



RESEARCH ARTICLE

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Moisture Decomposition With Normal Modes in Global Data: Balanced and Unbalanced Components

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Key Points:

- A balanced-unbalanced decomposition of water vapor is introduced and applied to ERA5 reanalysis data
- Balanced and unbalanced contributions to moisture variance are examined for different regions and different spatial scales
- Balanced moisture variance has significant contributions from both the Rossby/vortical mode and an additional balanced moist mode

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Abstract Decompositions with normal modes have been useful for numerous purposes, such as theoretical understanding of balanced and unbalanced circulations, and applications to data assimilation. However, normal mode decompositions have typically been formulated for dry dynamics without moisture. Here, a normal mode decomposition is investigated, including an eigenmode decomposition of moisture, on the sphere. One main aim is to investigate how much moisture is associated with each of the different eigenmodes, for different latitudes and different spatial scales. Another main aim is to investigate the additional moist eigenmode that is not present in a dry system. The moist decomposition was applied to ERA5 reanalysis data for August 2018. The results of this decomposition suggest that, on sub-synoptic scales, the balanced component of moisture has the largest contribution to the variance, in contrast to velocity and temperature which are known to be dominated by unbalanced inertia-gravity wave mode contributions on sub-synoptic scales. Furthermore, on sub-synoptic scales, among the two balanced eigenmodes of moisture (Rossby/vortical mode and slow moist eigenmode), it is the slow moist eigenmode that has the dominant contribution. Lastly, a case study is examined of an extreme rainfall event in the extratropics over North America, and the moisture anomalies on synoptic scales have appreciable contributions from both the Rossby/vortical mode and slow moist eigenmode. As other topics, a possible extension is discussed on including cloud latent heating into the eigenmode definitions in future work, and the vertical basis functions for moisture are connected with a Sturm–Liouville problem and orthogonality condition.

Plain Language Summary In order to better understand atmospheric dynamics, it is often useful to decompose atmospheric quantities into different components. A commonly used decomposition is a slow–fast or balanced–unbalanced decomposition. This decomposition is typically performed with an approximation that the atmosphere is “dry” in order to simplify the decomposition. Here it is shown how moisture can be included in this type of framework for global atmospheric decomposition. In the moisture decomposition, an additional physically relevant piece arises: an additional balanced component that arises in moist systems. This decomposition is then applied to ERA5 reanalysis data, a global atmospheric data set, during the month of August 2018. An extreme rainfall event during this time is also examined using this decomposition, to analyze whether the moisture variability is mainly in the balanced or unbalanced components. It is found that the additional balanced portion of the moisture makes an appreciable contribution, in concert with other balanced and unbalanced components. A better understanding of moisture variability, as well as the role of its balanced and unbalanced components, can help us to better understand and predict atmospheric dynamics.

1. Introduction

Decompositions, such as normal mode function (NMF) decompositions, have been useful for numerous applications in atmospheric science. Applications include analysis of observational or model data (e.g., Hoskins et al., 1985; Žagar, Kasahara, et al., 2015), data assimilation (e.g., Bannister, 2021; Lorenc, 1981; Žagar et al., 2004), predictability (e.g., Žagar, Buizza, & Tribbia, 2015), and nonlinear wave interactions (e.g., Ferguson et al., 2009; Gottwald & Grimshaw, 1999; Majda & Biello, 2003). Moreover, decompositions are applied for a variety of scales and regions, including globally (e.g., Kasahara & Puri, 1981; Madden, 1979; Žagar, 2017), regionally (e.g., Bretherton & Smolarkiewicz, 1989; Fulton & Schubert, 1985; Stechmann & Majda, 2009), in the tropics (e.g., Back & Bretherton, 2009; Gill, 1980; Khouider et al., 2012; Mapes, 1993), and in the midlatitudes/extratropics (e.g., Daley, 1993; Molemaker et al., 2005).

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However, decompositions are commonly applied to the dry portions (e.g., velocity, temperature) of the atmospheric state, and not to moisture. Given that slow-fast and NMF decompositions have been fundamental for understanding many concepts of atmospheric dynamics, it would be desirable to have similar decompositions that include moisture and moist processes. To apply NMF decompositions to moisture and moist processes, it would be desirable to put dry and moist variables on the same footing, in the form of moist dynamics and its associated eigenmodes.

Fundamental understanding of moisture variability can potentially be gained via NMF decompositions, to help fill the gap in understanding between the moist and dry variables of the atmosphere. For example, it is well-known that, in the extratropics, the dry atmospheric variables are approximately in geostrophically balanced modes on planetary and synoptic scales, and they are mostly in divergent, inertia-gravity (IG) wave modes on sub-synoptic scales. This well-known fact can be seen, for instance, by decomposing kinetic energy in rotational and divergent components in limited-area or global domains (e.g., Blažica et al., 2013; Burgess et al., 2013), and is illustrated in Figure 7 of Žagar, Kasahara, et al. (2015) using a global normal mode decomposition of reanalysis data. Furthermore, it is a slow-fast decomposition: the geostrophically balanced circulations represent a slowly evolving portion of atmospheric motions, and the IG wave modes represent a fast evolution of atmospheric motions. On the other hand, such properties are unknown for moisture.

The main aim of this study is to investigate the concepts from the previous paragraph, in reanalysis data, now for moisture. What are the characteristics of moisture and its slow and fast components, or balanced and unbalanced components? How do these characteristics vary on different scales, and in different regions?

As an example of a practical application, the investigation here can potentially have important implications for data assimilation and forecasting. Understanding the concepts of balanced-unbalanced decompositions can have an impact on forecast accuracy, and can sometimes even change a forecast from a failure to a success (Daley, 1993; Kalnay, 2003; Platzman, 1967; Richardson, 1922). However, past work has largely focused on a dry dynamics perspective, without a decomposition arising from moist dynamics. In a recent investigation of moist dynamics and data assimilation, Ogrosky et al. (2025) found that, by using a moist eigenmode decomposition, errors in data assimilation were reduced. To put these concepts into practice, a better understanding is needed of a balanced-unbalanced decomposition of moisture. Such a moisture decomposition is the main aim here, and in Section 6 additional applications are discussed for other possible future work on moisture variability.

We note that it is sometimes not precisely true that the Rossby-IG decomposition is also a slow-fast decomposition. On the f -plane, the Rossby-IG decomposition is, indeed, also a slow-fast decomposition. On the sphere, however, this is not precisely the case, for several reasons. First, the Kelvin and mixed Rossby-gravity (MRG) modes do not have a clear characterization as only slow or only fast. Second, some large-scale IG modes can be slow, or even quasi-stationary, especially in the tropics (Žagar, Kasahara, et al., 2015). The linear decomposition considered here also projects a part of the midlatitude, slow ageostrophic flows onto IG modes, otherwise considered as a part of the balanced dynamics (Stephan et al., 2021; Žagar et al., 2023). Third, if clouds and phase changes of water are present, the IG component can include some portions that are slow, even on an f -plane (Remond-Tiedrez et al., 2026; Zhang et al., 2021). Despite these three examples of caveats which should be kept in mind, it is nevertheless convenient to label groups of modes as slow or fast for simplicity of exposition.

Some past work has approached moist phenomena using eigenmode decompositions, typically using dry eigenmodes but sometimes using moist eigenmodes. By adapting or modifying the dry linear eigenmodes, some studies have analyzed observational and model data for understanding moisture-coupled phenomena such as convectively coupled equatorial waves (Gehne & Kleeman, 2012; Holube et al., 2025; Knippertz et al., 2022; Marques & Castanheira, 2018; Ogrosky & Stechmann, 2016; Yang et al., 2003), tropical intraseasonal oscillations such as the Madden-Julian Oscillation (MJO) (Žagar & Franzke, 2015), and the Walker circulation (Stechmann & Ogrosky, 2014). Moist eigenmodes, obtained by linearization of moist dynamics, have also been used for analyzing observational and model data for the MJO (Ogrosky & Stechmann, 2015; Stechmann & Majda, 2015), and for asymptotic analysis of weakly nonlinear wave interactions (Chen et al., 2015, 2016). Balanced and unbalanced moisture decompositions, in a midlatitude/extratropical setting, have been investigated with a type of linearization that applies with clouds present (Wetzel et al., 2019, 2020). Applications to data

assimilation have also investigated linear eigenmodes that decompose moisture into its balanced and unbalanced components (Ogrosky et al., 2025). Recently a decomposition has been introduced that does not rely on linearization, so that it may potentially include a more faithful representation of the nonlinearity and the physics associated with clouds. In particular, Remond-Tiedrez et al. (2026) introduce a nonlinear eigenspace decomposition that includes the effects of cloud latent heating.

Also, note a distinction in that there is a long history of theoretical studies that identify eigenmodes in moist dynamical models, typically with the aim of identifying unstable versus damped eigenmodes (e.g., Adames & Ming, 2018; Emanuel et al., 1987; Khouider & Majda, 2006; Kohl & O’Gorman, 2022; Mapes, 2000; Neelin & Yu, 1994; Stechmann & Hottovy, 2017). These theoretical studies are aimed at understanding a variety of observed phenomena. However, there appears to be less work (as in the relatively small number of existing examples mentioned in previous paragraphs) in applying moist eigenmodes for the purpose of analyzing and decomposing observational data. This latter purpose is the focus of the present paper.

In setting up the investigation here, different types of moisture effects can potentially be included into the definition of the moist eigenmodes. A linear effect is the vertical advection of background moisture. Nonlinear effects include cloud latent heating, precipitation, etc. While it would be desirable to include as many effects as possible, the presence of complicated processes will make the decompositions complicated. With these considerations in mind, here we investigate a linear moisture evolution equation for defining the eigenmodes. While the moisture evolution equation is linear, many non-trivial properties arise—such as a non-diagonal inner product and an additional slow eigenmode associated with moisture—so it is a non-trivial step beyond the dry case.

Lastly, we note that the small amount of past work on moisture decompositions has been in regional settings, such as a midlatitude f -plane or β -plane or an equatorial β -plane. Here, we formulate a moist multivariate decomposition on the sphere, and apply it to global reanalysis data. The global setting is valuable for numerous reasons; for instance, it allows a seamless view across different regions (tropics, sub-tropics, midlatitudes, and polar regions), and it is the natural setting for data that is available globally. To the author’s knowledge, this is the first decomposition of moisture into its balanced and unbalanced components on the sphere.

In what follows, Section 2 reviews the normal mode decomposition for the global atmosphere in a dry setting. Section 3 presents the normal mode decomposition for moist dynamics. Sections 4 and 5 show the decomposition of moisture into different eigenmode contributions in global reanalysis data. Section 4 presents statistical features, and Section 5 describes individual snapshots from a case study. A concluding discussion is in Section 6.

2. Normal Mode Decomposition for a Dry Global Atmosphere

To set the stage for the moisture decomposition in Section 3, we will begin here by briefly reviewing the derivation of the normal mode functions for a dry system in the pressure vertical coordinate in Section 2.1, and how to do projections of data onto these normal mode functions for discrete data sets in Section 2.2. We will then review the current treatment of the vertical velocity ω within this system in Section 2.3.1, and then in Section 2.3.2 we describe the Sturm–Liouville problem that is satisfied by the vertical basis functions of the vertical velocity.

2.1. Dry Normal Modes

We consider the normal modes of the linearized, hydrostatic, adiabatic, inviscid evolution equations for a global atmosphere on a rotating sphere. The normal mode functions for a dry system in the pressure vertical coordinate were described by Tanaka (1985). For other in-depth derivations, see also Kasahara (2020), Žagar and Tribbia (2020), and especially Žagar et al. (2023) as the notation we will use also closely follows the notation used in that work.

In a pressure coordinate system the linear evolution equations are given by

$$\frac{\partial u}{\partial t} - 2\Omega \sin(\varphi) v + \frac{1}{a \cos \varphi} \left(\frac{\partial \phi}{\partial \lambda} \right) = 0, \quad (1a)$$

$$\frac{\partial v}{\partial t} + 2\Omega \sin(\varphi) u + \frac{1}{a} \left(\frac{\partial \phi}{\partial \varphi} \right) = 0, \quad (1b)$$

$$\frac{\partial}{\partial t} \left(\frac{1}{\Gamma_o} \left(\frac{\partial \phi}{\partial p} \right) \right) + \omega = 0, \quad (1c)$$

$$\frac{\partial \omega}{\partial p} + \nabla \cdot \mathbf{V} = \frac{\partial \omega}{\partial p} + \frac{1}{a \cos \varphi} \left(\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \varphi} (v \cos \varphi) \right) = 0. \quad (1d)$$

Here, Equations 1a and 1b are the evolution equations for the zonal and meridional velocities, u and v . Equation 1c describes the evolution of the geopotential ($\phi = gh$ where h is the geopotential height) (i.e., this is the thermodynamic equation), and Equation 1d is the continuity equation expressed in pressure coordinates with $\mathbf{V} = (u, v)$. The vertical velocity ω is the pressure velocity. The constants Ω , a , g , R , and c_p are the rotation rate of Earth, radius of Earth, average acceleration due to gravity on the surface of Earth, gas constant for dry air, and the specific heat capacity at constant pressure of dry air, respectively. The coordinates in this system of equations are longitude (λ), latitude (φ), time (t), and the vertical coordinate is given by pressure (p). The function $\Gamma_o(p)$ is the static stability parameter defined as

$$\Gamma_o(p) = \frac{R}{p} \left(\frac{RT_{v,o}(p)}{c_p p} - \frac{dT_{v,o}(p)}{dp} \right), \quad (2)$$

where $T_{v,o}(p)$ is a globally averaged virtual temperature profile. Note that the evolution equations in Equation 1 are for perturbations about a static base state ($\phi_o(p), T_{v,o}(p)$) that satisfies the hydrostatic balance condition,

$$\frac{d\phi_o}{dp} = -\frac{RT_{v,o}(p)}{p}. \quad (3)$$

Finally, the boundary conditions for the surface ($p = p_s$) and the top ($p = p_t$) of the atmosphere are

$$\frac{\partial}{\partial t} \left(\frac{\partial \phi}{\partial p} + \frac{p^2 \Gamma_o}{R^2 T_o} \phi \right) = 0 \quad \text{at } p = p_s, \quad (4a)$$

$$\omega \phi \rightarrow 0 \quad \text{as } p \rightarrow 0, \quad (4b)$$

which concludes the specification of the system. Note, the boundary condition specified in Equation 4b is a condition that guarantees conservation of energy since it requires that ϕ is bounded for all time. For more details on this boundary condition see Cohn and Dee (1989).

Note now that ω can be eliminated from Equations 1c and 1d by taking a partial derivative of Equation 1c with respect to p and subtracting Equation 1d. The result is in Equation 5c as

$$\frac{\partial u}{\partial t} - 2\Omega \sin(\varphi) v + \frac{1}{a \cos \varphi} \left(\frac{\partial \phi}{\partial \lambda} \right) = 0, \quad (5a)$$

$$\frac{\partial v}{\partial t} + 2\Omega \sin(\varphi) u + \frac{1}{a} \left(\frac{\partial \phi}{\partial \varphi} \right) = 0, \quad (5b)$$

$$\frac{\partial}{\partial t} \frac{\partial}{\partial p} \left(\frac{1}{\Gamma_o} \left(\frac{\partial \phi}{\partial p} \right) \right) - \nabla \cdot \mathbf{V} = 0, \quad (5c)$$

where Equations 5a and 5b are the evolution equations for the two horizontal velocity components, which are carried along unchanged. Hence the system of equations has been reduced to a set of evolution equations for u , v , and ϕ alone.

