Supplemental Document

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Scintillation minimization versus intensity maximization in optimal beams: supplement

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This document contains supporting content for the main article 'Scintillation Minimization versus Intensity Maximization in Optimal Beams'.

1. INTRODUCTION

For a beam with mutual intensity function J, the total intensity at the transmitter region is given by

$$I_0 = \int_{X \in \mathcal{A}} J(X, X) \mathrm{d}X \,, \tag{S1}$$

where $\mathcal{A}$ denotes the transmitter region. Let $\mathcal{R}$ denote the receiver region. Then this beam has a total received intensity given by

$$I = \int_{X_1, X_2 \in \mathcal{A}} H(X_1, X_2) J(X_1, X_2) \mathrm{d}X_1 \mathrm{d}X_2 \,. \tag{S2}$$

Here H is a Hermitian semi positive kernel given by

$$H(X_1, X_2) = \int_{X' \in \mathcal{R}} h(X_1, X') h^*(X_2, X') \mathrm{d}X' \,, \tag{S3}$$

where $h(X, X')$ is the propagator function from a point $X \in \mathcal{A}$ on the transmitter to a point $X' \in \mathcal{R}$ on the receiver. Then the total average intensity and scintillation of the received signal are given by

$$\mathbb{E}[I] = \int_{X_1, X_2 \in \mathcal{A}} \mathbb{E}[H](X_1, X_2) J(X_1, X_2) \mathrm{d}X_1 \mathrm{d}X_2 \,. \tag{S4}$$

and

$$\mathcal{S} = \frac{\int\limits_{X_1, X_2, X_3, X_4 \in \mathcal{A}} J(X_1, X_2) \mathbb{E}[H(X_1, X_2) H^*(X_3, X_4)] J^*(X_3, X_4) \mathrm{d}X_1 \mathrm{d}X_2 \mathrm{d}X_3 \mathrm{d}X_4}{\left(\int\limits_{X_1, X_2 \in \mathcal{A}} \mathbb{E}[H](X_1, X_2) J(X_1, X_2) \mathrm{d}X_1 \mathrm{d}X_2 \right)^2} - 1 \,. \tag{S5}$$

2. REFORMULATION OF SCINTILLATION MINIMIZATION AS A CONVEX CONSTRAINED OPTIMIZATION PROBLEM

Now we show that the problem of minimizing scintillation can be reformulated as an equivalent problem of minimizing a quadratic quantity with a constraint. To see this, first note that scintillation involves a ratio,

$$\mathcal{S} = \frac{\mathbb{E}[I^2]}{\mathbb{E}[I]^2} - 1 \tag{S6}$$

and recall that I is linearly related to J, see Eq. (S2). Therefore, rescaling J by a constant $\alpha \neq 0$ is **not** going to change the value of $\mathcal{S}$; i.e.,

$$\tilde{J} = \alpha J \quad \Rightarrow \quad \tilde{I} = \alpha I \quad \Rightarrow \quad \tilde{\mathcal{S}} = \mathcal{S} \,.$$

Since a constant rescaling of J gives the same scintillation index, there is no need to consider all possible J. Instead, one can consider a constrained class of J, so that any J can be written as a rescaling of an element in this class. There are many choices to select such a constrained class, and a convenient choice is to select the constraint so that $\mathbb{E}[I] = 1$, which renormalizes the denominator in the definition of scintillation. Hence we can define an equivalent reformulation of the scintillation index minimization problem into a constrained optimization problem in Eqn. (7) of the main text, where we require the constraint $\mathbb{E}[I] = 1$.

One advantage of this reformulation is that the original non-convex optimization problem (minimizing scintillation index directly) is now turned into a convex problem, where many optimization packages are readily applicable.

3. ANALYTICAL MINIMIZER

Consider a beam with the mutual intensity given by the Dirac delta function:

$$J(X_1, X_2) = \delta(X_1 - X_2)\,. \tag{S7}$$

Suppose the turbulence depends only on the phase of the propagator as

$$h(X_1, X) = h_0(X_1, X)e^{i\psi(X_1)}\,, \tag{S8}$$

where h_0 is the propagator in uniform medium and ψ denotes a random function of turbulence. Such a form of the propagator arises, for instance, for a single phase screen model of a turbulent medium. Plugging Eq. (S7) into the above equation gives the expression of received intensity as

$$\begin{aligned} I &= \int\limits_{X_1, X_2 \in \mathcal{A}} \delta(X_1 - X_2)e^{i\left(\psi(X_1)-\psi(X_2)\right)}\mathrm{d}X_1\mathrm{d}X_2 \int\limits_{X\in\mathcal{R}} h_0(X_1, X)h_0^*(X_2, X)\mathrm{d}X \\ &= \int\limits_{X_1\in\mathcal{A}}\int\limits_{X\in\mathcal{R}} |h_0(X_1, X)|^2\mathrm{d}X_1\mathrm{d}X\,. \end{aligned} \tag{S9}$$

This shows that the total received intensity is a deterministic quantity, yielding an intensity variance of zero.

