Supplemental Material for "Energy Decompositions for Moist Boussinesq and Anelastic Equations with Phase Changes"

David H. Marsico, Leslie M. Smith, and Samuel N. Stechmann

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Energy Principle for the Kessler Scheme

In this section, we will derive the energy principle given in section 4a of the main text. First, we obtain an expression for the material derivative of Π. Using the fact that

$$\frac{D\Pi}{Dt} = \frac{\partial \Pi}{\partial b_u^{\text{tot}}}\frac{Db_u^{\text{tot}}}{Dt} + \frac{\partial \Pi}{\partial b_s^{\text{tot}}}\frac{Db_s^{\text{tot}}}{Dt} + \frac{\partial \Pi}{\partial z}\frac{Dz}{Dt}, \tag{1}$$

we have

$$\begin{aligned}\frac{D}{Dt}\Pi(b_u^{\text{tot}}, b_s^{\text{tot}}, z) = &-\left[(b_u^{\text{tot}} - \tilde{b}_u)H_u + (b_s^{\text{tot}} - \tilde{b}_s)H_s\right]\frac{Dz}{Dt} \\ &- \int_a^z \frac{\partial}{\partial b_u^{\text{tot}}}\left[(b_u^{\text{tot}} - \tilde{b}_u)H_u + (b_s^{\text{tot}} - \tilde{b}_s)H_s\right]dz'\,\frac{Db_u^{\text{tot}}}{Dt} \\ &- \int_a^z \frac{\partial}{\partial b_s^{\text{tot}}}\left[(b_u^{\text{tot}} - \tilde{b}_u)H_u + (b_s^{\text{tot}} - \tilde{b}_s)H_s\right]dz'\,\frac{Db_s^{\text{tot}}}{Dt}.\end{aligned} \tag{2}$$

The second and third terms on the right hand side of (2) can be simplified further:

$$\begin{aligned}&- \int_a^z \frac{\partial}{\partial b_u^{\text{tot}}}\left[(b_u^{\text{tot}} - \tilde{b}_u)H_u + (b_s^{\text{tot}} - \tilde{b}_s)H_s\right]dz' \\ &= -\int_a^z H_u + (b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u(z') - \tilde{b}_s(z')))\frac{\partial}{\partial b_u^{\text{tot}}}H_u\,dz' \\ &= -\int_a^z H_u\,dz' - \int_a^z (b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u(z') - \tilde{b}_s(z')))\frac{\partial}{\partial b_u^{\text{tot}}}H_u\,dz' \\ &= -\int_a^z H_u\,dz'\end{aligned} \tag{3}$$

where we have used the fact that

$$\int_a^z (b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u(z') - \tilde{b}_s(z')))\frac{\partial}{\partial b_u^{\text{tot}}}H_u\,dz' = 0$$

A similar procedure yields

$$- \int_a^z \frac{\partial}{\partial b_s^{\text{tot}}}\left[(b_u^{\text{tot}} - \tilde{b}_u)H_u + (b_s^{\text{tot}} - \tilde{b}_s)H_s\right]dz' = -\int_a^z H_s\,dz'. \tag{4}$$

We can integrate the terms $\int_a^z H_u\,dz'$ and $\int_a^z H_s\,dz'$ that appear (3) and (4) exactly. Using an integration by parts we have:

$$
\begin{aligned}
-\int_a^z H_u\,dz' &= -zH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u(z') - \tilde{b}_s(z')))\Big|_a^z \\
&\quad - \int_a^z z'\left(N_u^2 - N_s^2\right)(z')\delta(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(z'))\,dz' \\
&= -zH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(z)) + aH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) \\
&\quad - \int_{-\infty}^{\infty} \big[H(z'-a) - H(z'-z)\big] z'(N_u^2 - N_s^2)(z')\delta(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(z'))\,dz' \\
&= -zH_u + aH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) - z_rH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) + z_rH_u.
\end{aligned}
\tag{5}
$$

A similar procedure results in

$$
\begin{aligned}
-\int_a^z H_s\,dz' = -(z-a) + zH_u - aH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) \\
+ z_rH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) - z_rH_u.
\end{aligned}
\tag{6}
$$

