot\left[-\frac{\sqrt{2}}{\tau_{mid}} - \frac{\sqrt{2}}{2}b_{mid} + \sqrt{2}\left(\frac{\sqrt{2}}{\tau_{mid}} + \frac{5}{2}b_{mid}\right)^{-1}\frac{\sqrt{2}}{2}b_{mid}\right]^{-1}\partial_x r_0)$$
$$+b_{mid}2\tilde{Q}_{mid}\left[-\frac{\sqrt{2}}{\tau_{mid}} - \frac{\sqrt{2}}{2}b_{mid} + \sqrt{2}\left(\frac{\sqrt{2}}{\tau_{mid}} + \frac{5}{2}b_{mid}\right)^{-1}\frac{\sqrt{2}}{2}b_{mid}\right]^{-1}\partial_x^3 r_0.$$

Note from (9b) that $q_{mid,0} = c\partial_x r_0$. This gives the evolutions equation for the asymptotic model

$$\frac{\partial}{\partial t}q_{mid,0} + A\frac{\partial}{\partial x}q_{mid,0} + B\frac{\partial^3}{\partial x^3}q_{mid,0} = -Cq_{mid,0}. \tag{13}$$

Substituting the parameters of Table 1 into equation (13) above, gives $A = 0.2909$, $B = 1.0132$, $C = 0.04$.

Text S2. In terms of these rescaled variables, the full model in (4) becomes

$$\epsilon^2\frac{\partial u}{\partial t'} - \epsilon y v' - \epsilon\frac{\partial \theta}{\partial x'} = -\epsilon^2\frac{1}{\tau}u, \tag{32a}$$

$$\epsilon^3\frac{\partial v'}{\partial t'} + yu - \frac{\partial \theta}{\partial y} = -\epsilon^3\frac{1}{\tau}v', \tag{32b}$$

$$\epsilon^2\frac{\partial \theta}{\partial t'} - \epsilon\nabla'\cdot\mathbf{u}' = \epsilon\frac{1}{\tau_l}q_l + \epsilon^2\frac{1}{\tau_m}q_m - \epsilon^2\frac{1}{\tau}\theta, \tag{32c}$$

$$\epsilon^3\frac{\partial q_l}{\partial t'} + \epsilon\tilde{Q}_l\nabla'\cdot\mathbf{u}' = -\epsilon\frac{1}{\tau_l}q_l + \epsilon^4 b_l\partial_{x'}^2 q_l + \epsilon^2 b_l\partial_y^2 q_l, \tag{32d}$$

$$\epsilon^3\frac{\partial q_m}{\partial t'} + \epsilon^2\tilde{Q}_m\nabla'\cdot\mathbf{u}' = -\epsilon^2\frac{1}{\tau_m}q_m + \epsilon^4 b_m\partial_{x'}^2 q_m + \epsilon^2 b_m\partial_y^2 q_m, \tag{32e}$$

To obtain the evolution of moist static energy, $h = \theta + q_l + q_m$ equations (32c), (32d), and (32e) are added together to give,

$$\begin{aligned}\epsilon^2\frac{\partial \theta}{\partial t'} + \epsilon^3\left(\frac{\partial q_l}{\partial t'} + \frac{\partial q_m}{\partial t'}\right) - \epsilon(1-\tilde{Q}_l-\epsilon\tilde{Q}_m)\nabla'\cdot\mathbf{u}'\\ = -\epsilon^2\frac{1}{\tau}\theta + \epsilon^4 b_l\partial_{x'}^2 q_l + \epsilon^2 b_l\partial_y^2 q_l + \epsilon^4 b_m\partial_{x'}^2 q_m + \epsilon^2 b_m\partial_y^2.\end{aligned}$$

Note that $(1-\tilde{Q}_l) = O(\epsilon)$, thus retaining only the order ϵ^2 terms yields,

$$\begin{aligned}\frac{\partial \theta}{\partial t} - (1-\tilde{Q}_{low}-\tilde{Q}_{mid})\nabla\cdot\mathbf{u}\\ = -\frac{1}{\tau_\theta}\theta + b_{low}\partial_y^2 q_{low} + b_{mid}\partial_y^2 q_{mid},\end{aligned} \tag{33d}$$

which arises as the leading-order equation for the dynamics of $h = \theta + q_{low} + q_{mid} \approx \theta$.

Table S1.

A table of the synoptic scalings used to non-dimensionalize equation (4) is given below.

Table S1. Physical constants, reference scales, and model parameters.

Parameter	Derivation	Value	Description
β		2.28×10^{-11} m^{-1} s^{-1}	Variation of Coriolis parameter with latitude
g		9.8 m s^{-2}	Gravitational acceleration
c_p		1006 J kg^{-1} K^{-1}	Specific heat of dry air at constant pressure
L_v		2.5×10^6 J kg^{-1}	Latent heat of vaporization
θ_0		300 K	Potential temperature at surface
P_e		40,000 km	Circumference of Earth at the equator
H		16 km	Tropopause height
N^2	$(g/\theta_0)d\bar{\theta}/dz$	10^{-4} s^{-2}	Buoyancy frequency squared
c	NH/π	50.9 m s^{-1}	Velocity scale
L	$\sqrt{c/\beta}$	1490 km	Equatorial length scale
T	L/c	8.15 hrs	Equatorial time scale
$\tilde{\alpha}$	$HN^2\theta_0/(\pi g)$	15.6 K	Potential temperature scale
	H/π	5.09 km	Vertical length scale
	$H/(\pi T)$	0.174 m s^{-1}	Vertical velocity scale
	c^2	2590 m^2 s^{-2}	(Density-scaled) Pressure anomaly scale
	c^2/g	265 m	Geopotential height scale
	$c_p\tilde{\alpha}/L_v$	6.27 g kg^{-1}	Water vapor scale