However, it has to be noted that, if the mutual intensity above is $J(X_1, X_2) = \delta(X_1 - X_2)$, then the initial intensity I_0 involves $J(X, X) = \delta(0)$. Since the Dirac delta function is not well defined at 0, the initial intensity of this beam is an indeterminate quantity.

In order to avoid this ambiguity of $\delta(0)$, we use a mutual intensity function with a finite width, specified by ϵ:

$$J(X_1, X_2) = \frac{\mathbb{I}_\epsilon(X_1 - X_2)}{H_0(0)\epsilon|\mathcal{A}|}\,, \tag{S10}$$

where $\mathbb{I}_\epsilon$ is the indicator function given by

$$\mathbb{I}_\epsilon(X) = \begin{cases} 1, |X| \le \epsilon \\ 0, |X| > \epsilon\,. \end{cases} \tag{S11}$$

$H(X_1, X_2)$ in uniform medium is a function of only the difference $X_1 - X_2$ [1] which we denote by $H_0(X_1 - X_2)$. Using Eq. (S1) and Eq. (S2), this beam has a total intensity of

$$I_0 = \frac{1}{H_0(0)\epsilon}\,, \tag{S12}$$

at the transmitter and a received intensity of

$$I = \frac{1}{H_0(0)\epsilon|\mathcal{A}|}\int\limits_{X_1, X_2\in\mathcal{A}, |X_1 - X_2|<\epsilon} H(X_1, X_2)\mathrm{d}X_1\mathrm{d}X_2\,. \tag{S13}$$

Using the single phase screen model for turbulence, H takes the form

$$H(X_1, X_2) = H_0(X_1 - X_2)e^{i[\psi(X_1)-\psi(X_2)]}\,, \tag{S14}$$

where ψ s a random phase such that $\psi(X_1') - \psi(X_2')$ is a stationary random process with mean 0 and covariance D_ψ [2], the structure function of turbulence. This gives the expected value of received intensity as

$$\mathbb{E}[I] = \frac{1}{H_0(0)\epsilon|\mathcal{A}|}\int\limits_{\substack{|X_1-X_2|<\epsilon \\ X_1, X_2\in\mathcal{A}}} H_0(X_1 - X_2)e^{-\frac{D_\psi(X_1, X_2)}{2}}\mathrm{d}X_1\mathrm{d}X_2\,. \tag{S15}$$

In a small region $|X_1 - X_2| < \epsilon$, we have that

$$H_0(X_1 - X_2) = H_0(0) + (X_1 - X_2)H_0'(0) + \mathcal{O}(|X_1 - X_2|^2) = H_0(0) + \mathcal{O}(\epsilon) \tag{S16}$$

Plugging this into Eq. (S15) gives us

$$\mathbb{E}[I] = 1 + \mathcal{O}(\epsilon)\,. \tag{S17}$$

This means that the beam in Eq. (S10) has a received intensity of 1 at the leading order. Similarly, we can compute the expectation of I^2 as [2]:

$$\mathbb{E}[I^2] = \frac{1}{(H_0(0)\epsilon|\mathcal{A}|)^2} \int_{\substack{|X_1-X_2|,|X_3-X_4|<\epsilon \\ X_1,X_2,X_3,X_4\in\mathcal{A}}} H_0(X_1 - X_2)H_0^*(X_3 - X_4)e^{-\frac{C(X_1,X_2,X_3,X_4)}{2}} \mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3\mathrm{d}X_4\,, \tag{S18}$$

where C is given by

$$\begin{aligned} C(X_1, X_2, X_3, X_4) = D_\psi(X_1, X_2) + D_\psi(X_3, X_4) - D_\psi(X_1, X_4) \\ - D_\psi(X_2, X_3) + D_\psi(X_1, X_3) + D_\psi(X_2, X_4)\,. \end{aligned} \tag{S19}$$

Using Eq. (S15) and Eq. (S18) gives the variance of total intensity as

$$\mathbb{E}[I^2] - (\mathbb{E}[I])^2 = \frac{1}{(H_0(0)\epsilon|\mathcal{A}|)^2} \int_{\substack{|X_1-X_2|,|X_3-X_4|<\epsilon \\ X_1,X_2,X_3,X_4\in\mathcal{A}}} H_0(X_1 - X_2)H_0^*(X_3 - X_4)\Big(e^{-\frac{C(X_1,X_2,X_3,X_4)}{2}} - e^{-\frac{D_\psi(X_1,X_2)+D_\psi(X_3,X_4)}{2}}\Big)\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3\mathrm{d}X_4\,. \tag{S20}$$