We have thus shown, so far, that

$$
\begin{aligned}
-\int_a^z \frac{\partial}{\partial b_u^{\text{tot}}}\left[(b_u^{\text{tot}} - \tilde{b}_u)H_u + (b_s^{\text{tot}} - \tilde{b}_s)H_s\right]dz' = -zH_u + aH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) \\
- z_rH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) + z_rH_u
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
-\int_a^z \frac{\partial}{\partial b_s^{\text{tot}}}\left[(b_u^{\text{tot}} - \tilde{b}_u)H_u + (b_s^{\text{tot}} - \tilde{b}_s)H_s\right]dz' = -(z-a) + zH_u - aH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) \\
+ z_rH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) - z_rH_u,
\end{aligned}
\tag{8}
$$

in which case,

$$
\begin{aligned}
&-\int_a^z \frac{\partial}{\partial b_u^{\text{tot}}}\left[(b_u^{\text{tot}} - \tilde{b}_u)H_u + (b_s^{\text{tot}} - \tilde{b}_s)H_s\right]dz'\,\frac{Db_u^{\text{tot}}}{Dt} \\
&= \Big[-zH_u + aH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) - z_rH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) + z_rH_u\Big] \\
&\quad \left(\frac{g}{\tilde{\rho}(z)}\left(R_{vd} - \frac{L_v}{c_p\theta_0}\right)\frac{\partial}{\partial z}(\tilde{\rho}(z)V_Tq_r) - g\left(R_{vd} - \frac{L_v}{c_p\theta_0} + 1)\right)S_r\right),
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
&-\int_a^z \frac{\partial}{\partial b_s^{\text{tot}}}\left[(b_u^{\text{tot}} - \tilde{b}_u)H_u + (b_s^{\text{tot}} - \tilde{b}_s)H_s\right]dz'\,\frac{Db_s^{\text{tot}}}{Dt} = \\
&-\Big[-(z-a) + zH_u - aH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) + z_rH(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) - z_rH_u\Big] \\
&\quad \frac{g}{\tilde{\rho}(z)}\frac{\partial}{\partial z}(\tilde{\rho}(z)V_Tq_r),
\end{aligned}
\tag{10}
$$

where $S_r = A_r + C_r - E_r$.

Combining (10) and (9), and a great deal of simplifying we obtain the material derivative of Π:

$$\frac{D}{Dt}\Pi(b_u^{\rm tot}, b_s^{\rm tot}, z) = -\big[(b_u^{\rm tot} - \tilde{b}_u)H_u + (b_s^{\rm tot} - \tilde{b}_s)H_s\big]w + (z-a)\frac{gV_T}{\tilde{\rho}(z)}\frac{\partial}{\partial z}(\tilde{\rho}(z)q_r)$$
$$-g(z-z_r)\left(R_{vd} + \frac{L_v}{c_p\theta_0} + 1\right)\left[\frac{1}{\tilde{\rho}(z)}\frac{\partial}{\partial z}\left(\tilde{\rho}(z)V_T q_r\right) - S_r\right]H_u \tag{11}$$
$$-g(z_r - a)\left(R_{vd} - \frac{L_v}{c_p\theta_0} + 1\right)\left[\frac{1}{\tilde{\rho}(z)}\frac{\partial}{\partial z}\left(\tilde{\rho}(z)V_T q_r\right) - S_r\right]H(b_u^{\rm tot} - b_s^{\rm tot} - (\tilde{b}_u - \tilde{b}_s)(a))$$

Using the momentum equations, it is straightforward to verify that

$$\tilde{\rho}(z)\frac{D}{Dt}\left(\frac{1}{2}|\mathbf{u}|^2\right) = -\tilde{\rho}(z)\nabla\phi + \tilde{\rho}(z)w(b_uH_u + b_sH_s). \tag{12}$$

Adding $\tilde{\rho}(z)D\Pi/Dt$ to (12), and using incompressibility, we obtain the desired equation:

$$\frac{\partial}{\partial t}\left(\tilde{\rho}(z)\left(\frac{|\mathbf{u}|^2}{2} + \Pi\right)\right) + \nabla\cdot\left[\tilde{\rho}(z)\mathbf{u}\left(\frac{|\mathbf{u}|^2}{2} + \Pi + \phi\right)\right] = g(z-a)\frac{\partial}{\partial z}(V_T\tilde{\rho}(z)q_r)$$
$$-g(z-z_r)\left(R_{vd} - \frac{L_v}{c_p\theta_0} + 1\right)\left[\frac{\partial}{\partial z}\left(V_T\tilde{\rho}(z)q_r\right) - \tilde{\rho}(z)S_r\right]H_u \tag{13}$$
$$-g(z_r - a)\left(R_{vd} - \frac{L_v}{c_p\theta_0} + 1\right)\left[\frac{\partial}{\partial z}\left(V_T\tilde{\rho}(z)q_r\right) - \tilde{\rho}(z)S_r\right]H(b_u^{\rm tot} - b_s^{\rm tot} - (\tilde{b}_u - \tilde{b}_s)(a)).$$