We now perform a separation of variables in the form of

$$(u, v, \phi)^T(\lambda, \varphi, p, t) = G(p) \times (u', v', gh')^T(\lambda, \varphi, t), \quad (6)$$

where $G(p)$ is the vertical structure function (VSF). To proceed, it is helpful to note that Equation 5c is the only one of the three evolution equations that contains a pressure derivative. This means that G is easily decoupled from Equations 5a and 5b, and then we obtain the following second-order ODE for G from Equation 5c:

$$\frac{1}{G} \frac{d}{dp} \left(\frac{1}{\Gamma_o} \frac{dG}{dp} \right) = \frac{-1}{gD}, \quad (7)$$

where $-1/gD$ is the separation constant (e.g., Staniforth et al., 1985). The notation D is chosen because D is an equivalent depth, with units of length. For the values of D which are eigenvalues, which we denote D_m (i.e., $D = D_m$), the corresponding G which satisfies Equation 7 is denoted G_m , and we say that m is the “vertical mode number.” We can also note that, because Equation 7 along with the boundary conditions given in 4 constitutes a Sturm-Liouville problem, the VSFs must be orthogonal in the following sense:

$$\frac{1}{p_s - p_t} \int_{p_t}^{p_s} G_i(p') G_j(p') dp' = \delta_{ij}, \quad (8)$$

where δ_{ij} is the standard Kronecker-delta.

The vertical modes are ordered such that

$$D_1 > D_2 > D_3 > \dots > D_M, \quad (9)$$

where the finite number of vertical modes, M , is associated with the vertical resolution of the computational implementation. The VSF $G_1(p)$, which corresponds with the largest equivalent depth D_1 , will not have any points at which it changes sign, and is referred to as the “external mode” or the Lamb mode. The remaining modes are referred to as “internal modes,” and the number of sign changes is $m - 1$ for a given vertical mode index m .

As the horizontal portion of the separation of variables, one finds from Equation 5c and Equation 6 the evolution equation for h' ,

$$\nabla \cdot \mathbf{V}'(\lambda, \varphi, t; m) = \frac{-1}{D_m} \frac{\partial}{\partial t} (h'(\lambda, \varphi, t; m)), \quad (10)$$

which is associated with Equation 7 through the same separation constant D_m . The notation $\mathbf{V}' = (u', v')$ is used here.

For two additional evolution equations for u' and v' , we note that G can be immediately canceled in Equations 5a and 5b using Equation 6. Hence we now have a system of three evolution equations for the three quantities (u', v', h') with (λ, φ, t) dependence for each vertical mode, m . We can then create a nondimensionalized version of the remaining Equations 5a, 5b, and 10 by defining the following rescaled variables:

$$\tilde{u} = \frac{u'}{\sqrt{gD}}, \quad \tilde{v} = \frac{v'}{\sqrt{gD}}, \quad \tilde{h} = \frac{h'}{D}, \quad \tilde{t} = 2\Omega t, \quad \tilde{\nabla} = a\nabla. \quad (11)$$

Using this rescaling, the set of nondimensionalized equations for $\tilde{u}$, $\tilde{v}$, and $\tilde{h}$ can be arranged into a form known as Laplace's tidal equations without forcing. In vector form, the equations are

$$\frac{\partial}{\partial \tilde{t}} \mathbf{W} + \mathbf{LW} = 0, \quad (12)$$

where the vector $\mathbf{W}$ and the linear operator $\mathbf{L}$ are given by

$$\mathbf{W} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{h} \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 0 & -\sin \varphi & \frac{\gamma}{\cos \varphi} \frac{\partial}{\partial \lambda} \\ \sin \varphi & 0 & \gamma \frac{\partial}{\partial \varphi} \\ \frac{\gamma}{\cos \varphi} \frac{\partial}{\partial \lambda} & \frac{\gamma}{\cos \varphi} \frac{\partial}{\partial \varphi} (\cos \varphi \cdot) & 0 \end{pmatrix}, \quad (13)$$

in which $\gamma = (gD)^{1/2}/(2a\Omega)$ is a nondimensional parameter. Note that we really have M copies of this set of equations to solve, one set for each vertical mode index m . As shown by Longuet-Higgins (1968), Kasahara (1976), and Swartrauber and Kasahara (1985), the solutions to the equation can be expressed as a sum of harmonics, where each harmonic has the form

$$\mathbf{W}_n^k(\lambda, \varphi, \tilde{t}) = e^{-i\tilde{\nu}_n^k \tilde{t}} \boldsymbol{\Theta}_n^k(\varphi) e^{ik\lambda}. \quad (14)$$

Here k and n are the zonal wavenumber and meridional index, and $\mathbf{H}_n^k(\lambda, \varphi) = \boldsymbol{\Theta}_n^k(\varphi) e^{ik\lambda}$ is known as the Hough harmonic (Hough, 1898). The components of $\boldsymbol{\Theta}_n^k$ are expressed as

$$\boldsymbol{\Theta}_n^k(\varphi) = (U_n^k(\varphi), iV_n^k(\varphi), Z_n^k(\varphi))^T, \quad (15)$$

where the factor of $i = \sqrt{-1}$ accounts for a phase difference of $V_n^k(\varphi)$ compared to $U_n^k(\varphi)$ and $Z_n^k(\varphi)$. The functions $\boldsymbol{\Theta}_n^k(\varphi)$ and the Hough Harmonics $\mathbf{H}_n^k(\varphi)$ also obey orthogonality conditions,

$$\int_{-1}^1 \boldsymbol{\Theta}_n^k(\varphi') \cdot (\boldsymbol{\Theta}_{n'}^{k'}(\varphi'))^* d\mu' = \delta_{kk'} \delta_{nn'}, \quad (16)$$

$$\frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^1 \mathbf{H}_n^k(\lambda', \varphi') \cdot (\mathbf{H}_{n'}^{k'}(\lambda', \varphi'))^* d\mu' d\lambda' = \delta_{kk'} \delta_{nn'}, \quad (17)$$

where $\mu' = \sin(\varphi')$ and $*$ denotes the complex conjugate. We can see that this second separation of variables, from Equation 14, can be plugged into Equation 12 to yield the eigenvalue equation,

$$\mathbf{LW} = i\tilde{\nu}\mathbf{W}, \quad (18)$$

with eigenvalues of $i\tilde{\nu}$. Since this is a 3×3 system there will be three separate eigenvalues for each choice of n , k , and m .

These three frequencies have been identified as the eastward inertio-gravity (EIG), westward inertio-gravity (WIG), and Rossby, or rotational (ROT), modes. To provide some intuition into the eigenvalues, one can derive analytical expressions under the assumption that D is large (Kasahara, 2020):

$$\tilde{\nu}_{ROT} \approx \frac{-k}{n(n+1)}, \quad (19a)$$

$$\tilde{\nu}_{IG} \approx \frac{-k}{2n(n+1)} \pm \left[\frac{k^2}{4n^2(n+1)^2} + \frac{n(n+1)}{\epsilon} \right]^{\frac{1}{2}}, \quad (19b)$$

where $\epsilon = 1/\gamma^2$ is the non-dimensional Lamb's parameter. The two inertio-gravity (IG) solutions in Equation 19b are the EIG and WIG solutions where they refer to the $+$ and $-$ parts of the $\pm$ in the solution, respectively. A sample of the frequencies for one equivalent depth is shown in Figure 1.

One noteworthy feature in Figure 1 is that the frequencies of the IG modes are higher than the ROT frequencies, that is, the IG modes are fast wave oscillations whereas the ROT modes are slow, low-frequency modes.

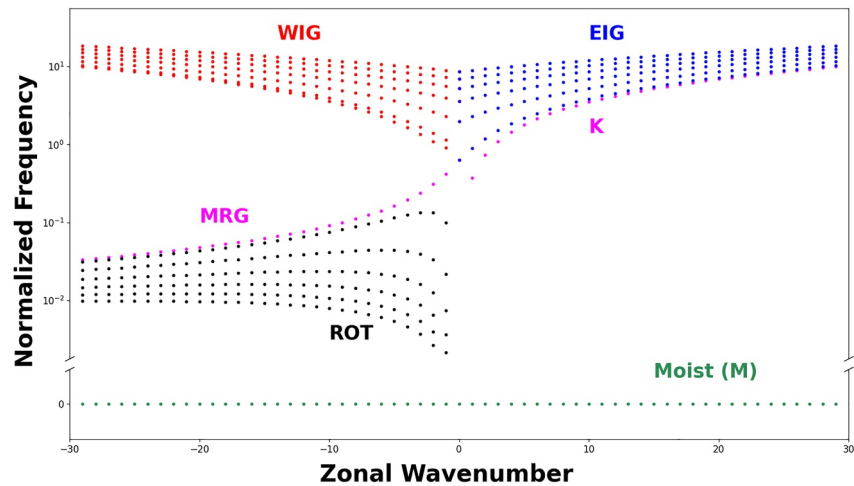


Figure 1. Dispersion curves of several eigenmodes on the sphere, plotted as normalized frequency versus zonal wavenumber. The WIG, EIG, and ROT frequencies (colored red, blue, and black respectively) are shown for various n and k combinations for the first equivalent depth $D_1 = 10$ km on a semi-log scale. Dispersion relation plots for different equivalent depths have the same qualitative features. The $n = 0$ ROT and EIG modes, known as the mixed Rossby–gravity and Kelvin modes, are shown in pink. The Moist eigenmode (colored green) arises in a moist system, and is a slow eigenmode with a frequency of zero, for all zonal wavenumbers. Adapted from Žagar, Kasahara, et al. (2015).

In Figure 1, while most of the modes have a clear classification as high-frequency or low-frequency, also notice that the $n = 0$ ROT and EIG modes are highlighted for their unique properties. These modes are known as the mixed Rossby–gravity (MRG) and Kelvin modes, respectively, and they play a central role in tropical dynamics as they represent an important subset of the equatorially trapped wave modes. A unique property of the MRG and Kelvin waves is that they do not fit into a binary classification as only slow or only fast, since these modes have frequencies that span the range from the low frequencies of ROT modes to the high frequencies of typical IG modes. It is sometimes appropriate, then, to give the MRG and Kelvin modes special consideration such as in studies of tropical dynamics. It would be interesting in the future to investigate a more nuanced viewpoint of the MRG and Kelvin modes. Here, in a global setting of global variability, we will use an approximate classification of the MRG and Kelvin modes as slow and fast, respectively, in accordance with their classification as the $n = 0$ ROT and EIG modes, respectively. Such a classification is in line with the bulk of the modes' flow being in vorticity (MRG) and divergence (Kelvin) (Neduh et al., 2024), and also in line with the bulk of these modes' frequencies illustrated in Figure 1. In what follows, the IG modes will be referred to as fast, high-frequency modes, and the ROT modes will be referred to as slow, low-frequency modes, in line with the broad aspects of the mode frequencies illustrated in Figure 1.

2.2. Projection of Data Onto the Normal Mode Functions

The orthogonality conditions from Equations 8 and 17 in the previous section allow us to decompose a state (u, v, h) into a sum of all of the (m, n, k) modes. As described by Kasahara and Puri (1981) for the terrain-following coordinate sigma and by Tanaka (1985) for the pressure vertical coordinate, this can be done for a discrete data set for a single time value in the following way.

First the VSFs are constructed by using for example, the finite difference method outlined in Kasahara and Puri (1981). Using their method, the discretized VSFs then satisfy a discrete version of the orthogonality condition,

$$\sum_{j=1}^J G_m(j) G_{m'}(j) = \delta_{mm'}, \quad (20)$$

where j is the vertical level index which takes the values $j = 1, 2, 3, \dots, J$. Then the state (u, v, h) can be projected onto vertical mode m to produce the quantity $\mathbf{X}_m(\lambda, \varphi)$:

$$\mathbf{X}_m(\lambda, \varphi) = (\tilde{u}_m, \tilde{v}_m, \tilde{h}_m)^T = \mathbf{S}_m^{-1} \sum_{j=1}^J (u, v, h)_j^T G_m(j), \quad (21)$$

$$\mathbf{S}_m = \begin{pmatrix} \sqrt{gD_m} & 0 & 0 \\ 0 & \sqrt{gD_m} & 0 \\ 0 & 0 & D_m \end{pmatrix}, \quad (22)$$

where $\mathbf{S}_m$ is an appropriate scaling matrix. Now, using the orthogonality condition of the Hough harmonics, the vector $\mathbf{X}_m(\lambda, \varphi)$ can be projected onto the Hough functions to compute the complex-valued “Hough coefficients” $\chi_n^k(m)$:

$$\mathbf{X}_m(\lambda, \varphi) = \sum_{n=1}^R \sum_{k=-K}^K \chi_n^k(m) \mathbf{H}_n^k(\lambda, \varphi; m), \quad (23)$$

$$\chi_n^k(m) = \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^1 \mathbf{X}_m(\lambda, \varphi) \cdot (\mathbf{H}_n^k(\lambda, \varphi; m))^* d\mu d\lambda, \quad (24)$$

where K represents the total number of zonal wavenumbers. The meridional index n takes values from 1 to R , where $R = N_E + N_W + N_R$ (where N_E, N_W and N_R are the number of EIG, WIG, and ROT modes) to account for the three solutions that we obtain for each (n, k, m) combination.

Using these definitions we can express our final decomposition of $\mathbf{X} = (u, v, h)^T$ as

$$\mathbf{X}(\lambda, \varphi, p_j) = \sum_{m=1}^M \mathbf{S}_m \mathbf{X}_m(\lambda, \varphi) G_m(j), \quad (25)$$

$$= \sum_{m=1}^M \sum_{n=1}^R \sum_{k=-K}^K \mathbf{S}_m \chi_n^k(m) \mathbf{H}_n^k(\lambda, \varphi; m) G_m(j). \quad (26)$$

This decomposition provides a useful tool for analyzing atmospheric dynamics and is included in the MODES software (Žagar, Kasahara, et al., 2015) used for data analysis in this study.

2.3. Modal Decomposition of the Vertical Velocity

During the decomposition procedure in Section 2.1, we eliminated one of the variables of the system: the vertical velocity ω . This was done to make the system of equations easier to solve.

However, it can also be valuable to have a decomposition for the vertical velocity. In Section 2.3.1 we will review a recently presented method for decomposing the vertical velocity (Žagar et al., 2023). Then, in Section 2.3.2, we review the Sturm–Liouville equation that is satisfied by the vertical basis functions for vertical velocity, $G_\omega(p)$, and also the associated orthogonality condition.

2.3.1. Vertical Velocity Decomposition via Horizontal Divergence

We now review how to decompose ω using the modes that were presented in the previous sections. We review the approach of Žagar et al. (2023) which utilizes the horizontal divergence (Neduhel et al., 2024).