Using Eq. (S19), the exponential terms in the integral can be written as

$$e^{-\frac{C(X_1,X_2,X_3,X_4)}{2}} - e^{-\frac{D_\psi(X_1,X_2)+D_\psi(X_3,X_4)}{2}} = e^{-\frac{D_\psi(X_1,X_2)+D_\psi(X_3,X_4)}{2}}\Big(e^{-\frac{-D_\psi(X_1,X_4)-D_\psi(X_2,X_3)+D_\psi(X_1,X_3)+D_\psi(X_2,X_4)}{2}} - 1\Big)\,. \tag{S21}$$

Note that $D_\psi(X_1, X_2) = D_\psi(X_1 - X_2)$ is an increasing function of $|X_1 - X_2|$, often a power law with $D_\psi(X_1, X_1) = 0$ [2]. So when $|X_1 - X_2|$ and $|X_3 - X_4|$ are confined to small regions of size ϵ, we have that

$$D_\psi(X_1, X_2) = D_\psi(X_1, X_1) + (X_2 - X_1)D_\psi'(X_1, X_1) + \mathcal{O}(|X_2 - X_1|^2) = \mathcal{O}(\epsilon)\,. \tag{S22}$$

Similarly,

$$D_\psi(X_3, X_4) = \mathcal{O}(\epsilon)\,. \tag{S23}$$

Using Eq. (S22) and Eq. (S23), we get the estimate

$$e^{-\frac{D_\psi(X_1,X_2)+D_\psi(X_3,X_4)}{2}} = 1 + \mathcal{O}(\epsilon) \tag{S24}$$

Similarly, we have

$$\begin{aligned} & - D_\psi(X_1, X_4) - D_\psi(X_2, X_3) + D_\psi(X_1, X_3) + D_\psi(X_2, X_4) \\ & = -D_\psi(X_1, X_3) - (X_4 - X_3)D_\psi'(X_1, X_3) + \mathcal{O}(|X_4 - X_3|^2) - D_\psi(X_2, X_3) \\ & + D_\psi(X_1, X_3) + D_\psi(X_2, X_3) + (X_4 - X_3)D_\psi'(X_2, X_3) + \mathcal{O}(|X_2 - X_3|^2) \\ & = (X_4 - X_3)\Big(D_\psi'(X_2, X_3) - D_\psi'(X_1, X_3)\Big) + \mathcal{O}(\epsilon^2) \\ & = \mathcal{O}(\epsilon^2)\,, \end{aligned} \tag{S25}$$

and we obtain

$$e^{-\frac{-D_\psi(X_1,X_4)-D_\psi(X_2,X_3)+D_\psi(X_1,X_3)+D_\psi(X_2,X_4)}{2}} - 1 = \mathcal{O}(\epsilon^2) \tag{S26}$$

Using Eq. (S24) and Eq. (S26) in Eq. (S20), we obtain the variance of the total received intensity as

$$\mathbb{E}[I^2] - (\mathbb{E}[I])^2 = \frac{\mathcal{O}(\epsilon^2)}{(H_0(0)\epsilon|\mathcal{A}|)^2} \int_{\substack{|X_1-X_2|,|X_3-X_4|<\epsilon \\ X_1,X_2,X_3,X_4\in\mathcal{A}}} H_0(X_1 - X_2)H_0^*(X_3 - X_4)\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3\mathrm{d}X_4 \tag{S27}$$

Performing a similar approximation as Eq. (S16) in the region $|X_3 - X_4| < \epsilon$ and plugging into Eq. (S27) gives

$$\mathbb{E}[I^2] - (\mathbb{E}[I])^2 = \mathcal{O}(\epsilon^2)\,. \tag{S28}$$

This shows that the beam Eq. (S10) minimizes scintillation as $\epsilon \to 0$. However, such a beam should also be transmitted using a very high intensity as is evident from Eq. (S12).

Multiple phase screen model

The result above that a mutual intensity function in the form of a Dirac delta minimizes scintillation can be extended to a multiple phase screen model of turbulence as well. Also, since the splitting method for the paraxial wave equation [3] can be interpreted as a multiple phase screen model, this result is potentially relevant for a general class of refractive indices.

Consider a multiple phase screen model with N phase screens placed at locations z_i, with phase $\{\psi_i\}$ where $i = 0, \cdots N-1$. Let $u_{i_{\psi,\omega}}(X_i), X_i \in \mathbb{R}^d$ denote the total field arriving at the ith phase screen. ψ denotes the randomness wrt the medium and ω denotes the randomness wrt the source. Let $J_i(X_i, Y_i), X_i, Y_i \in \mathbb{R}^d$ denote the corresponding mutual intensity function. Without loss of generality, let $z_0 = 0$. Then $u_{i_{\psi,\omega}}(X_i)$ is given by

$$u_{i_{\psi,\omega}}(X_i) = \int\limits_{X_{i-1}} G(X_i - X_{i-1}, z_i - z_{i-1}) e^{i\psi_{i-1}(X_{i-1})} u_{i-1_{\psi,\omega}} \mathrm{d}X_{i-1}\,, \tag{S29}$$

where G is the propagator function defined by

$$G(X, z) = \left(\frac{k}{2\pi i z}\right)^{d/2} \exp\left(\frac{ik\|X\|^2}{2z}\right). \tag{S30}$$