Integrating Π for the Anelastic Equations

In this appendix, we provide the details of the decomposition of Π described in section 3b of the main text. We integrate the term $\int_a^z (b_u^{\rm tot} - \tilde{b}_u)H_u\,dz'$ by parts:

$$-\int_a^z (b_u^{\rm tot} - \tilde{b}_u(z'))H_u\,dz' = -H(b_u^{\rm tot} - b_s^{\rm tot} - (\tilde{b}_u - \tilde{b}_s)(z'))\int_a^{z'}(b_u^{\rm tot} - \tilde{b}_u(s))\,ds\Big|_{z'=a}^{z'=z}$$
$$-\int_a^z (N_u^2(z') - N_s^2(z'))\delta(b_u^{\rm tot} - b_s^{\rm tot} - (\tilde{b}_u - \tilde{b}_s)(z'))\int_a^{z'}(\tilde{b}_u^{\rm tot} - \tilde{b}_u(s))\,ds\,dz'. \tag{14}$$

The first term in (14) is straightforward to evaluate:

$$-H(b_u^{\rm tot} - b_s^{\rm tot} - (\tilde{b}_u - \tilde{b}_s)(z'))\int_a^{z'}(b_u^{\rm tot} - \tilde{b}_u(s))\,ds\Big|_{z'=a}^{z'=z}$$
$$= -H(b_u^{\rm tot} - b_s^{\rm tot} - (\tilde{b}_u - \tilde{b}_s))\int_a^{z}(b_u^{\rm tot} - \tilde{b}_u(s))\,ds \tag{15}$$

For the second term in (14), we multiply the integrand by the characteristic function of $[a, z]$ and integrate over the entire real line. Doing this ensures that the zero of the delta function's argument lies in the region of integration.

$$-\int_a^z (N_u^2(z') - N_s^2(z'))\delta(b_u^{\rm tot} - b_s^{\rm tot} - (\tilde{b}_u - \tilde{b}_s)(z'))\int_a^{z'}(\tilde{b}_u^{\rm tot} - \tilde{b}_u(s))\,ds\,dz' =$$
$$-\int_{-\infty}^{\infty}\Bigg[\big[H(z'-a) - H(z-z')\big](N_u^2(z') - N_s^2(z')) \tag{16}$$
$$\delta(b_u^{\rm tot} - b_s^{\rm tot} - (\tilde{b}_u - \tilde{b}_s)(z'))\int_a^{z'}(\tilde{b}_u^{\rm tot} - \tilde{b}_u(s))\,ds\Bigg]\,dz'$$

Now make the substitution $u = b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(z')$. Then $du = -(N_u^2(z') - N_s^2(z'))\,dz'$,

$$\lim_{z'\to\infty} b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(z')) = -\infty, \tag{17}$$

$$\lim_{z'\to-\infty} b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(z')) = \infty, \tag{18}$$

and the integral on the right hand side of (16) becomes

$$\begin{gathered} -\int_{-\infty}^{\infty} \Big[\big[H(z'-a) - H(z-z')\big](N_u^2(z') - N_s^2(z')) \\ \delta(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(z')) \int_a^{z'} (\tilde{b}_u^{\text{tot}} - \tilde{b}_u(s))\,ds \Big] dz' = \\ \int_{\infty}^{-\infty} \Big(H((\tilde{b}_u - \tilde{b}_s)^{-1}(u') - a) - H((\tilde{b}_u - \tilde{b}_s)^{-1}(u') - z) \Big)\, \delta(u) \int_a^{(\tilde{b}_u - \tilde{b}_s)^{-1}(u')} (b_u^{\text{tot}} - \tilde{b}_u(s))\,ds\,du, \end{gathered} \tag{19}$$

where $u' = b_u^{\text{tot}} - b_s^{\text{tot}} - u$. The integral on the right hand side can be evaluated by setting $u = 0$, yielding