By noting that ω is related to the horizontal wind velocities through the continuity equation, we can derive a relationship between h' and ω using Equation 10:

$$\frac{1}{D_m} \left(\frac{\partial h'}{\partial t} \right) G = -\nabla \cdot \mathbf{V} = \frac{\partial \omega}{\partial p} \quad (27)$$

Now, if we integrate both sides of Equation 27 from the top of the atmosphere, and use our original separation of variables for h' , from Equation 14, we find that for an arbitrary mode (m, n, k) we have:

$$\omega_n^k(\lambda, \varphi, p; m) = -i\nu_n^k e^{i(k\lambda - \nu_n^k t)} \chi_n^k(m) Z_n^k(\varphi; m) \int_0^p G_m(p') dp', \quad (28)$$

where Z_n^k is the component of Θ_n^k which corresponds with $\tilde{h}$.

The expression in Equation 28 gives the (m, n, k) modes for omega, which also have EIG, WIG, and ROT components, and correspond with the modes previously found for $\tilde{u}$, $\tilde{v}$, and $\tilde{h}$. Similar to the other modes, ω can be written as a sum of all of these modes as

$$\omega(\lambda, \varphi, p) = -i \sum_{m=1}^M \sum_{n=1}^R \sum_{k=-K}^K \bar{p}_s \nu_n^k e^{i(k\lambda - \nu_n^k t)} \chi_n^k(m) Z_n^k(\varphi; m) \frac{1}{\bar{p}_s} \int_0^p G_m(p') dp', \quad (29)$$

where $\bar{p}_s$ is the average surface pressure. This also allows us to identify the ω -VSFs (G_ω) as

$$G_{\omega, m} = \frac{1}{\bar{p}_s} \int_0^p G_m(p') dp', \quad (30)$$

where the factor of $\bar{p}_s$ is introduced to keep the ω -VSFs dimensionless. Note that another corresponding multiplicative factor of $\bar{p}_s$ is also included in Equation 29 to keep the dimensions of ω consistent.

When examining global data it is often convenient to examine the relative contributions of ω in a given latitudinal band. To this end we can equivalently express ω as

$$\omega(\lambda, \varphi, p) = \sum_{k=-K}^K \hat{\omega}_k(\varphi, p) e^{ik\lambda}, \quad (31)$$

$$\hat{\omega}_k(\varphi, p) = -i \sum_{m=1}^M \sum_{n=1}^R \nu_n^k \chi_n^k(m) Z_n^k(\varphi; m) \int_0^p G_m(p') dp', \quad (32)$$

where $\hat{\omega}_k$ are the Fourier coefficients of omega when taking the Fourier transform at a single latitude at a particular pressure level. We have also suppressed the time dependence as these decompositions can be computed at any given time instant.

2.3.2. Sturm–Liouville Problem and Orthogonality Conditions for VSFs of Vertical Velocity

While the ω decomposition in Equations 29 and 30 has a natural connection with the decomposition for u , v , and h , it is not clear from the derivation presented if the vertical basis functions $G_\omega(p)$ for ω are an orthogonal basis.

To establish orthogonality, we now review a Sturm–Liouville problem for $G_\omega(p)$. A version of this is presented by Staniforth et al. (1985) and can also be found in the oceanography literature (see, e.g., Early et al., 2020; Gill & Clarke, 1974; Kelly, 2016, and references therein), and here we present a version that is tailored to the atmospheric setting.

The discussion of G_ω here is also helpful as a prelude to the discussion of vertical structure functions for moisture, in Section 3.2.

To derive a Sturm–Liouville problem for $G_\omega(p)$, we return to the derivation in Section 2.1. Recall that we had eliminated ω from Equation 1 in an early stage of the derivation there. However, we could have instead chosen to eliminate the geopotential φ and to retain ω . We will now examine how this can be done.

We begin by using the same decompositions for u , v , and φ (as shown in Equation 6), and performing the following additional separation of variables for ω :

$$\omega(\lambda, \varphi, p, t) = G_{\omega}(p) \times \omega'(\lambda, \varphi, t). \quad (33)$$

Plugging both of these separation of variables into Equation 1c and Equation 1d yields

$$g \left(\frac{\partial h'}{\partial t} \right) \left(\frac{1}{\Gamma_o} \frac{dG}{dp} \right) + G_{\omega} \omega' = 0, \quad (34)$$

$$\omega' \frac{dG_{\omega}}{dp} + G \nabla \cdot \mathbf{V}' = 0. \quad (35)$$

Now if we divide both equations by ω' and then to the second equation we take a p derivative and multiply by gD/Γ_o we obtain

$$\frac{1}{\omega'} \left(g \frac{\partial h'}{\partial t} \right) \left(\frac{1}{\Gamma_o} \frac{dG}{dp} \right) + G_{\omega} = 0, \quad (36)$$

$$\frac{gD}{\Gamma_o} \frac{d^2 G_{\omega}}{dp^2} + \frac{1}{\omega'} (gD \nabla \cdot \mathbf{V}') \left(\frac{1}{\Gamma_o} \frac{dG}{dp} \right) = 0. \quad (37)$$

From here we note that these two equations each contain a term with a single p derivative, and these terms are the same (but with opposite signs) because of Equation 10. Adding these two equations to cancel these terms yields the following equation for G_{ω} :

$$\frac{1}{\Gamma_o} \frac{d^2 G_{\omega}}{dp^2} = \frac{-1}{gD} G_{\omega}. \quad (38)$$

This is the differential equation of a Sturm-Liouville Problem (see, e.g., Early et al., 2020; Gill & Clarke, 1974; Kelly, 2016; Staniforth et al., 1985, and references therein). There is also an associated orthogonality condition for the vertical structure functions (G_{ω}) of the vertical velocity ω , with respect to the weight function $\Gamma_o(p)$:

$$\frac{1}{p_s} \int_0^{p_s} \Gamma_o(p') G_{\omega,i}(p') G_{\omega,j}(p') dp' = \delta_{ij} + \text{boundary terms}. \quad (39)$$

While it is common for the right-hand side of an orthogonality condition to appear as only δ_{ij} , here there are extra terms in the orthogonality condition arising from the free surface boundary condition; we refer to the literature for further information (e.g., Early et al., 2020; Gill & Clarke, 1974; Kelly, 2016; Staniforth et al., 1985, and references therein).

Note that one can take the derivative of (38) with respect to pressure and transform the differential equation for G_{ω} into the differential equation for G in Equation 7, or vice versa with an anti-derivative. Alternatively, it can also be valuable to note that G and G_{ω} have their own separate origins in the coupled system in Equations 34 and 35, and the steps that lead to Equation 38 are essentially a diagonalization of the coupled system into two independent equations, one for G and one for G_{ω} . Now that we have described how a decomposition can be performed for each of the dry variables of interest (u , v , h , and ω), we will turn our attention on how to include a decomposition of moisture within this framework.

3. Decomposing Moisture With Global Normal Modes

In this section, we consider the inclusion of moisture, and a resulting moist multivariate decomposition on the sphere. The setup for including moisture is discussed in Section 3.1, and the eigenmodes and associated decomposition are described in Section 3.2.

3.1. Moisture Evolution Equation for Defining Eigenmodes

To include moisture, a first step is to decide which processes to include in its evolution equation for defining the eigenmodes. On the one hand, one may want to include various moist processes, such as cloud latent heating and

precipitation, so that the influence of these processes are included within the eigenmode definitions. On the other hand, the inclusion of complicated processes will make the eigenmode decomposition complicated, especially given that cloud latent heating and precipitation involve nonlinear thresholds that are non-smooth and hence are not readily amenable to standard techniques of linearization.

In light of these considerations, and given that a moisture decomposition on the sphere has not been previously conducted (to the authors' knowledge), it is sensible to begin with the simplest setting here. Hence, for including moisture, we use a moisture evolution equation that is of the same type as the temperature evolution or geopotential height evolution in Equation 1c, as in traditional frameworks for normal mode decompositions on the sphere. In particular, the specific humidity q_v evolution equation for the eigenmode definitions will be

$$\frac{D}{Dt}(q_v) = 0. \quad (40)$$

Similar to what was done in the previous section for the dry framework, we will express the moisture in terms of background and perturbation terms as

$$q_v(\lambda, \varphi, p, t) = q_o(p, \Phi) + q(\lambda, \varphi, p, t), \quad (41)$$

where $q_o(p, \Phi)$ is the pressure-dependent background function, which also varies slowly in latitude, and q is the perturbation from this background quantity. It is natural to include a latitude-dependence in the background moisture q_o , since the vertical moisture profile can vary considerably from the tropics to polar regions. Note that, in writing $q_o(p, \Phi)$, we are explicitly using Φ rather than φ to serve as a reminder that q_o has latitude variations on a long horizontal spatial scale which we changed one denote with large-scale coordinate Φ . It would be interesting in the future to consider the influence of gradients of the reference state q_o with respect to latitude, although this large-scale gradient will be neglected in what follows. The latitude-dependence of q_o will instead enter through the appearance of Φ in the latitude-dependence of $\partial_p q_o(p, \Phi)$ in what follows.

In what follows we will refer to q as the specific humidity rather than the perturbation of the specific humidity, for brevity. Plugging this into Equation 40, and ignoring the $\partial q_o / \partial \varphi$ term due to the q_o 's slow variation in latitude yields

$$\frac{\partial q}{\partial t} + \Gamma_q \omega = 0, \quad (42)$$

where we have introduced the notation

$$\Gamma_q := \frac{\partial}{\partial p} q_o(p, \Phi) = \Gamma_q(p, \Phi). \quad (43)$$

Figure 2 shows a typical profile for the background moisture as well as its derivative.

The background moisture quantities (q_o and Γ_q) that are used in the moisture decomposition are computed using the ECMWF data. The background moisture values ($q_o(p, \Phi)$) are found by averaging zonally, and one vertical profile is computed for each latitude. We have computed a $q_o(p, \Phi)$ for each day for simplicity although the changes are small from one day to another. The pressure derivatives of $q_o(p, \Phi)$ are then computed to obtain the $\Gamma_q(p, \Phi)$ values. It is worth noting that, when a pressure level is below the Earth's surface (which can occur when topography has a high elevation), we use the extrapolated values that are provided by default in the ERA5 data set. The extrapolated values are simply the moisture value at the lowest model level that is the first value at or above the surface. We have verified that, if the extrapolated values are excluded from the averages to produce $q_o(p, \Phi)$, there is little difference in the profile in most regions, and the noticeable differences are at locations that are very near to the surface and in the vicinity of Antarctica. Since these regions are not considered in Section 4, the default extrapolated values are included in all calculations of q_o and Γ_q for ease of application.

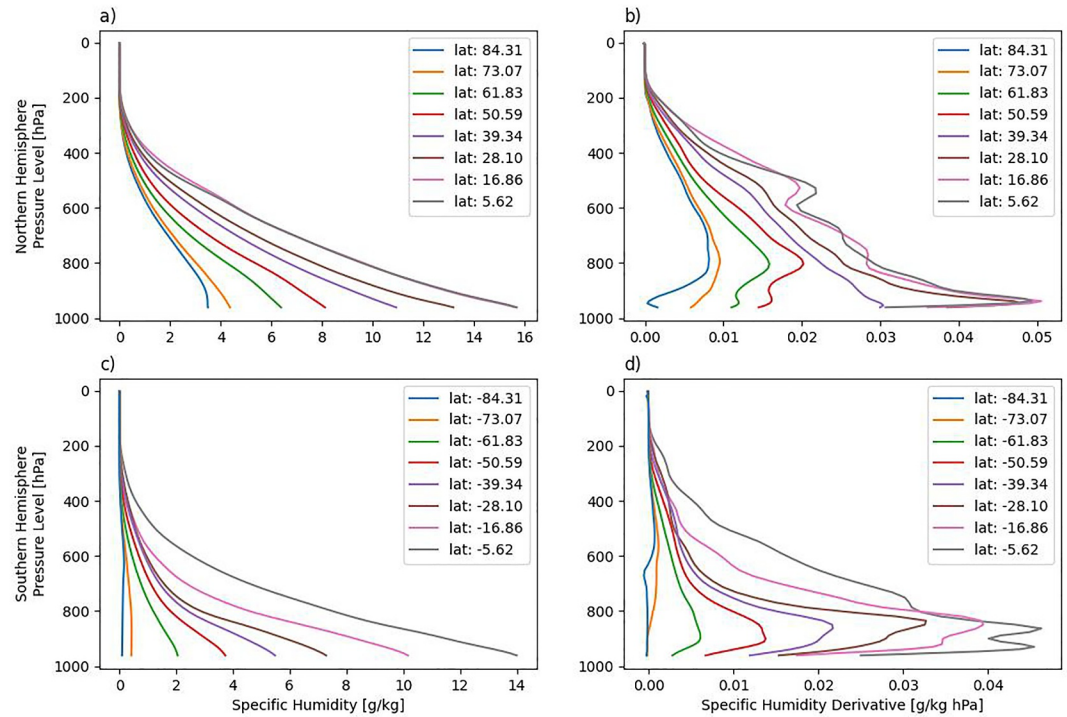


Figure 2. Sample profiles of the reference moisture profile $q_o(p, \Phi)$ and its derivative $\partial_p q_o(p, \Phi)$. These profiles were calculated using global ECMWF analysis data that was zonally averaged at each of 137 pressure levels closest to the model levels for the month of August 2018. Panels (a) and (c) show the $q_o(p, \Phi)$ profiles in the northern and southern hemispheres respectively, and (b) and (d) show their respective pressure derivatives.

In what follows, we will proceed with Equation 42 as a moisture equation that is in the same spirit as the temperature or geopotential height equation in Equation 1c, as in traditional frameworks for normal mode decompositions on the sphere.

In this setting, the dry variables (u', v', h') still obey the same equations as in a dry framework, and hence their eigenmode decomposition is unchanged and is described by the WIG, EIG, and ROT modes,

so that any additional moist eigenmode will have $u' = v' = h' = 0$ associated with the eigenmode's moisture variations. Even though it is a simple addition of moisture, many non-trivial properties arise for this type of treatment of moisture and its decomposition, such as a non-diagonal inner product and an additional slow eigenmode associated with moisture (Ogrosky et al., 2025). Hence, this system is a non-trivial step beyond the dry case.

3.2. Moist Eigenmodes and Moist Decomposition

We now aim to find the moist eigenmodes of the moisture-coupled equations, using the moisture evolution equation from Equation 42 coupled with the evolution equations for the dry variables u, v, ω , and ϕ from Equations 1a to 1d. In order to find the eigenmodes, it appears at first glance that we must begin anew and solve a new eigenvalue problem, since moisture coupling has now been introduced. However, three key observations will allow us to avoid the need to solve an entirely new eigenvalue problem.