Let $\mathbb{E}[\cdot]_{\psi,\omega}$ denote an expectation wrt ψ and ω. Using the definition of the mutual intensity function, this gives

$$\begin{aligned} J(X_i, Y_i) &= \mathbb{E}[u_{i_{\psi,\omega}}(X_i) u^*_{i_{\psi,\omega}}(Y_i)]_{\psi,\omega} \\ &= \int\limits_{X_{i-1}, Y_{i-1}} G(X_i - X_{i-1}, z_i - z_{i-1}) G^*(Y_i - Y_{i-1}, z_i - z_{i-1}) \\ &\quad \times \mathbb{E}[e^{i[\psi_{i-1}(X_{i-1}) - \psi_{i-1}(Y_{i-1})]} u_{i-1_{\psi,\omega}}(X_{i-1}) u^*_{i-1_{\psi,\omega}}(Y_{i-1})]_{\psi,\omega} \mathrm{d}X_{i-1} \mathrm{d}Y_{i-1}\,. \end{aligned} \tag{S31}$$

Note that $u_{i-1_{\psi,\omega}}(X_{i-1})$ depends only on $\{\psi_m(X_m)\}_{m=1}^{i-2}$ and ω. This means $\psi_{i-1}(X_{i-1})$ and $u_{i-1_{\psi,\omega}}(X_{i-1})$ are independent random variables. This gives us

$$\begin{aligned} J(X_i, Y_i) &= \int\limits_{X_{i-1}, Y_{i-1}} G(X_i - X_{i-1}, z_i - z_{i-1}) G^*(Y_i - Y_{i-1}, z_i - z_{i-1}) \\ &\quad \times \mathbb{E}[e^{i[\psi_{i-1}(X_{i-1}) - \psi_{i-1}(Y_{i-1})]}]]_{\psi,\omega} \mathbb{E}[u_{i-1_{\psi,\omega}}(X_{i-1}) u^*_{i-1_{\psi,\omega}}(Y_{i-1})]_{\psi,\omega} \mathrm{d}X_{i-1} \mathrm{d}Y_{i-1} \\ &= \int\limits_{X_{i-1}, Y_{i-1}} G(X_i - X_{i-1}, z_i - z_{i-1}) G^*(Y_i - Y_{i-1}, z_i - z_{i-1}) \\ &\quad \times \mathbb{E}[e^{i[\psi_{i-1}(X_{i-1}) - \psi_{i-1}(Y_{i-1})]}]]_{\psi,\omega} J_{i-1}(X_{i-1}, Y_{i-1}) \mathrm{d}X_{i-1} \mathrm{d}Y_{i-1}\,. \end{aligned} \tag{S32}$$

Now suppose $J_{i-1}(X_{i-1}, Y_{i-1})$ is of the form

$$J_{i-1}(X_{i-1}, Y_{i-1}) = \delta(X_{i-1}, Y_{i-1})\,. \tag{S33}$$

Plugging the form of J_{i-1} in Eq. S33 into Eq. S32 gives

$$J_i(X_i, Y_i) = \int\limits_{X_{i-1}} G(X_i - X_{i-1}, z_i - z_{i-1}) G^*(Y_i - X_{i-1}, z_i - z_{i-1}) \mathrm{d}X_{i-1}\,. \tag{S34}$$

Expanding G according to S30 and substituting in S34 gives

$$\begin{aligned} J_i(X_i, Y_i) &= G(X_i, z_i - z_{i-1}) G^*(Y_i, z_i - z_{i-1}) \int\limits_{X_{i-1}} \exp\left(-\frac{ikX_{i-1} \cdot (X_i - Y_i)}{z_i - z_{i-1}}\right) \mathrm{d}X_{i-1} \\ &= \left(\frac{2\pi(z_i - z_{i-1})}{k}\right)^d G(X_i, z_i - z_{i-1}) G^*(Y_i, z_i - z_{i-1}) \delta(X_i - Y_i) \\ &= \delta(X_i - Y_i) \end{aligned} \tag{S35}$$

This means that mutual intensity function which is a delta function continues to stay a delta function as it propagates through multiple layers of phase screens. Finally at the last phase screen, we can use the argument of a single phase screen case to conclude that the received intensity is deterministic, and has a scintillation of zero.