$$\begin{gathered} \int_{\infty}^{-\infty} \Big[\Big(H((\tilde{b}_u - \tilde{b}_s)^{-1}(u') - a) - H((\tilde{b}_u - \tilde{b}_s)^{-1}(u') - z) \Big)\, \delta(u) \\ \int_a^{(\tilde{b}_u - \tilde{b}_s)^{-1}(u')} (b_u^{\text{tot}} - \tilde{b}_u(s))\,ds \Big] du = \\ -\big[H((\tilde{b}_u - \tilde{b}_s)^{-1}(b_u^{\text{tot}} - b_s^{\text{tot}}) - a) - H((\tilde{b}_u - \tilde{b}_s)^{-1}(b_u^{\text{tot}} - b_s^{\text{tot}}) - z))\big] \times \\ \int_a^{(\tilde{b}_u - \tilde{b}_s)^{-1}(b_u^{\text{tot}} - b_s^{\text{tot}})} (b_u^{\text{tot}} - \tilde{b}(s))\,ds = \\ -\big[H(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) - H((b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(z))\big] \int_a^{z_r} (b_u^{\text{tot}} - \tilde{b}(s))\,ds \end{gathered} \tag{20}$$

where $z_r = (\tilde{b}_u - \tilde{b}_s)^{-1}(b_u^{\text{tot}} - b_s^{\text{tot}})$, and we have used the facts that

$$H((\tilde{b}_u - \tilde{b}_s)^{-1}(b_u^{\text{tot}} - b_s^{\text{tot}}) - a) = H(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) \tag{21}$$

and

$$H((\tilde{b}_u - \tilde{b}_s)^{-1}(b_u^{\text{tot}} - b_s^{\text{tot}}) - z) = H(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(z)). \tag{22}$$

To see why equations (21) and (22) are true, note that $\tilde{b}_u - \tilde{b}_s$, and $(\tilde{b}_u - \tilde{b}_s)^{-1}$ are both monotone increasing functions. A consequence of this is that $(\tilde{b}_u - \tilde{b}_s)^{-1}(b_u^{\text{tot}} - b_s^{\text{tot}}) - z \geq 0$ if and only if $b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(z) \geq 0$, which ensures that $H((\tilde{b}_u - \tilde{b}_s)^{-1}(b_u^{\text{tot}} - b_s^{\text{tot}}) - z) = H(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(z))$. Therefore,

$$\begin{aligned} -\int_a^z (b_u^{\text{tot}} - \tilde{b}_u(z'))H_u\,dz' = -H_u \int_a^z (b_u^{\text{tot}} - \tilde{b}_u(z'))\,dz' + H_u \int_a^{z_r} (b_u^{\text{tot}} - \tilde{b}_u(z'))\,dz' \\ - H(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) \int_a^{z_r} (b_u^{\text{tot}} - \tilde{b}(z'))\,dz' \end{aligned} \tag{23}$$

A similar procedure yields

$$\begin{aligned} -\int_a^z (b_s^{\text{tot}} - \tilde{b}_s(z'))H_s\,dz' = -H_s \int_a^z (b_s^{\text{tot}} - \tilde{b}_s(z'))\,dz' - H_u \int_a^{z_r} (b_s^{\text{tot}} - \tilde{b}_s(z'))\,dz' \\ + H((b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) \int_a^{z_r} (b_s^{\text{tot}} - \tilde{b}_s(z'))\,dz' \end{aligned} \tag{24}$$

Adding these two expression and making further simplifications we obtain

$$
\begin{aligned}
\Pi = &-H_u \int_a^z (b_u^{\text{tot}} - \tilde{b}_u(z'))\, dz' - H_s \int_a^z (b_s^{\text{tot}} - \tilde{b}_s(z'))\, dz' \\
&+ H_u \int_a^{z_r} (b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u(z') - \tilde{b}_s(z'))\, dz' \\
&- H(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) \int_a^{z_r} (b_u^{\text{tot}} - \tilde{b}_u(z'))\, dz' \\
&+ H(b_u^{\text{tot}} - b_s^{\text{tot}} - (\tilde{b}_u - \tilde{b}_s)(a)) \int_a^{z_r} (b_s^{\text{tot}} - \tilde{b}_s(z'))\, dz'.
\end{aligned}
\tag{25}
$$