A first key observation comes from separation of the time variable. Assigning the same time dependence to q as for the dry variables ($e^{-i\nu_n^k t}$), and separating the time variable t in the moisture evolution equation in Equation 42, we obtain

$$i\nu_n^k q_n^k(\lambda, \varphi, p, t; m) = \Gamma_q(p, \Phi) \omega_n^k(\lambda, \varphi, p, t; m) \quad (44)$$

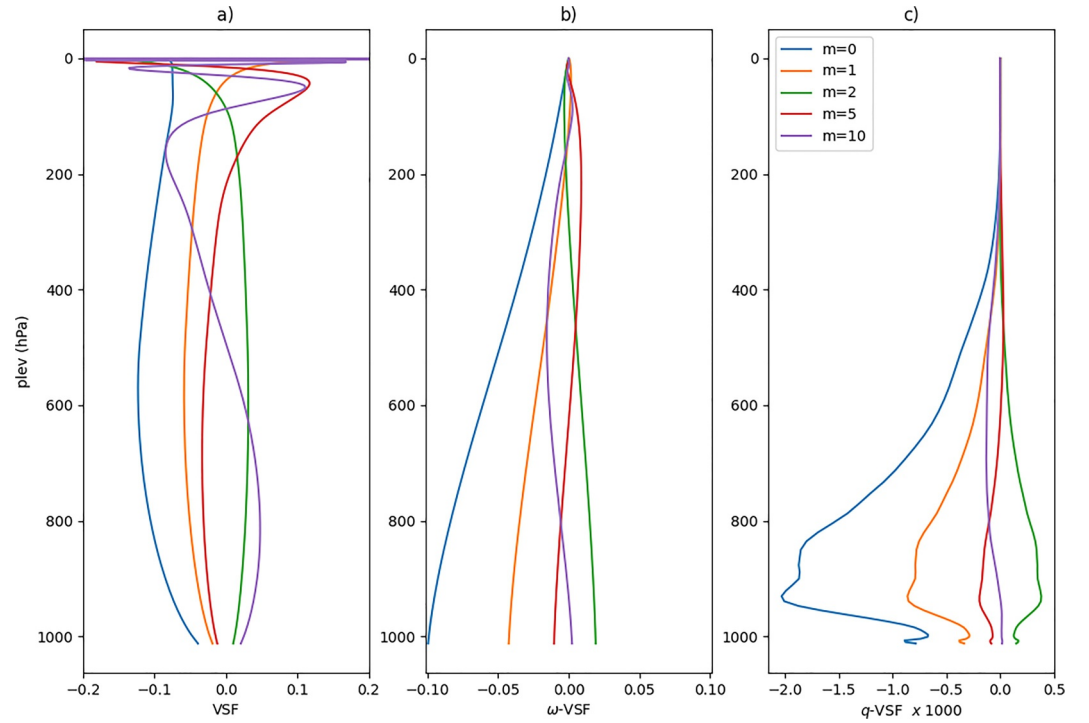


Figure 3. Panels (a), (b), and (c) show the vertical structure functions G [VSF], G_ω [ω -VSF], and G_q [q -VSF], for the $m = 0, 1, 2, 5, 10$ modes.

for a single (m, k, n) mode. Hence the first key observation here is that the q_n^k can be found, if the modes for ω are known. We can then leverage the recent results of Žagar et al. (2023) that provide the modes for ω , as reviewed here in Section 2.3. (A subtlety will arise, though, and it is overcome by a third key observation, as described below.).

A second key observation is that Equation 42 and Equation 44 can also help to find the vertical structure functions for moisture, which we call $G_q(p)$. To see this, we follow the methods used in the previous section, and we perform the following separation of variables for the moisture:

$$q(\lambda, \varphi, p, t) = q'(\lambda, \varphi, t) \times G_q(p). \quad (45)$$

Plugging this separation of variables into Equation 42, along with our previous separation of variables for ω from Equation 33, we obtain

$$\left(\frac{\partial q'}{\partial t}\right) G_q = \left(\frac{-\omega'}{\bar{p}_s}\right) \times (\bar{p}_s \Gamma_q G_\omega), \quad (46)$$

where we have once again introduced the factor of $\bar{p}_s$ to keep the VSFs dimensionless. Equating the pressure-dependent factors on both sides of the equation brings us to the second key observation: we can deduce the vertical structure functions of moisture, G_q , from the VSFs of vertical velocity, G_ω . We can summarize this relationship, and the relations between all of the vertical structure functions, as

$$G_{q,m}(p) = \bar{p}_s \Gamma_q G_{\omega,m}(p) = \Gamma_q \int_0^p G_m(p') dp'. \quad (47)$$

Figure 3 shows examples of the VSFs. Given that G_q and G_ω are proportional to each other, we also obtain a Sturm–Liouville problem for G_q . The quantity $G_q/(\bar{p}_s \Gamma_q)$ satisfies the same Sturm–Liouville problem as G_ω , which was presented earlier in Section 2.3.2. The differential equation takes the form of

$$\frac{1}{\Gamma_o} \frac{d^2}{dp^2} \left(\frac{G_q}{\bar{p}_s \Gamma_q} \right) = \frac{-1}{gD} \frac{G_q}{\bar{p}_s \Gamma_q}, \quad (48)$$

along with boundary conditions that follow from Equation 4 and the associated orthogonality condition is

$$\frac{1}{p_s} \int_0^{p_s} \frac{\Gamma_o(p')}{(\bar{p}_s \Gamma_q)^2} G_{q,i}(p') G_{q,j}(p') dp' = \delta_{ij} + \text{boundary terms}. \quad (49)$$

While it is common for the right-hand side of an orthogonality condition to appear as only δ_{ij} , here there are extra terms in the orthogonality condition arising from the free surface boundary condition, as in the case of vertical velocity; we refer to the literature for further information on these extra terms (e.g., Early et al., 2020; Gill & Clarke, 1974; Kelly, 2016; Staniforth et al., 1985, and references therein).

For the horizontal portion of the separation of variables, we can obtain a system of evolution equations for $(\tilde{u}, \tilde{v}, \tilde{h}, q')$, as follows. For the moisture, we find from Equation 45 and 46, that

$$\frac{\partial q'}{\partial t} = -\frac{\omega'}{\bar{p}_s} = \nabla \cdot \mathbf{V}', \quad (50)$$

where the latter equality follows from inserting the separation of variables for $\mathbf{V}$ and ω (Equations 6 and 33) into the continuity equation and equating the pressure-independent factors. By coupling the dry system with Equation 50 for moisture, we can now express our system of pressure-independent perturbations $(\tilde{u}, \tilde{v}, \tilde{h}, q')$ as a linear operator equation as

$$\frac{\partial}{\partial t} \mathbf{W}_q + \mathbf{L}_q \mathbf{W}_q = 0, \quad (51)$$

where the vector $\mathbf{W}_q$ and the linear operator $\mathbf{L}_q$ are given by

$$\mathbf{W}_q = \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{h} \\ q' \end{pmatrix}, \quad \mathbf{L}_q = \begin{pmatrix} 0 & -\sin \varphi & \frac{\gamma}{\cos \varphi} \frac{\partial}{\partial \lambda} & 0 \\ \sin \varphi & 0 & \gamma \frac{\partial}{\partial \varphi} & 0 \\ \frac{\gamma}{\cos \varphi} \frac{\partial}{\partial \lambda} & \frac{\gamma}{\cos \varphi} \frac{\partial}{\partial \varphi} (\cos \varphi \cdot) & 0 & 0 \\ \frac{-\gamma}{\cos \varphi} \frac{\partial}{\partial \lambda} & \frac{-\gamma}{\cos \varphi} \frac{\partial}{\partial \varphi} (\cos \varphi \cdot) & 0 & 0 \end{pmatrix}. \quad (52)$$

Since q' is already nondimensional (with units of kg kg^{-1}), we simply leave it as q' in the $\mathbf{W}_q$ vector, and the other quantities $(\tilde{u}, \tilde{v}, \tilde{h})$ have been nondimensionalized in the same way as they were in the dry case.

The third key observation arises as follows. Since we have moved from a (dry) 3×3 system to a (moist) 4×4 system, we must have an extra eigenmode. The extra eigenmode could potentially be found if it is possible to adapt knowledge of the f -plane case (Ogrosky et al., 2025; Smith & Stechmann, 2017) to apply here on the sphere. In particular, it is reasonable to expect that the extra eigenmode is associated with moisture, since moisture is the new variable that has been added to the system. Hence we should examine the evolution equation for moisture, in the bottom row of Equations 51 and 52. Doing so, one can see that the q' evolution equation and the $\tilde{h}$ equation are linearly dependent as $\partial q' / \partial t = -\partial \tilde{h} / \partial t$. As a result, since the third and fourth rows of $\mathbf{L}_q$ in Equation 52 differ by only an overall minus sign, those rows are linearly dependent, which implies, by the invertible matrix theorem of linear algebra, that the extra mode that we obtain must be a zero eigenmode (or equivalently a zero frequency mode). This is the third key observation. It allows us to identify the fourth eigenmode, and what we know about it, at the least, is that it is a zero-frequency, moist eigenmode.

Table 1

Summary of the Different Types of Modes for the Moist System, and Their Characterization as Balanced or Unbalanced

Mode type	Balanced or unbalanced
IG	Unbalanced
ROT	Balanced
<i>MOIST (M)</i>	<i>Balanced</i>

Note. For the moist system, a new *balanced MOIST* mode type arises (shown in italics), in addition to the balanced Rossby/rotational (ROT) modes and the unbalanced inertia-gravity (IG) modes of the dry system. The full set of modes also includes Kelvin modes and mixed Rossby-gravity modes that do not have a clear characterization as only balanced or only unbalanced. See Figure 1 for a plot of the dispersion curves.

We then find the eigenmode decomposition of moisture as follows. We can calculate the contribution of the EIG, WIG, and ROT modes by examining the relationship between q and ω in Equation 44. This allows us to calculate each of these q modes which correspond with previously found modes for ω . We can explicitly solve for each of these (m, k, n) moist eigenmodes in terms of the coefficients, Hough functions, and vertical structure functions, as

$$q_n^k(\lambda, \varphi, p, t; m) = -\chi_n^k(m) e^{i(k\lambda - \nu_n^k t)} Z_n^k(\varphi; m) \Gamma_q(p) \int_0^p G_m(p') dp'. \quad (53)$$

Note, though, a subtlety of applying Equation 53: it is based on Equation 44, and hence it only provides information about q_n^k if $\nu_n^k \neq 0$. This is the subtlety that was alluded to when Equation 44 was introduced. For any q_n^k modes with zero frequency, which were just found to exist in the previous paragraph,

Equation 44 and Equation 53 are not used to identify the mode. Instead, for the zero-frequency moist modes, we will compute the slow, moist eigenmode contribution, q_M , as the residual, $q_M = q - q_{IG} - q_{ROT}$, where q_{IG} and q_{ROT} represent the total contribution of the EIG, WIG, and ROT modes by summing over all of the (m, k, n) combinations for each mode, respectively. An alternative would be to calculate q_M based on the associated eigenvectors, which is a possible route but which would introduce additional computational cost; here we will follow the residual approach which utilizes the information of the EIG, WIG, and ROT modes that have already been computed in order to take advantage of this cost savings. We note here that this definition of q_M is based on a modal representation of circulation that is a complete representation, while discretizations will introduce a certain amount of incompleteness when working with real data (e.g., Žagar et al., 2009).

As a final remark, we note that, in order to isolate the moisture anomaly associated with one of the individual eigenmodes EIG, WIG, or ROT, we follow the same procedure that was mentioned for the dry variable decomposition in Equations 23 to 26. In more detail, recall that the meridional index n takes values from 1 to R , where $R = N_E + N_W + N_R$ (where N_E, N_W and N_R are the number of EIG, WIG, and ROT modes) to account for the three solutions that we obtain for each (n, k, m) combination. Therefore, an individual eigenmode contribution can be isolated by restricting the n index summation to include only the n values associated with the individual eigenmode.

To summarize, the moisture decomposition has four components:

$$q = q_{EIG} + q_{WIG} + q_{ROT} + q_M, \quad (54)$$

where q_M represents the moist, zero-frequency eigenmode that arises as a new eigenmode in moving from the dry system to the moist system. Table 1 gives a summary of these modes, as well as their characterization as either balanced or unbalanced. In particular, we note that moisture has two eigenmode contributions that are slow, or balanced: the Rossby/rotational (ROT) mode and the slow, moist eigenmode.

4. Analysis of Global Moisture Data Using a Normal Mode Decomposition

In this section we will examine global moisture data using the framework outlined above in Section 3.

As topics of interest, we will investigate properties that are well known for dry variables (velocity, temperature, geopotential) but which are not well understood for moisture. In particular, here we investigate questions such as: How much moisture variability is attributed to the balanced (ROT and MOIST) modes versus the unbalanced (IG) modes? Do balanced or unbalanced modes provide a dominant contribution, or are the contributions approximately equal? Do the answers change for different latitudes or different spatial scales?

The data for specific humidity is from the ERA5 data from the European Center for Medium-Range Weather Forecasts, of August, 2018, with one value each day at 00:000 UTC. These data are on a global grid with 0.25° resolution in longitude and latitude on 137 pressure levels. The data for the (u, v, h) coefficients, Hough Functions, vertical structure functions and frequencies, were obtained using the MODES software package, following Žagar et al. (2023). The decomposition of divergence and pressure vertical velocity for the same data set can be found in

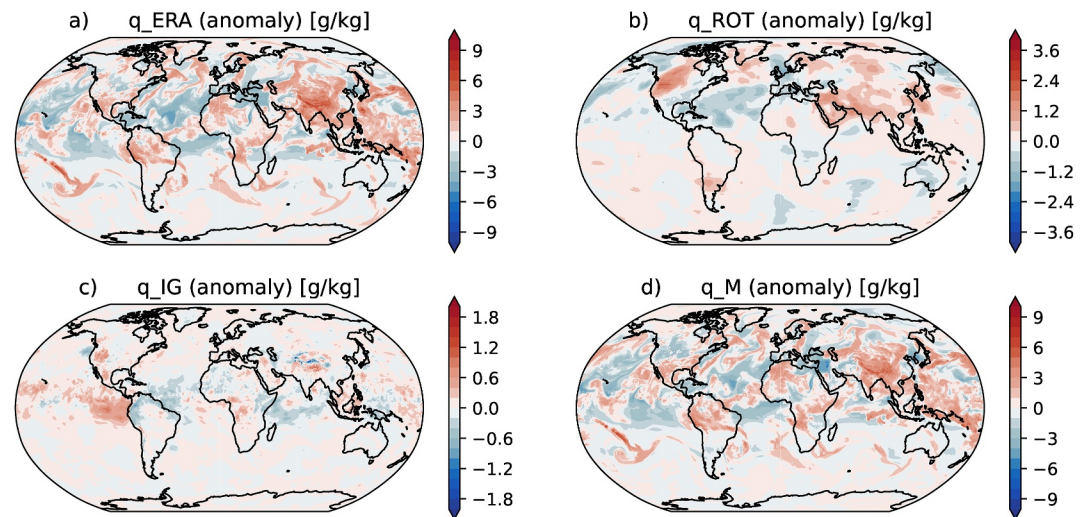


Figure 4. Contour plot of specific humidity at a pressure level of 610.66 hPa at 00:00 UTC on 11 August 2018. Panel (a) shows the specific humidity perturbations as calculated from the ERA data. Panels (b), (c), and (d) show the q_{ROT} , q_{IG} , and q_M modes respectively. The plots are displayed using a Miller Cylindrical Projection, and the units for all plots are g/kg.