Remark: It has to be noted that in the proof above, we have considered the domain of the source to be all of $\mathbb{R}^d$. This is not the case in practical applications; however, for sufficiently large transmitter sizes, a Dirac delta mutual intensity function should potentially still have near zero scintillation. In more detail, if a finite transmitter size is used, then Eq. (S35) is modified as:

$$J_i(X_i, Y_i) = G(X_i, z_i - z_{i-1})G^*(Y_i, z_i - z_{i-1}) \int_{X_{i-1}\in\mathcal{A}} \exp\Big(-\frac{ikX_{i-1}\cdot(X_i - Y_i)}{z_i - z_{i-1}}\Big)\mathrm{d}X_{i-1}\,. \tag{S36}$$

For a one dimensional transmitter region, say $\mathcal{A} := (-r, r)$, this becomes:

$$J_i(X_i, Y_i) = 2rG(X_i, z_i - z_{i-1})G^*(Y_i, z_i - z_{i-1})\mathrm{sinc}\Big(\frac{rk(X_i - Y_i)}{z_i - z_{i-1}}\Big)\,, \tag{S37}$$

where $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$ denotes the sinc function. Using the definition of G in Eq. (S30), we have

$$J_i(X_i, Y_i) = \underbrace{\Big(\frac{rk}{\pi(z_i - z_{i-1})}\Big)\mathrm{sinc}\Big(\frac{rk(X_i - Y_i)}{z_i - z_{i-1}}\Big)}_{f_\gamma}\exp\Big(\frac{ik}{2(z_i - z_{i-1})}(\|X_i\|^2 - \|Y_i\|^2)\Big)\,, \tag{S38}$$

where we used the notation $f_\gamma(x) = \frac{\gamma}{\pi}\mathrm{sinc}(\gamma x)$ with $\gamma = \frac{rk}{z_i - z_{i-1}}$. Compared with the exponential factor in Eq. (S38) which has length scale $l_{ref} = ((z_i - z_{i-1})/k)^{1/2}$, the $f_\gamma(x)$ factor will be approximately a delta function if $\gamma l_{ref} \gg 1$. I.e.,

$$f_\gamma(x) \approx \delta(x) \quad \text{as } \gamma l_{ref} \gg 1\,, \tag{S39}$$

so that one can regard $J_i(X_i, Y_i)$ as a delta function provided that the relation $r \gg \frac{z_i - z_{i-1}}{kl_{ref}}$, or equivalently $r \gg l_{ref}$, approximately holds.

4. DISCRETIZATION

Let the transmitter plane be discretized as $X_i, i = 1, \cdots, n$. Then in discrete form, J is a Hermitian semi positive definite matrix of dimension $n \times n$ such that $J_{i,j} = J(X_{1_i}, X_{2_j}), i, j = 1, \cdots n$. Similarly, H is a Hermitian semi positive definite matrix of size $n \times n$ such that $H_{i,j} = H(X_{1_i}, X_{2_j})$ and A is a tensor of dimensions $n \times n \times n \times n$ such that $A_{i,j,k,l} = \mathbb{E}[H_{i,j}H^*_{k,l}]$. Then constraint on the received intensity takes the form of a trace operator upon discretization:

$$\mathbb{E}[I] = \int_{X_1, X_2 \in \mathcal{A}} \mathbb{E}[H](X_1, X_2)J(X_1, X_2)\mathrm{d}X_1\mathrm{d}X_2 \xrightarrow{\textit{discretization}} \mathrm{Tr}(\mathbb{E}[H]J^\top)\,. \tag{S40}$$

Then the scintillation minimization problem has the form

$$\min_J f(J) \quad \text{s.t. } J \in \mathcal{C}, \tag{S41}$$

where the cost function f is given by the quadratic form

$$f(J) = J^*AJ = \sum_{i,j,k,l} J_{ij}A_{ijkl}J^*_{kl}\,, \tag{S42}$$

and the constraint is given by the set

$$\mathcal{C} := \{J : \mathrm{Tr}(\mathbb{E}[H]J^\top) = 1\} \cap \{J : J \in S^n_+\}\,. \tag{S43}$$

S^n_+ denotes the set of Hermitian semi positive definite matrices of size $n \times n$.

5. CONVEXITY OF THE PROBLEM

It can be shown that the cost function $f(J)$ is convex for any complex valued J. Let $J \in \mathbb{C}^{n\times n}$. Using the definition of A,

$$\begin{aligned} J^* A J &= \mathbb{E}\Big[\sum_{i,j,k,l} J_{i,j} H_{i,j} H^*_{k,l} J^*_{k,l}\Big] \\ &= \mathbb{E}\Big[\Big|\sum_{i,j} J_{i,j} H_{i,j}\Big|^2\Big] \geq 0 \quad \forall J \in \mathbb{C}^{n\times n} \end{aligned} \tag{S44}$$

Now we have to prove that the constraint forms a convex set as well. First consider the set $\mathcal{C}_1 = \{J : \mathrm{Tr}(\mathbb{E}[H]J^\top) = 1\}$. Then if $J_1, J_2 \in \mathcal{C}_1$, for any $\alpha \in [0,1]$,

$$\mathrm{Tr}(\mathbb{E}[H][\alpha J_1^\top + (1-\alpha)J_2^\top]) = 1 \tag{S45}$$

So the constraint on the trace gives a convex set when $J \in \mathbb{C}^{n\times n}$. Since the set of semi positive definite matrices is also is a convex set in $C^{n\times n}$, we have that the set of constraints $\mathcal{C}$ also form a convex set.

6. OPTIMIZATION ALGORITHM

We have to minimize a convex function on a convex set. In this case we can use a projected gradient descent(PGD) algorithm [4]. For a step size α_m, the PGD algorithm updates as follows:

$$J_{m+1} = P_{\mathcal{C}}(J_m - \alpha_m \nabla f(J_m)), \quad m = 0, 1, \cdots \tag{S46}$$

where $P_{\mathcal{C}}$ is the projection onto the set of constraints given by

$$P_{\mathcal{C}}(J_m - \alpha_m \nabla f(J_m)) = \arg\min_{J\in\mathcal{C}} \|J_m - \alpha_m \nabla f(J_m) - J\|_F^2\,, \tag{S47}$$

where $\|\cdot\|_F$ denotes the Frobenious norm. The gradient of the cost function can be computed using the formula:

$$\nabla_J f(J) = J^* A := \{\sum_{k,l} A_{i,j,k,l} J^*_{k,l}\}_{i,j}\,. \tag{S48}$$

The projection problem Eq. (S47) has a unique solution since $\mathcal{C}$ is closed and convex[4]. Eq. (S47) seeks to find the nearest semi positive definite Hermitian matrix to the update from regular gradient descent, with received intensity equal to 1. In other words, we need an algorithm to solve the following constrained optimization problem:

$$\begin{aligned} &\min_J \|J_m - \alpha_m \nabla f(J_m) - J\|_F^2 \\ &\text{s.t. } J \in S^n_+ \text{ and } \mathrm{Tr}(\mathbb{E}[H]J^\top) = 1 \end{aligned} \tag{S49}$$

Let $(X)_+$ denote the positive part of X given by

$$(X)_+ = \frac{(X + (X^*)^\top)}{2} \tag{S50}$$

Then for a properly chosen α, we can use an adaption of the algorithm in [5]:

Algorithm S1. Projected gradient descent algorithm

while $m \leq M$ **do**
  $J_{m,1} \leftarrow J_m$
  while $\mathrm{Tr}(\mathbb{E}[H]J_{m,l}) - 1 > tol$ and $l \leq L$ **do**
    $J_{m,l+1} \leftarrow (X - \nu_{m,l}\mathbb{E}[H])_+$
    $\nu_{m,l+1} \leftarrow \nu_{m,l} + \alpha[\mathrm{Tr}(\mathbb{E}[H]J_{m,l}) - 1]$
    $J_m \leftarrow J_{m,l}$
    $l \leftarrow l + 1$
  $J_{m+1} \leftarrow (J_m - \alpha_m \nabla f(J_m) - \nu_{*m}\mathbb{E}[H])_+$

7. RECONSTRUCTION OF H AND A

When H is low rank, we could use the randomized singular value decomposition(rSVD) algorithm [6] to find H. A can also be reconstructed in a similar manner by reconstructing H multiple times for different realizations of the random medium.

A. Randomized SVD algorithms

Here, we describe the adaptation to the randomized SVD algorithms we use to construct H and A. The rSVD algorithm requires only the knowledge of H and A viewed as matrices acting on random Gaussian vectors. The reconstruction algortithm for H is a straightforward adaptation of the algorithm in [6] for a Hermitian matrix. The action of H on a Gaussian random vector is interpreted through forward and adjoint solves of the propagation model.

Algorithm S2. rSVD algorithm for reconstruction of H

$\Omega := n \times k_1 + p_1$ Gaussian random matrix.
Compute $Y_{n\times k+p} = H\Omega$ using propagation model.
Find $Q_{n\times k+p} :=$ QR decomposition of Y.
Form $B_{k+p\times k+p} = Q^* H Q$.
Find the eigenvalue decomposition of the small matrix $B : B = U\Sigma U^*$.
$\tilde{U} = QU$
$H \approx \tilde{U}\Sigma\tilde{U}^*$

The reconstruction of A requires an additional element of Monte Carlo averaging. A is interpreted as $\sum_\omega (H^*_\omega) H^\top_\omega$ for many realizations of the random variable ω representing different configurations of the random medium. H_ω denotes the profile of H (viewed as a vector) for each realization of the random variable ω and this is constructed every time using the rSVD algorithm for H. The action of each of these $H_\omega (H^*_\omega)^\top$ on a Gaussian random vector is captured using the propagation model and finally averaged to compute an approximation for the action of A on a Gaussian random vector.