Neduhai et al. (2024) and Žagar et al. (2023) respectively. The decomposition of specific humidity is then computed based on the framework outlined in Section 3, where a final summary is described in Equations 53 and 54 and the surrounding text. In analyzing the decomposed data, the data will mostly be analyzed in the mid-troposphere (approximately 500–600 hPa). In part it is helpful to avoid the pressure levels in the lower troposphere since those pressure levels can correspond to topographic features such as high mountains at some locations, and we would like to analyze data over the entire globe. In part it is also helpful to focus on the mid-troposphere since the expansions in vertical structure functions can require detailed treatment for resolving features in the lower troposphere and closer to the Earth's surface. We leave additional details of the vertical structure of moisture for future investigation, and we proceed with analyzing the mid-troposphere for a viewpoint of the bulk of the troposphere across the globe.

4.1. General Features of the Different Eigenmode Components

We will begin by examining some of the general features of the various modes by examining a single snapshot of the moisture anomalies. Figure 4 shows the ERA5 moisture anomaly, as well as the moisture anomalies that are associated with the ROT, IG, and M modes in the midtroposphere on the 610.66 hPa pressure level. In the tropics almost all of the moisture is contained within the q_M mode. Looking into the mid-latitudes and into the polar regions, the Rossby (ROT) mode also begins to have a substantial contribution. This is associated with the prevalence of Rossby wave activity in mid-latitudes and polar regions. The Inertio-Gravity (IG = EIG + WIG) mode has a smaller contribution among the different mode contributions. We note, though, that a global plot will emphasize features on the largest scales, and this plot is only meant to provide a first look. In what follows we will calculate statistics to provide a quantitative evaluation.

4.2. Latitude-Dependence of Eigenmode Contributions

To better quantify the contribution of each mode at different latitudes, we calculate the variance of each of the modes over all longitude values, at each particular latitude, and create a profile of how this variance changes for all latitudes. The plots in Figure 5 show that the zero frequency M mode has its largest contribution in the tropics, and the Rossby mode has its largest contribution in the midlatitudes. The IG variance seems to be approximately constant across all latitudes. We note that these plots show the variance over all spatial scales, and a more detailed view of the variance on different spatial scales will be shown below.

Also, we note a subtlety about the moisture decomposition: it is not an orthogonal decomposition with respect to the standard L^2 Euclidean inner product. We refer to Ogrosky et al. (2025) for a more detailed discussion of the non-diagonal inner product, in the setting of an f -plane, and the appendix herein describes the non-diagonal inner

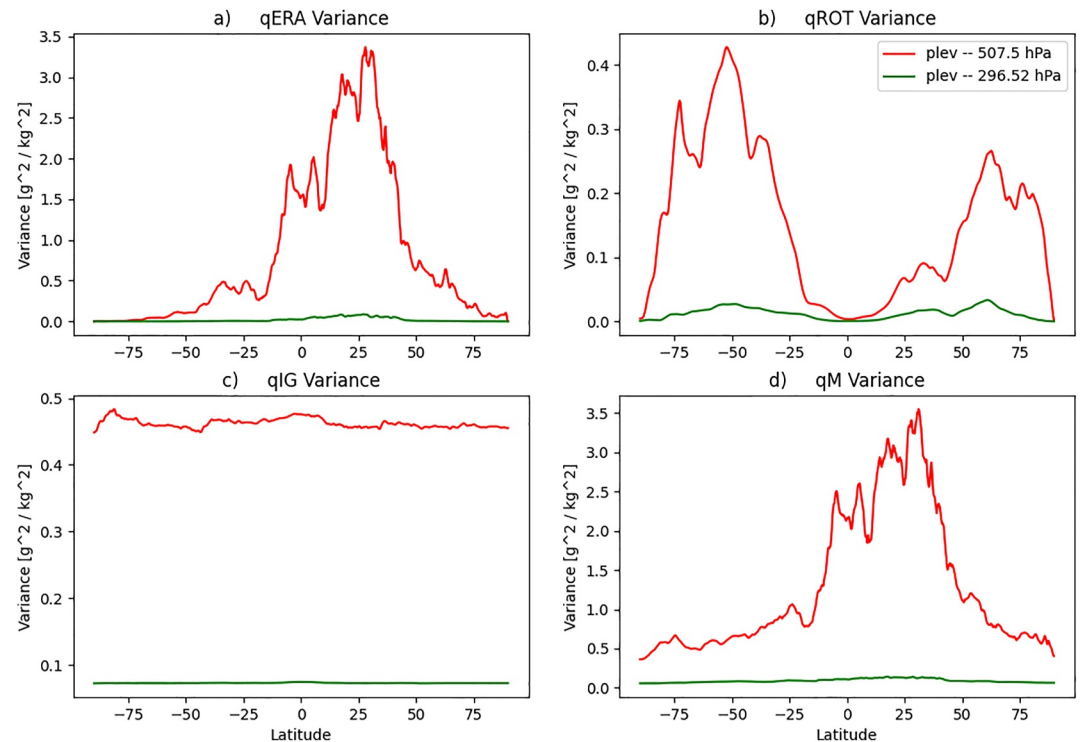


Figure 5. Moisture variance at each latitude. Panel (a) shows the variance of the ERA data, and panels (b), (c), and (d) show the variances of the *ROT*, *IG*, and *M* modes respectively. The data for this plot is from the date 11 August 2018, at 00:00 UTC, and the two profiles shown are for pressure levels of 507.50 hPa (red) and 296.52 hPa (green). In the tropics, the mode with the largest contribution to the moisture variance is the *M* mode, while in the midlatitudes, the *M*, *IG*, and *ROT* modes all have substantial contributions.

product for the case of the sphere that is considered here. Also, another example of a non-diagonal inner product was described by Stechmann and Majda (2015) for a model of tropical intraseasonal oscillations, where again it was moisture that was the cause of the non-diagonal property of the inner product. Here we note one aspect that is related to the non-diagonal nature of the inner product: the total variance of moisture in Figure 5 is not the sum of the variances of the different eigenmode contributions. That is, whereas the decomposition of q as $q = q_{IG} + q_{ROT} + q_M$ is true as an equality, the sum of the variances is not equal to the variance of the moisture: $q^2 \neq q_{IG}^2 + q_{ROT}^2 + q_M^2$.

4.3. Contributions of Moist Eigenmodes Across Different Spatial Scales

In the previous plots we have examined the relative contribution of each of the modes, and we have examined all scales together. However, if we examine a particular spatial scale, the relative importance of each of these modes may be different. To this end, we now compute the Fourier Transform in different regions to examine the impact of each mode on different scales, in different regions.

The Fourier transform, or more specifically the fast Fourier transform, is computed on the anomaly data at each latitude, and then averaged across a regional band, at a single pressure level. Three regional bands are specified as the northern mid-latitudes (60.28° to 29.93°), tropics (15.04° to -15.04°), and the southern mid-latitudes (-29.93° to -60.28°). This is done for the moisture anomaly data from the ERA5 data and each eigenmode.

The Fourier spectrum of each eigenmode, and in each latitude band, is shown in Figure 6. First we consider the largest spatial scales—i.e., the smallest wavenumbers k , from approximately $k \approx 1$ to $k \approx 40$. For the smallest wavenumbers, we see the following. In the tropics, the slow moist eigenmode makes the dominant contribution. In the extratropics, on the other hand, both the slow moist eigenmode and the Rossby eigenmode have substantial contributions. Note that these statements agree with the indications from the earlier Figures 4 and 5, which were

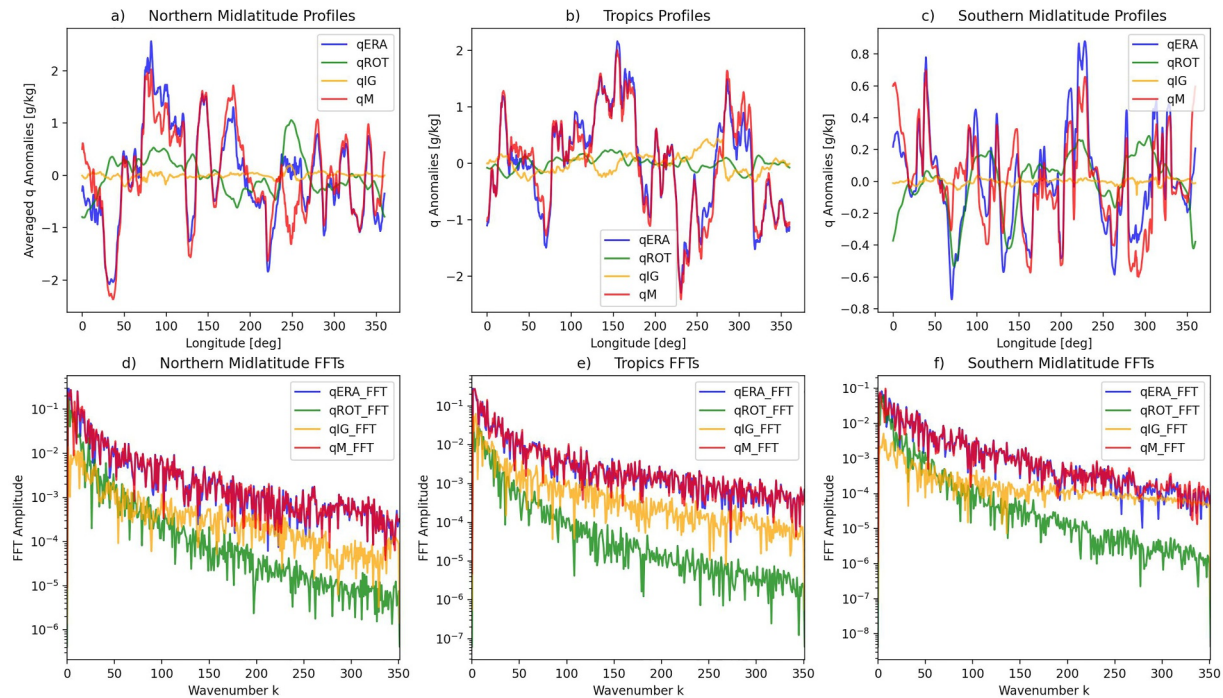


Figure 6. Moisture variance contributions on different spatial scales. Panels (a), (b), and (c) show the averaged profiles for each of the eigenmodes in the northern mid-latitude, tropic, and southern mid-latitude regions respectively for data on 11 August 2018 at 00:00 UTC. Panels (d), (e), and (f) show the fast Fourier transform amplitude (normalized by the number of longitude values) for the corresponding profiles shown in panels (a), (b), and (c). The M mode has the largest contribution, and in the midlatitudes the ROT mode also makes a substantial contribution on synoptic scales ($k \lesssim 40$). On sub-synoptic scales ($k \gtrsim 40$), where dry variables are known to be dominated by IG modes, the moisture is dominated by the balanced M mode.

figures that consider all scales together, since the moisture has a red spectrum with the largest variance at the lowest wavenumbers.

Next we consider the sub-synoptic scales, that is, wavenumbers $k \gtrsim 40$. In the tropics, on sub-synoptic scales, from largest to smallest amplitude, we see the slow moist eigenmode, the IG modes, and the Rossby mode contributions to moisture. In the mid-latitudes, there is a large change across scales in the Rossby mode contribution, which is one of two leading contributors on synoptic and planetary scales, but which transitions to the smallest contributor on sub-synoptic scales. Such a transition is similar to what is seen for the dry variables (velocity, geopotential height) (e.g., Žagar, Kasahara, et al., 2015). However, another feature of the moist eigenmodes is different from the dry variables. In particular, in contrast to the dry variables (velocity, geopotential height) which are dominated by unbalanced, IG contributions on sub-synoptic scales, the moisture variance on sub-synoptic scales is dominated by a balanced eigenmode contribution, due to the additional, slow eigenmode that is present in the moist system.

5. Case Study of an Extreme Rainfall Event

In the previous section, the moisture decomposition was analyzed from a statistical perspective. In this section we now examine an extreme rainfall event as a case study. The case study will provide an illustration to accompany the statistical analyses from Section 4. The moisture NMF decomposition will be used to investigate the moisture variability of the event and the different contributions of different moist eigenmodes.

For the case study, we consider an extreme rainfall event on August 20–21, 2018, in the region surrounding Madison, Wisconsin. This event was a record breaking (or near record breaking) event for total rainfall over a 24 hr period, with some rainfall measurements reaching over 11 inches of rainfall, the majority of which was accumulated in only a few hours. This event caused flash floods and as a result there was much damage to local homes and infrastructure. While this event had significant local impact, we are not considering here the local,

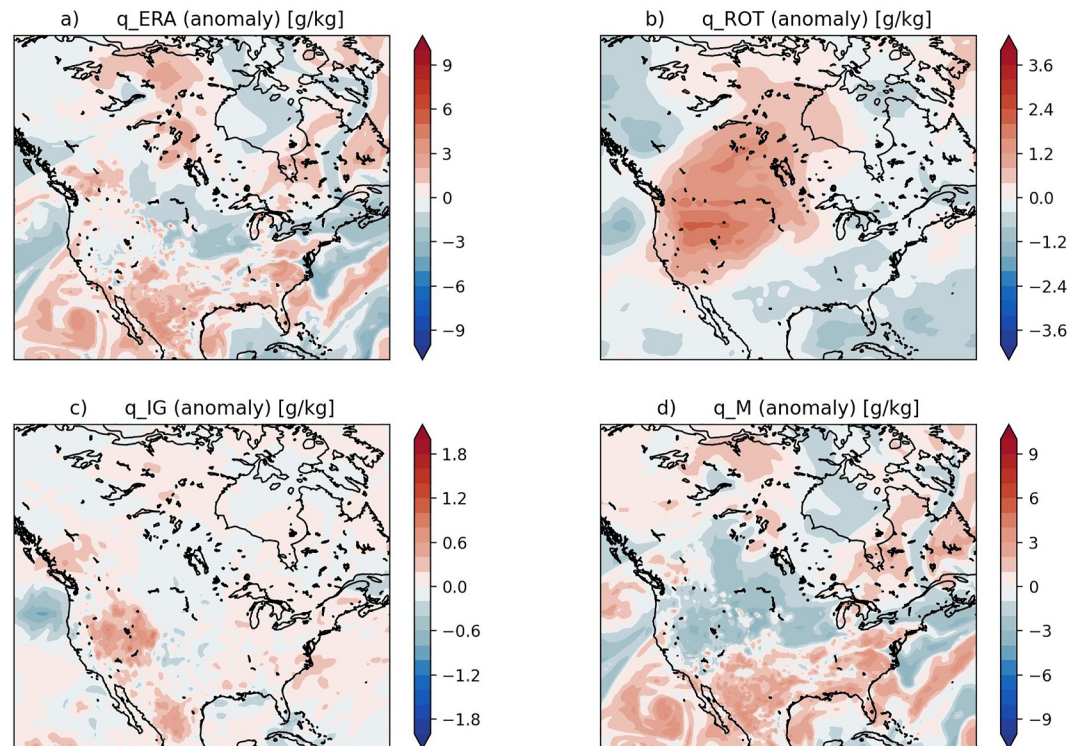


Figure 7. Large-scale conditions of moisture, preceding an extreme rainfall event in Wisconsin by approximately 10 days. Panel (a) shows the ERA5 data for the specific humidity anomaly, and panels (b), (c), and (d) show the *ROT*, *IG*, and *M* modes, respectively. These contour plots are made on a pressure level of 610.66 hPa at 00:00 UTC on 11 August 2018, and cover latitudes from 20° to 70°. The *M* mode makes the dominant contribution at lower latitudes, and both the *ROT* and *M* modes have significant contributions over the northwestern portion of the United States and the western portion of Canada.

small-scale features of the event. In what follows, we will consider the large-scale conditions of moisture associated with this event.

A priori, there are various possibilities for the physical mechanisms of this event's moisture variability. To gain some insight by using the mode decomposition, a question to investigate here is: Which eigenmode is responsible for the dominant contribution of the moisture variability? Given that it is a midlatitude event, one might suspect that the dominant moisture contribution is from one or both of the balanced eigenmodes. However, a priori, it is not clear which proportion might be in the *ROT* versus *M* eigenmodes. A priori, since midlatitude synoptic circulations are typically associated with the *ROT* eigenmode, it is possible that these synoptic circulations in the *ROT* eigenmode are also playing a dominant role in the moisture variations. On the other hand, some precipitation events in midlatitudes are associated with incursions of tropical moisture into the midlatitudes, which could be associated with the *M* eigenmode, given the amount of *M* eigenmode variance in the tropics. It is also possible that the *ROT* and *M* eigenmodes both have appreciable contributions to the moisture variability in this event, given that the both of these eigenmodes were seen to have appreciable contributions to moisture variance in the midlatitudes in the statistical analyses in Section 4. Viewing the spatial variability and time evolution of this midlatitude event will provide some illustration and context to accompany the statistical analysis from Section 4.

Figure 7 shows the moisture at 10 days prior to the event. The figure shows contour plots for moisture in the vicinity of North America on August 11. We can notice that 10 days prior to the event there are substantial contributions from both the *ROT* and *M* modes to the total moisture, over the northwestern portion of the United States and also in the western portion of Canada. This fits with our observations from the previous section since the majority of region shown (35°–70°) is within the mid-latitude region examined in the previous section, which is where the Rossby (*ROT*) mode makes its strongest contribution. There is also little contribution from the *IG* modes during this time, and the *M* mode has its strongest contribution over the southern half of the United States.

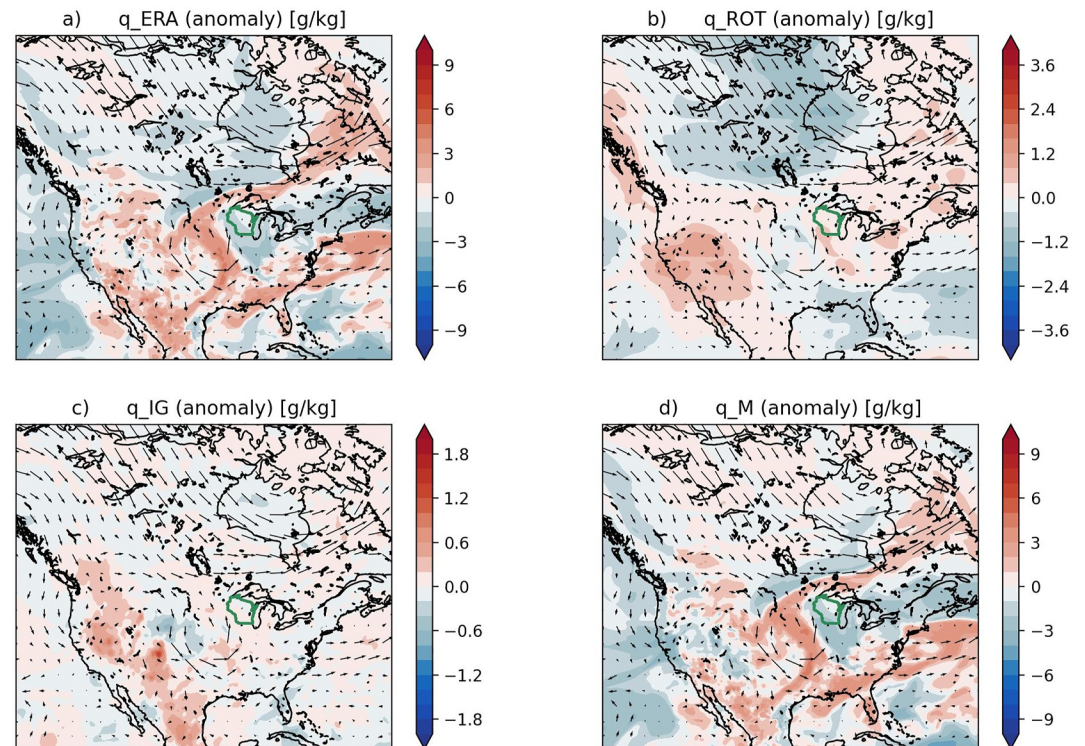


Figure 8. Large-scale conditions of moisture and circulation, 1 day prior to the extreme rainfall event. Panel (a) shows the ERA5 data for the specific humidity anomaly, and panels (b), (c), and (d) show the *ROT*, *IG*, and *M* modes, respectively. These contour plots are made on a pressure level of 610.66 hPa at 00:00 UTC on 20 August 2018, and cover latitudes from 20° to 70°. Wind vectors are also included in this plot to show the circulation immediately preceding the event, and the outline of Wisconsin is shown in green.

In the north-central portion of the United States, where the extreme rainfall event will occur, there is a negative moisture anomaly at this time (10 days preceding the event).

The moisture and circulation are shown 1 day prior to the event in Figure 8. The plot is over the same region and at the same pressure level as Figure 7, with wind vectors overlaid so that we gain a large-scale picture of the moisture flux prior to the event. We can see that, in the region south of Wisconsin, there is a poleward flux of moisture in the raw ERA5 data. The poleward flux of moisture is indicated by the overlap of a positive moisture anomaly and a northward or northeastward velocity vector. We can notice that some of this poleward flux appears to come from the *IG* mode, and the *ROT* mode also appears to have a small positive contribution, while the majority of this poleward flux is coming from the *M* mode.

At the same time of 1 day prior to the event, the vertical structure of the moisture anomaly is shown in Figure 9. The variations with respect to latitude and pressure are shown at a fixed longitude of -91.41° . This longitude is near the western edge of Wisconsin (shown in green in Figure 8), and the range of latitudes is the same as in Figure 8, extending from the Gulf of Mexico in the south to northern Canada in the north. In terms of the vertically integrated moisture anomaly, the largest amount is still south of Wisconsin and located at approximately 30°N to 40°N latitude, at this time of 1 day prior to the event. A prominent feature of the vertical structure is a relatively moist lower troposphere (700–1,000 hPa) and a relatively dry mid-troposphere (500–700 hPa) in the vicinity of Wisconsin (at latitudes from approximately 42.5°N to 47°N). However, the lower altitudes have not been emphasized in this study in part because a more detailed treatment of expansions in vertical structure functions should be undertaken for resolving features in the lower troposphere in future work. Keeping this in mind, there is nevertheless the possibility of a more substantial contribution of the *ROT* and *IG* modes in lower altitudes in Figure 9 in comparison to higher altitudes.

At the time of the extreme rainfall event on August 20–21, 2018, the moisture state is shown in Figure 10. A large positive anomaly of moisture can be seen over Wisconsin (i.e., west of Lake Michigan and south of Lake

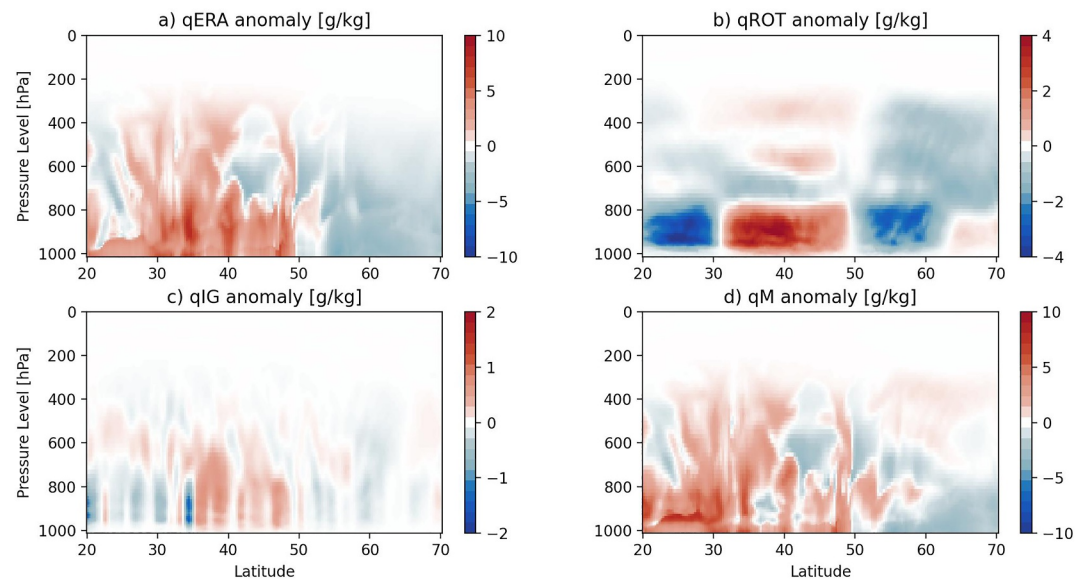


Figure 9. Large-scale conditions of the vertical and meridional structure of the moisture anomaly, 1 day prior to the extreme rainfall event. Panel (a) shows the ERA5 data for the specific humidity anomaly, and panels (b), (c), and (d) show the *ROT*, *IG*, and *M* modes, respectively. These contour plots are made at a fixed longitude of -91.41° at 00:00 UTC on 20 August 2018, and cover latitudes from 20° to 70° . This particular longitude value is approximately near the western boundary of Wisconsin (see the green outline of Wisconsin in Figure 8), and Wisconsin extends from latitudes of approximately 42.5° to 47° . Note that the lower altitudes have not been emphasized in this study in part because a more detailed treatment of expansions in vertical structure functions should be undertaken for resolving features in the lower troposphere in future work; nevertheless, this figure provides an initial view of the vertical structures based on the present methods and approach, and it indicates the possibility of a more substantial contribution of the *ROT* and *IG* modes in lower altitudes in comparison to higher altitudes and could be examined further in future work.

Superior). Almost all of this positive moisture anomaly is associated with the zero-frequency *M* mode. Hence, for this particular extreme rainfall event, it appears that the large-scale moisture is not due to a constructive coupling of multiple eigenmodes; instead, it is due to a northward shift of the low-latitude moisture that is associated with the zero-frequency *M* mode.

It may seem counterintuitive that a transient event of moisture variability could be attributed to a stationary (i.e., zero-frequency) eigenmode. An analogy can shed some light on this type of behavior. In particular, such behavior can also be seen in the more familiar and analogous situation of potential vorticity (PV). PV is associated with the low-frequency *ROT* eigenmode on the sphere and with the zero-frequency (i.e., stationary) vortical eigenmode on a midlatitude *f*-plane. Yet PV can still have transient behavior and can evolve substantially on synoptic time scales. The moist *M* eigenmode is analogous (Wetzel et al., 2020): it is a zero-frequency eigenmode, yet it can still have transient behavior and can evolve substantially on synoptic time scales.

Figure 10 also helps to provide an illustration of a conclusion from the statistical analysis from Figures 5 and 6. In particular, the *ROT* and *M* modes both have substantial contributions to moisture anomalies in midlatitudes and/or higher latitudes. The contribution of the *ROT* mode occurs at higher latitudes on the date of August 21 in Figure 10, such as the negative moisture anomaly that occurs in the region surrounding Hudson Bay in Canada. Hence, the different eigenmodes can have different contributions at different latitudes and different locations, with changes from day-to-day and week to week.

6. Concluding Discussion

In this study we were motivated by dry atmospheric decompositions into balanced and unbalanced components, which have been valuable for understanding global atmospheric dynamics. Here we have investigated if similar decompositions can be performed for moisture in the hope that such a decomposition could lead to an improved understanding of the variability of moisture. For instance, we were motivated by questions such as: What are the

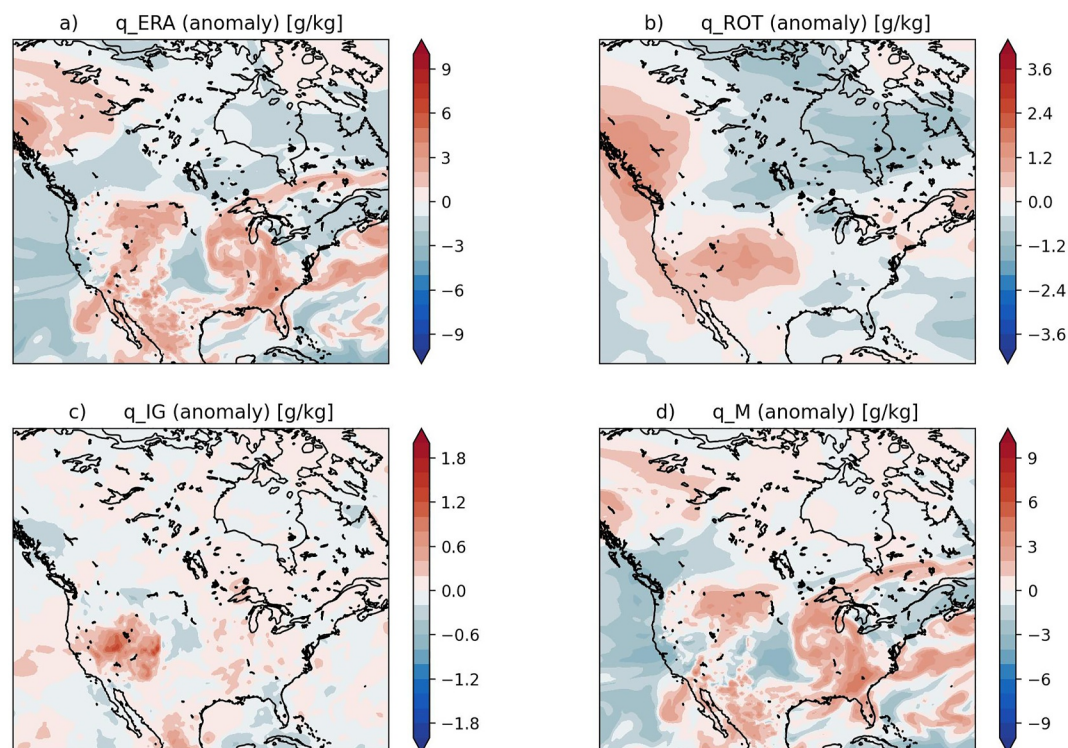


Figure 10. Large-scale conditions of moisture, at the time of an extreme rainfall event in Wisconsin. Panel (a) shows the ERA5 data for the specific humidity anomaly, and panels (b), (c), and (d) show the *ROT*, *IG*, and *M* modes, respectively. These contour plots are made on a pressure level of 610.66 hPa at 00:00 UTC on 21 August 2018, and cover latitudes from 20° to 70°. The *M* mode makes the dominant contribution at lower latitudes and extending northward over most of the United States on this date, and both the *ROT* and *M* modes have significant contributions at higher latitudes in the vicinity of Hudson Bay in Canada.

characteristics of moisture and its slow and fast components, or balanced and unbalanced components? How do these characteristics vary on different scales, and in different regions?