Algorithm S3. rSVD algorithm for reconstruction of A

while $i \leq I$ **do**
  Form H_i using Algorithm S2 for a realization of random medium i
  $\Omega := n^2 \times k_2 + p_2$ Gaussian random matrix.
  Compute $Y_{i,n\times k_2+p_2} = H_i H_i^* \Omega$
  Find the average: $\sum_i Y_{i,n\times k_2+p_2}/I$
$Q_{n\times k_2+p_2} :=$ QR decomposition of Y
while $i \leq I$ **do**
  Form H_i using Algorithm S2 for a realization of random medium i
  $B_{k+p\times k_2+p_2} = Q^* H_i H_i^* Q$
  Find the average: $\sum_i Y_{i,n\times k_2+p_2}/I$
Find the eigenvalue decomposition of the small matrix $B : B = U\Sigma U^*$
$\tilde{U} = QU$
$A \approx \tilde{U}\Sigma\tilde{U}^*$

B. Computing $H\Omega$ and the adjoint equation

To implement the rSVD algorithm, we need an efficient way to compute the matrix vector product $H\Omega$, where Ω is a random vector. From the definition of H,

$$H\Omega = \sum_{X_2} H(X_1, X_2)\Omega(X_2) = \int\limits_{X'\in\mathcal{R}} h(X_1, X') \sum_{X_2} h^*(X_2, X')\Omega(X_2) \tag{S51}$$

Note that $\tilde{\phi}(X') = \sum\limits_{X_2} h^*(X_2, X')\Omega(X_2) = \big(\sum\limits_{X_2} h(X_2, X')\Omega(X_2)^*\big)^*$ denotes the conjugate of the solution to the propagtion model with a source given by Ω^*, at a point X' on the receiver. The action of $h(X_1, X')$ on this term is equivalent to solving the adjoint solution to the propagtion model with a source given by $\sum_{X_2} h^*(X_2, X')\Omega(X_2)$. Finally we can do an empirical average over many such realizations to compute the expectation over the random medium $\mathbb{E}[\cdot]$.

Suppose we use the paraxial wave equation(PWE) as the propagation model. Then the complex amplitude of the signal $A(X,z)$ is given by:

$$\begin{aligned} \nabla_X^2 A + 2ik\partial_z A + k^2 VA = 0 \\ A(X,0) = \phi(X)\,. \end{aligned} \tag{S52}$$

Here, $V = n^2 - 1$, where n is the random refractive index of the medium. The propagator function $h(X_1, X)$ maps a delta function source located at the point $(X_1, 0)$ to the signal at a point (X, z). We introduce the z dependence on h as $h = h(X, X', z)$. Related discussion can be found in [7]. Now, for fixed point X_1 on the transmitter, $h(X_1, X, z)$ solves the PWE:

$$\begin{aligned} \nabla_X^2 h + 2ik\partial_z h + k^2 Vh = 0 \\ h(X_1, X, 0) = \delta(X - X_1) \end{aligned} \tag{S53}$$

Let $h_Z^\dagger(X, X_2, z)$ denote the solution to the adjoint equation. Drop the Z subscript for convenience. Conjecture is that $h(X_1, X_2, Z) = h^\dagger(X_1, X_2, 0)$, with the adjoint equation being

$$\begin{aligned} \nabla_X^2 h^\dagger - 2ik\partial_z h^\dagger + k^2 Vh^\dagger = 0 \\ h^\dagger(X, X_2, Z) = \delta(X - X_2) \end{aligned} \tag{S54}$$

Proof: Multiply Eq. (S53) with $h^\dagger(X, X_2, z)$ and vice versa and subtracting gives

$$\begin{aligned} &h^\dagger(X, X_2, z)\nabla_\perp^2 h(X_1, X, z) - h(X_1, X, z)\nabla_\perp^2 h^\dagger(X, X_2, z) \\ &+ 2ik[h(X_1, X, z)\partial_z h^\dagger(X, X_2, z) + h^\dagger(X, X_2, z)\partial_z h(X_1, X, z)] = 0 \end{aligned} \tag{S55}$$

Integrating the above equation in X, z gives

$$\begin{aligned} &\int_{X\in\mathbb{R}^d} \int_{z=0}^{Z} h^\dagger(X, X_2, z)\nabla_\perp^2 h(X_1, X, z) - h(X_1, X, z)\nabla_\perp^2 h^\dagger(X, X_2, z) = \\ &- 2ik \int_{X\in\mathbb{R}^d} \int_{z=0}^{Z} [h(X_1, X, z)\partial_z h^\dagger(X, X_2, z) + h^\dagger(X, X_2, z)\partial_z h(X_1, X, z)] \end{aligned} \tag{S56}$$

The integral on the RHS reduces to

$$-2ik \int_{X\in\mathbb{R}^d} h(X_1, X, Z)h^\dagger(X, X_2, Z) - h(X_1, X, 0)h^\dagger(X, X_2, 0) = -2ik[h(X_1, X_2, Z) - h^\dagger(X_1, X_2, 0)]$$