We have presented a moisture decomposition that can be performed on the sphere, and it includes the additional zero-frequency eigenmode that arises for the moisture. This additional, slow eigenmode is necessary for a complete description of the moisture and is not present for a dry system. In carrying out the decomposition here, we followed the strategy of past dry decompositions which typically do not include the effects of cloud latent heating on the temperature. In this spirit we did not include the effects of cloud latent heating on the moisture in defining the eigenmodes. Further discussion of this is included in Section 3. This is also a natural approach to consider in a first investigation of these topics, given that, to the author's knowledge, this is the first decomposition of moisture into its balanced and unbalanced components on the sphere. Recent work by Remond-Tiedrez et al. (2026) has shown how a balanced–unbalanced decomposition can be defined even with the inclusion of phase changes of water and cloud latent heating. However, the decomposition of Remond-Tiedrez et al. (2026) is defined on an *f*-plane and would need to be extended to the sphere in order to be used for the purposes of the present paper. Such an extension would be an interesting topic for future research.

Using the framework described above to analyze global reanalysis data, we have investigated the balanced and unbalanced components of moisture on different scales and in different regions. On synoptic scales, some differences can be seen in the tropics versus the midlatitudes: in the tropics, the slow moist eigenmode makes the dominant contribution to moisture variance, whereas in the midlatitudes, appreciable contributions are made by both the Rossby eigenmodes and slow moist eigenmodes. On sub-synoptic scales, one notable feature was seen in midlatitudes: the moisture variability is dominated by the balanced mode contributions, in contrast to dry variability which is known to be dominated by unbalanced (*IG*) mode contributions.

From a big-picture perspective, a new viewpoint of moisture transports could potentially be probed in future work with moisture decompositions. Similar to decompositions of velocity into geostrophic and ageostrophic components, moisture can be decomposed into its different components (e.g., ROT, IG, and M, in the global setting here). Physical mechanisms can be gleaned from the association with well-known properties of geostrophic and ageostrophic components. Along these lines, in the results shown herein, moisture variability was often associated with both the ROT and M modes in the midlatitudes. The different roles of the ROT, M, and IG mode variability (similar to geostrophic and ageostrophic variability of velocity) can potentially provide insight into the physical processes of moisture evolution in various events and phenomena. Some phenomena that could be interesting to investigate in the future include atmospheric rivers (e.g., Edwards et al., 2020; Gimeno et al., 2014) and eddy moisture fluxes as in the concept of the stirring tropics (Adames Corraliza & Mayta, 2024), among many possibilities. Another valuable viewpoint of moisture transport, which has been in use in numerous studies, is the methodology of Lagrangian tracers or Lagrangian trajectories (e.g., Dahinden et al., 2023; Ramos et al., 2016; Sodemann, 2025; Zhao et al., 2021). Normal mode decompositions provide an additional, different perspective, and better understanding can potentially be obtained by considering multiple perspectives.

Also, from a big-picture perspective, tropical–extratropical interactions are particularly well-suited for investigation with the global framework here. Since some aspects of tropical–extratropical interactions can span from the tropics to the midlatitudes or polar regions, they can be difficult to investigate in regional frameworks such as the midlatitude f -plane or equatorial β -plane. For example, atmospheric rivers can involve processes across a wide range of latitudes (e.g., Baiman et al., 2023, 2024). Quasi-geostrophic concepts have been valuable in the past for investigations of atmospheric rivers (Baiman et al., 2023; Cordeira et al., 2013; Edwards et al., 2020; Lee & Mitchell, 2021). The methods herein extend beyond a geostrophic–ageostrophic decomposition of winds and allow a decomposition of moisture as well. The moisture evolution of atmospheric rivers could potentially be viewed in a new way through the perspectives mentioned in the previous paragraph. It would be interesting in the future to investigate atmospheric rivers using global moisture decompositions.

Finally, as a practical application, the methods presented here can potentially have important implications for data assimilation and numerical weather prediction. In such a setting, the concepts of balanced and unbalanced components have long played an important role. While attention has mostly been given to dry variables, the recent work of Ogrosky et al. (2025) has investigated a moist scenario, that is, data assimilation with balanced–unbalanced decompositions for moisture. In order to put these concepts into practice, further understanding is needed of the balanced and unbalanced components of moisture in observational and model data. The present study adds an important element by providing methods for global moisture decompositions and by applying these methods to global reanalysis data. It would be interesting as future work to utilize the methods and results herein for data assimilation and forecasting, and also for investigating moisture transports and tropical–extratropical interactions as mentioned above.

Appendix A: Non-Diagonal Inner Product

In this appendix we provide a note on a subtlety about the moisture decomposition: it is not an orthogonal decomposition with respect to the standard L^2 Euclidean inner product in terms of $\tilde{u}$, $\tilde{v}$, $\tilde{h}$, and q' . Instead, the inner product is diagonal in terms of $\tilde{u}$, $\tilde{v}$, $\tilde{h}$, and M , where $M = \tilde{h} + q'$ is a moist variable that is a function of q' and $\tilde{h}$. Specifically, the inner product is

$$\langle \mathbf{W}_{q,1}, \mathbf{W}_{q,2} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^1 \left[\tilde{u}_1 \tilde{u}_2 + \tilde{v}_1 \tilde{v}_2 + \tilde{h}_1 \tilde{h}_2 + c(\tilde{h}_1 + q'_1)(\tilde{h}_2 + q'_2) \right] d\mu d\lambda. \quad (\text{A1})$$

This inner product is associated with the moist evolution equations in Equation 51 and 52: the conserved energy of the moist evolution equations in Equation 51 and 52 is $\langle \mathbf{W}_q, \mathbf{W}_q \rangle$. Due to the $c(\tilde{h}_1 + q'_1)(\tilde{h}_2 + q'_2)$ term, the inner product is non-diagonal in terms of $\tilde{u}$, $\tilde{v}$, $\tilde{h}$, and q' . In the extra moist term of the energy, the constant c can be chosen to have any value here, but it can take a specific value in settings with cloud latent heating and/or precipitation (Marsico et al., 2019). We refer to Ogrosky et al. (2025) for a more detailed discussion of the non-diagonal inner product, in the setting of an f -plane.

Conflict of Interest

The authors declare no conflicts of interest relevant to this study.

Availability Statement

The ERA5 specific humidity data used in this study were obtained from Copernicus Climate Change Service (C3S) Climate Data Store (CDS) (Hersbach et al., 2023), downloaded in July, 2023. The VSF, Hough coefficient, Hough function, and frequency data from the MODES software for the month of August, 2018 used in this study is publicly available in these respective locations: vertical structure functions—(Neduh, 2024j), Hough coefficients—(Neduh, 2024a), Hough function and frequency data—(Neduh, 2024b, 2024c, 2024d, 2024e, 2024f, 2024g, 2024h, 2024i). The software used to perform calculations and generate figures that were used to analyze this data have been made available at <https://doi.org/10.5281/zenodo.17136513> (Kumm, 2025), which also includes a link to a Github repository.

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References

- Adames, Á. F., & Ming, Y. (2018). Interactions between water vapor and potential vorticity in synoptic-scale monsoonal disturbances: Moisture vortex instability. *Journal of the Atmospheric Sciences*, 75(6), 2083–2106. <https://doi.org/10.1175/jas-d-17-0310.1>
- Adames Corraliza, Á. F., & Mayta, V. C. (2024). The stirring tropics: Theory of moisture mode–Hadley cell interactions. *Journal of Climate*, 37(4), 1383–1401. <https://doi.org/10.1175/jcli-d-23-0147.1>
- Back, L. E., & Bretherton, C. S. (2009). A simple model of climatological rainfall and vertical motion patterns over the tropical oceans. *Journal of Climate*, 22(23), 6477–6497. <https://doi.org/10.1175/2009jcli2393.1>
- Baiman, R., Winters, A. C., Lenaerts, J., & Shields, C. A. (2023). Synoptic drivers of atmospheric river induced precipitation near Dronning Maud Land, Antarctica. *Journal of Geophysical Research: Atmospheres*, 128(7), e2022JD037859. <https://doi.org/10.1029/2022jd037859>
- Baiman, R., Winters, A. C., Pohl, B., Favier, V., Wille, J. D., & Clem, K. R. (2024). Synoptic and planetary-scale dynamics modulate Antarctic atmospheric river precipitation intensity. *Communications Earth & Environment*, 5(1), 127.
- Bannister, R. N. (2021). Balance conditions in variational data assimilation for a high-resolution forecast model. *Quarterly Journal of the Royal Meteorological Society*, 147(738), 2917–2934. <https://doi.org/10.1002/qj.4106>
- Blažica, V., Zagar, N., Strajnar, B., & Čedilnik, J. (2013). Rotational and divergent kinetic energy in the mesoscale model ALADIN. *Tellus*, 65A(1), 18918. <https://doi.org/10.3402/tellusa.v65i0.18918>
- Bretherton, C. S., & Smolarkiewicz, P. K. (1989). Gravity waves, compensating subsidence and detrainment around cumulus clouds. *Journal of the Atmospheric Sciences*, 46(6), 740–759. [https://doi.org/10.1175/1520-0469\(1989\)046<0740:gwcsad>2.0.co;2](https://doi.org/10.1175/1520-0469(1989)046<0740:gwcsad>2.0.co;2)
- Burgess, B. H., Erler, A. R., & Shepherd, T. G. (2013). The troposphere-to-stratosphere transition in kinetic energy spectra and nonlinear spectral fluxes as seen in ECMWF analyses. *Journal of the Atmospheric Sciences*, 70(2), 669–687. <https://doi.org/10.1175/jas-d-12-0129.1>
- Chen, S., Majda, A. J., & Stechmann, S. N. (2015). Multiscale asymptotics for the skeleton of the Madden-Julian oscillation and tropical–extratropical interactions. *Mathematics of Climate and Weather Forecasting*, 1(1). <https://doi.org/10.1515/mcw-2015-0003>
- Chen, S., Majda, A. J., & Stechmann, S. N. (2016). Tropical–extratropical interactions with the MJO skeleton and climatological mean flow. *Journal of the Atmospheric Sciences*, 73(10), 4101–4116. <https://doi.org/10.1175/jas-d-16-0041.1>
- Cohn, S. E., & Dee, D. P. (1989). An analysis of the vertical structure equation for arbitrary thermal profiles. *Quarterly Journal of the Royal Meteorological Society*, 115(485), 143–171. <https://doi.org/10.1002/qj.49711548508>
- Cordeira, J. M., Ralph, F. M., & Moore, B. J. (2013). The development and evolution of two atmospheric rivers in proximity to western North Pacific tropical cyclones in October 2010. *Monthly Weather Review*, 141(12), 4234–4255. <https://doi.org/10.1175/mwr-d-13-00019.1>
- Dahinden, F., Aemisegger, F., Wernli, H., & Pfahl, S. (2023). Unravelling the transport of moisture into the Saharan air layer using passive tracers and isotopes. *Atmospheric Science Letters*, 24(12), e1187. <https://doi.org/10.1002/asl.1187>
- Daley, R. (1993). *Atmospheric data analysis*. Cambridge University Press.
- Early, J. J., Lelong, M. P., & Smith, K. S. (2020). Fast and accurate computation of vertical modes. *Journal of Advances in Modeling Earth Systems*, 12(2), e2019MS001939. <https://doi.org/10.1029/2019ms001939>
- Edwards, T. K., Smith, L. M., & Stechmann, S. N. (2020). Atmospheric rivers and water fluxes in precipitating quasi-geostrophic turbulence. *Quarterly Journal of the Royal Meteorological Society*, 146(729), 1960–1975. <https://doi.org/10.1002/qj.3777>
- Emanuel, K. A., Fantini, M., & Thorpe, A. J. (1987). Baroclinic instability in an environment of small stability to slantwise moist convection. *Journal of the Atmospheric Sciences*, 44(12), 1559–1573. [https://doi.org/10.1175/1520-0469\(1987\)044<1559:biaeo>2.0.co;2](https://doi.org/10.1175/1520-0469(1987)044<1559:biaeo>2.0.co;2)
- Ferguson, J., Khouider, B., & Namazi, M. (2009). Two-way interactions between equatorially-trapped waves and the barotropic flow. *Chinese Annals of Mathematics, Series B*, 30(5), 539–568. <https://doi.org/10.1007/s11401-009-0102-9>
- Fulton, S. R., & Schubert, W. H. (1985). Vertical normal mode transforms: Theory and application. *Monthly Weather Review*, 113(4), 647–658. [https://doi.org/10.1175/1520-0493\(1985\)113<0647:vnmtta>2.0.co;2](https://doi.org/10.1175/1520-0493(1985)113<0647:vnmtta>2.0.co;2)
- Gehne, M., & Kleeman, R. (2012). Spectral analysis of tropical atmospheric dynamical variables using a linear shallow-water modal decomposition. *Journal of the Atmospheric Sciences*, 69(7), 2300–2316. <https://doi.org/10.1175/jas-d-10-05008.1>
- Gill, A. (1980). Some simple solutions for heat-induced tropical circulation. *Quarterly Journal of the Royal Meteorological Society*, 106(449), 447–462. <https://doi.org/10.1002/qj.49710644905>
- Gill, A., & Clarke, A. (1974). Wind-induced upwelling, coastal currents and sea-level changes. *Deep-Sea Research and Oceanographic Abstracts*, 21(5), 325–345. [https://doi.org/10.1016/0011-7471\(74\)90038-2](https://doi.org/10.1016/0011-7471(74)90038-2)
- Gimeno, L., Nieto, R., Vázquez, M., & Lavers, D. A. (2014). Atmospheric rivers: A mini-review. *Frontiers in Earth Science*, 2, 2. <https://doi.org/10.3389/feart.2014.00002>
- Gottwald, G., & Grimshaw, R. (1999). The formation of coherent structures in the context of blocking. *Journal of the Atmospheric Sciences*, 56(21), 3640–3662. [https://doi.org/10.1175/1520-0469\(1999\)056<3640:tfocsi>2.0.co;2](https://doi.org/10.1175/1520-0469(1999)056<3640:tfocsi>2.0.co;2)