Using Green's theorem, the integral on the LHS vanishes proving that

$$h(X_1, X_2, Z) = h^\dagger(X_1, X_2, 0)\,. \tag{S57}$$

C. Modified cost function

In order to find beams that give low values of scintillation but high enough received intensities, we consider a modified cost function of the form

$$\tilde{f}(J) = J^* AJ + \mu\mathcal{Q}(J)\,. \tag{S58}$$

Here, $\mathcal{Q}(J)$ denotes the square of ratio of intensity lost after propagation through the medium and the received intensity:

$$\mathcal{Q}(J) = \Big(\frac{I_0 - \mathbb{E}[I]}{\mathbb{E}[I]}\Big)^2 = \Big(\frac{\mathrm{Tr}(J) - \mathrm{Tr}(\mathbb{E}[H]J^*)}{\mathrm{Tr}(\mathbb{E}[H]J^*)}\Big)^2\,. \tag{S59}$$

Note that $\mathcal{Q}(J)$ is invariant to scalings of J, i.e, J and $\alpha J, \alpha > 0$ give the same value of $\mathcal{Q}$. For fixed received intensities, say $\mathbb{E}[I] = 1$, $\mathcal{Q}(J)$ takes the form of

$$\mathcal{Q}(J) = \big(\mathrm{Tr}(J) - 1\big)^2\,, \tag{S60}$$

which is convex in J.

D. Numerical examples

In this section, we discuss the numerical setup presented in the main article in detail. For the multiple phase screen model, we consider a two dimensional setup with reference frequency $k = 2\pi \times 10^6$ rad/m. We assume a transmitter region of $(-r, r)$ with $r = 0.04$ m and a point receiver placed at a distance of $Z = 2000$ m away from the transmitter. To simulate the statistics of the turbulent medium, we use the method in [8] with the power spectral density for each phase screen given by:

$$\Phi(K) = 0.023 r_0^{-5/3} \frac{\exp(-K^2/K_m^2)}{(K^2 + K_0^2)^{5/6}} \tag{S61}$$

with $C_n^2 = 10^{-13}$. The value of the Fried parameter for a plane wave source over the full sequence of phase screens is given by:

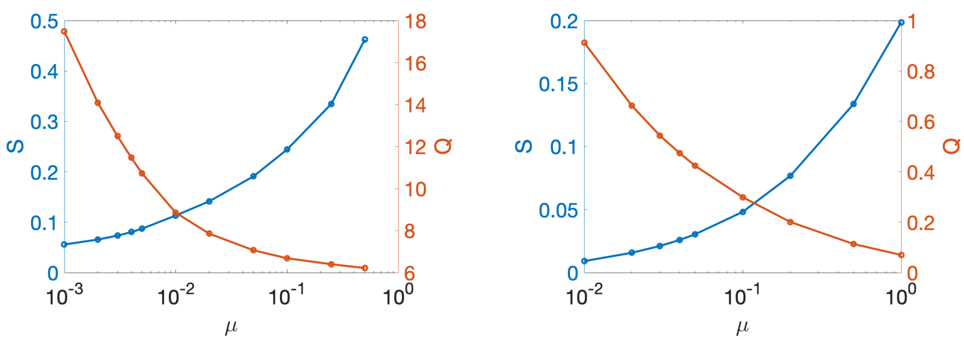
$$r_{0,pw} = (0.423 C_n^2 k^2 Z)^{-3/5} \tag{S62}$$

and is around 0.039 m in this example. We also use 15 phase screens placed at equal distances and use 50,000 Monte Carlo iterations to generate samples from the random medium. Discretization in X-direction $\Delta x = 0.002$ such that there are 40 grid points at the transmitter and the simulation takes place in a larger X domain $(-L, L)$ with $L = 2$ m.

As a second setup, we use the paraxial wave equation in two dimensions to simulate the field. Again, we set $k = 2\pi \times 10^6$ rad/m and use a point receiver. We use a transmitter size of 0.05 m and a propagation distance of $Z = 3000$ m in the z- direction. Suppose the random medium is given by

$$n^2 - 1 = V = \epsilon V_1 \sin(\omega_x x) \sin(\omega_z z), V_1 \sim \mathcal{N}(0, 1) \tag{S63}$$

with $\epsilon = 3.5 \times 10^{-8}, \omega_x = 2\pi/L, \omega_z = 10\pi/Z$ where $L = 0.8$ is the size of the larger simulation domain of X. We consider 15,000 Monte Carlo samples for the random medium. We also use $\Delta x = 0.002$ such that there are 50 grid point on the transmitter and $\Delta z = 93.75$ so that there are 33 grid point in z direction. For the convenience of the reader, we present a zoomed-in version of Fig. 4. from the main article.



(a) Phase screen model

(b) PDE model

Fig. S1. Scintillation $\mathcal{S}$ and intensity quotient $\mathcal{Q}$ of optimal J (zoomed-in version of Fig. 4. in main article). μ is presented on the log-scale.

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