- Hersbach, H., Bell, B., Berrisford, P., Biavati, G., Horányi, A., Muñoz Sabater, J., et al. (2023). Copernicus climate change service, climate data Store, (2023): ERA5 hourly data on pressure levels from 1940 to present [Dataset]. *copernicus climate change service (C3S) climate data store (CDS)*. <https://doi.org/10.24381/cds.143582cf>
- Holube, K. M., Lunkeit, F., Vasylykevych, S., & Žagar, N. (2025). Effects of dynamical and diabatic processes on stationary and transient Kelvin waves. *Journal of Advances in Modeling Earth Systems*, 17(12), e2025MS005225. <https://doi.org/10.1029/2025MS005225>
- Hoskins, B. J., McIntyre, M. E., & Robertson, A. W. (1985). On the use and significance of isentropic potential vorticity maps. *Quarterly Journal of the Royal Meteorological Society*, 111(470), 877–946. <https://doi.org/10.1002/qj.49711147002>
- Hough, S. S. (1898). V. On the application of harmonic analysis to the dynamical theory of the tides.—Part II. On the general integration of Laplace's dynamical equations. *Philosophical Transactions of the Royal Society of London - Series A: Containing Papers of a Mathematical or Physical Character*, 191, 139–185. <https://doi.org/10.1098/rsta.1898.0005>
- Kalnay, E. (2003). *Atmospheric modeling, data assimilation and predictability*. Cambridge University Press.
- Kasahara, A. (1976). Normal modes of ultralong waves in the atmosphere. *Monthly Weather Review*, 104(6), 669–690. [https://doi.org/10.1175/1520-0493\(1976\)104<0669:nmouwi>2.0.co;2](https://doi.org/10.1175/1520-0493(1976)104<0669:nmouwi>2.0.co;2)
- Kasahara, A. (2020). 3D normal mode functions (NMFS) of a global baroclinic atmospheric model. *Modal view of atmospheric variability: Applications of normal-mode function decomposition in weather and climate research*, 1–61.
- Kasahara, A., & Puri, K. (1981). Spectral representation of three-dimensional global data by expansion in normal mode functions. *Monthly Weather Review*, 109(1), 37–51. [https://doi.org/10.1175/1520-0493\(1981\)109<0037:srotgdg>2.0.co;2](https://doi.org/10.1175/1520-0493(1981)109<0037:srotgdg>2.0.co;2)
- Kelly, S. M. (2016). The vertical mode decomposition of surface and internal tides in the presence of a free surface and arbitrary topography. *Journal of Physical Oceanography*, 46(12), 3777–3788. <https://doi.org/10.1175/jpo-d-16-0131.1>
- Khouider, B., & Majda, A. J. (2006). A simple multicloud parameterization for convectively coupled tropical waves. Part I: Linear analysis. *Journal of the Atmospheric Sciences*, 63(4), 1308–1323. <https://doi.org/10.1175/jas3677.1>
- Khouider, B., Majda, A. J., & Stechmann, S. N. (2012). Climate science in the tropics: Waves, vortices and PDEs. *Nonlinearity*, 26(1), R1–R68. <https://doi.org/10.1088/0951-7715/26/1/r1>
- Knippertz, P., Gehne, M., Kiladis, G. N., Kikuchi, K., Rasheeda Satheesh, A., Roundy, P. E., et al. (2022). The intricacies of identifying equatorial waves. *Quarterly Journal of the Royal Meteorological Society*, 148(747), 2814–2852. <https://doi.org/10.1002/qj.4338>
- Kohl, M., & O'Gorman, P. A. (2022). The diabatic Rossby vortex: Growth rate, length scale, and the wave–vortex transition. *Journal of the Atmospheric Sciences*, 79(10), 2739–2755. <https://doi.org/10.1175/jas-d-22-0022.1>
- Kumm, B. (2025). Code for manuscript, moisture decomposition with normal modes in global data: Balanced and unbalanced components [Software]. *Zenodo*. <https://doi.org/10.5281/zenodo.17136513>
- Lee, H.-I., & Mitchell, J. L. (2021). The dynamics of quasi-stationary atmospheric rivers and their implications for monsoon onset. *Journal of the Atmospheric Sciences*, 78(8), 2353–2365. <https://doi.org/10.1175/jas-d-20-0262.1>
- Longuet-Higgins, M. S. (1968). The eigenfunctions of Laplace's tidal equation over a sphere. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 262(1132), 511–607. <https://doi.org/10.1098/rsta.1968.0003>
- Lorenc, A. (1981). A global three-dimensional multivariate statistical interpolation scheme. *Monthly Weather Review*, 109(4), 701–721. [https://doi.org/10.1175/1520-0493\(1981\)109<0701:agtdms>2.0.co;2](https://doi.org/10.1175/1520-0493(1981)109<0701:agtdms>2.0.co;2)
- Madden, R. A. (1979). Observations of large-scale traveling Rossby waves. *Reviews of Geophysics*, 17(8), 1935–1949. <https://doi.org/10.1029/rg017i008p01935>
- Majda, A. J., & Biello, J. A. (2003). The nonlinear interaction of barotropic and equatorial baroclinic Rossby waves. *Journal of the Atmospheric Sciences*, 60(15), 1809–1821. [https://doi.org/10.1175/1520-0469\(2003\)060<1809:tmioba>2.0.co;2](https://doi.org/10.1175/1520-0469(2003)060<1809:tmioba>2.0.co;2)
- Mapes, B. E. (1993). Gregarious tropical convection. *Journal of the Atmospheric Sciences*, 50(13), 2026–2037. [https://doi.org/10.1175/1520-0469\(1993\)050<2026:gte>2.0.co;2](https://doi.org/10.1175/1520-0469(1993)050<2026:gte>2.0.co;2)
- Mapes, B. E. (2000). Convective inhibition, subgrid-scale triggering energy, and stratiform instability in a toy tropical wave model. *Journal of the Atmospheric Sciences*, 57(10), 1515–1535. [https://doi.org/10.1175/1520-0469\(2000\)057<1515:cisste>2.0.co;2](https://doi.org/10.1175/1520-0469(2000)057<1515:cisste>2.0.co;2)
- Marques, C. A., & Castanheira, J. M. (2018). Diagnosis of free and convectively coupled equatorial waves. *Mathematical Geosciences*, 50(5), 585–606. <https://doi.org/10.1007/s11004-018-9729-y>
- Marsico, D. H., Smith, L. M., & Stechmann, S. N. (2019). Energy decompositions for moist Boussinesq and anelastic equations with phase changes. *Journal of the Atmospheric Sciences*, 76(11), 3569–3587. <https://doi.org/10.1175/jas-d-19-0080.1>
- Molemaker, M. J., McWilliams, J. C., & Yavneh, I. (2005). Baroclinic instability and loss of balance. *Journal of Physical Oceanography*, 35(9), 1505–1517. <https://doi.org/10.1175/jpo2770.1>
- Neduh, V. (2024a). Modes data - Hough coefficients v2 [Dataset]. *Zenodo*. <https://doi.org/10.5281/zenodo.12724196>
- Neduh, V. (2024b). Modes data - Hough functions and frequencies - 1 v3 [Dataset]. *Zenodo*. <https://doi.org/10.5281/zenodo.12749244>
- Neduh, V. (2024c). Modes data - Hough functions and frequencies - 2 v3 [Dataset]. *Zenodo*. <https://doi.org/10.5281/zenodo.12749316>
- Neduh, V. (2024d). Modes data - Hough functions and frequencies - 3 v3 [Dataset]. *Zenodo*. <https://doi.org/10.5281/zenodo.12749407>
- Neduh, V. (2024e). Modes data - Hough functions and frequencies - 4 v3 [Dataset]. *Zenodo*. <https://doi.org/10.5281/zenodo.12749482>
- Neduh, V. (2024f). Modes data - Hough functions and frequencies - 5 v3 [Dataset]. *Zenodo*. <https://doi.org/10.5281/zenodo.12751158>
- Neduh, V. (2024g). Modes data - Hough functions and frequencies - 6 v3 [Dataset]. *Zenodo*. <https://doi.org/10.5281/zenodo.12751242>
- Neduh, V. (2024h). Modes data - Hough functions and frequencies - 7 v3 [Dataset]. *Zenodo*. <https://doi.org/10.5281/zenodo.12751345>
- Neduh, V. (2024i). Modes data - Hough functions and frequencies - 8 v3 [Dataset]. *Zenodo*. <https://doi.org/10.5281/zenodo.12751416>
- Neduh, V. (2024j). Modes data - Vertical structure functions v2 [Dataset]. *Zenodo*. <https://doi.org/10.5281/zenodo.12726172>
- Neduh, V., Žagar, N., Lunkeit, F., Polichtchouk, I., & Zaplotnik, Ž. (2024). Decomposition of the horizontal wind divergence associated with the Rossby, mixed Rossby-gravity, inertia-gravity, and Kelvin waves on the sphere. *Journal of Geophysical Research: Atmospheres*, 129(9), e2023JD040427. <https://doi.org/10.1029/2023jd040427>
- Neelin, J. D., & Yu, J.-Y. (1994). Modes of tropical variability under convective adjustment and the Madden-Julian oscillation. Part I: Analytical theory. *Journal of the Atmospheric Sciences*, 51(13), 1876–1894. [https://doi.org/10.1175/1520-0469\(1994\)051<1876:motvuc>2.0.co;2](https://doi.org/10.1175/1520-0469(1994)051<1876:motvuc>2.0.co;2)
- Ogrosky, H. R., & Stechmann, S. N. (2015). The MJO skeleton model with observation-based background state and forcing. *Quarterly Journal of the Royal Meteorological Society*, 141(692), 2654–2669. <https://doi.org/10.1002/qj.2552>
- Ogrosky, H. R., & Stechmann, S. N. (2016). Identifying convectively coupled equatorial waves using theoretical wave eigenvectors. *Monthly Weather Review*, 144(6), 2235–2264. <https://doi.org/10.1175/mwr-d-15-0292.1>
- Ogrosky, H. R., Wetzel, A. N., Smith, L. M., & Stechmann, S. N. (2025). Data assimilation errors due to incomplete accounting of balanced versus unbalanced moisture. *Quarterly Journal of the Royal Meteorological Society*, 151(769), e4942. in press. <https://doi.org/10.1002/qj.4942>
- Platzman, G. W. (1967). A retrospective view of Richardson's book on weather prediction. *Bulletin of the American Meteorological Society*, 48(8), 514–551. <https://doi.org/10.1175/1520-0477-48.8.514>

- Ramos, A. M., Nieto, R., Tomé, R., Gimeno, L., Trigo, R. M., Liberato, M. L., & Lavers, D. A. (2016). Atmospheric rivers moisture sources from a Lagrangian perspective. *Earth System Dynamics*, 7(2), 371–384. <https://doi.org/10.5194/esd-7-371-2016>
- Remond-Tiedrez, A., Smith, L. M., & Stechmann, S. N. (2026). Beyond linear decomposition: A nonlinear eigenspace decomposition for a moist atmosphere with clouds. *Nonlinearity*, 39(3), 035014. <https://doi.org/10.1088/1361-6544/ae4730>
- Richardson, L. F. (1922). *Weather prediction by numerical process*. Cambridge University Press.
- Smith, L. M., & Stechmann, S. N. (2017). Precipitating quasigeostrophic equations and potential vorticity inversion with phase changes. *Journal of the Atmospheric Sciences*, 74(10), 3285–3303. <https://doi.org/10.1175/jas-d-17-0023.1>
- Sodemann, H. (2025). The Lagrangian moisture source and transport diagnostic WaterSip V3.2. *Geoscientific Model Development*, 18(22), 8887–8926. <https://doi.org/10.5194/gmd-18-8887-2025>
- Staniforth, A., Beland, M., & Côté, J. (1985). An analysis of the vertical structure equation in sigma coordinates. *Atmosphere-Ocean*, 23(4), 323–358. <https://doi.org/10.1080/07055900.1985.9649232>
- Stechmann, S. N., & Hottovy, S. (2017). Unified spectrum of tropical rainfall and waves in a simple stochastic model. *Geophysical Research Letters*, 44(20), 10–713. <https://doi.org/10.1002/2017gl075754>
- Stechmann, S. N., & Majda, A. J. (2009). Gravity waves in shear and implications for organized convection. *Journal of the Atmospheric Sciences*, 66(9), 2579–2599. <https://doi.org/10.1175/2009jas2976.1>
- Stechmann, S. N., & Majda, A. J. (2015). Identifying the skeleton of the Madden-Julian oscillation in observational data. *Monthly Weather Review*, 143(1), 395–416. <https://doi.org/10.1175/mwr-d-14-00169.1>
- Stechmann, S. N., & Ogrosky, H. R. (2014). The Walker circulation, diabatic heating, and outgoing longwave radiation. *Geophysical Research Letters*, 41(24), 9097–9105. <https://doi.org/10.1002/2014gl062257>
- Stephan, C. C., Žagar, N., & Shepherd, T. (2021). Waves and coherent flows in the tropical atmosphere: New opportunities, old challenges. *The Quarterly Journal of the Royal Meteorological Society*, 147(738), 2597–2624. <https://doi.org/10.1002/qj.4109>
- Swarztrauber, P. N., & Kasahara, A. (1985). The vector harmonic analysis of Laplace's tidal equations. *SIAM Journal on Scientific and Statistical Computing*, 6(2), 464–491. <https://doi.org/10.1137/0906033>
- Tanaka, H. (1985). Global energetics analysis by expansion into three-dimensional normal-mode functions during the FGGE winter. *J. Meteor. Soc. Japan*, 63(2), 180–200. https://doi.org/10.2151/jmsj1965.63.2_180
- Wetzel, A. N., Smith, L. M., Stechmann, S. N., & Martin, J. E. (2019). Balanced and unbalanced components of moist atmospheric flows with phase changes. *Chinese Annals of Mathematics, Series B*, 40(6), 1005–1038. <https://doi.org/10.1007/s11401-019-0170-4>
- Wetzel, A. N., Smith, L. M., Stechmann, S. N., Martin, J. E., & Zhang, Y. (2020). Potential vorticity and balanced and unbalanced moisture. *Journal of the Atmospheric Sciences*, 77(6), 1913–1931. <https://doi.org/10.1175/jas-d-19-0311.1>
- Yang, G.-Y., Hoskins, B., & Slingo, J. (2003). Convectively coupled equatorial waves: A new methodology for identifying wave structures in observational data. *Journal of the Atmospheric Sciences*, 60(14), 1637–1654. [https://doi.org/10.1175/1520-0469\(2003\)060<1637:ccewan>2.0.co;2](https://doi.org/10.1175/1520-0469(2003)060<1637:ccewan>2.0.co;2)
- Žagar, N., Buizza, R., & Tribbia, J. (2015). A three-dimensional multivariate modal analysis of atmospheric predictability with application to the ECMWF ensemble. *Journal of the Atmospheric Sciences*, 72(11), 4423–4444. <https://doi.org/10.1175/jas-d-15-0061.1>
- Žagar, N., & Franzke, C. L. (2015). Systematic decomposition of the Madden-Julian oscillation into balanced and inertia-gravity components. *Geophysical Research Letters*, 42(16), 6829–6835. <https://doi.org/10.1002/2015gl065130>
- Žagar, N., Gustafsson, N., & Källén, E. (2004). Variational data assimilation in the tropics: The impact of a background error constraint. *Quarterly Journal of the Royal Meteorological Society*, 130(596), 103–125. <https://doi.org/10.1256/qj.03.13>
- Žagar, N., Jelić, D., Blaauw, M., & Bechtold, P. (2017). Energy spectra and inertia-gravity waves in global analyses. *Journal of the Atmospheric Sciences*, 74(8), 2447–2466. <https://doi.org/10.1175/jas-d-16-0341.1>
- Žagar, N., Kasahara, A., Terasaki, K., Tribbia, J., & Tanaka, H. (2015). Normal-mode function representation of global 3-d data sets: Open-access software for the atmospheric research community. *Geoscientific Model Development*, 8(4), 1169–1195. <https://doi.org/10.5194/gmd-8-1169-2015>
- Žagar, N., Neduhaj, V., Vasylyevych, S., Zaplotnik, Ž., & Tanaka, H. L. (2023). Decomposition of vertical velocity and its zonal wavenumber kinetic energy spectra in the hydrostatic atmosphere. *Journal of the Atmospheric Sciences*, 80(11), 2747–2767. <https://doi.org/10.1175/jas-d-23-0090.1>
- Žagar, N., & Tribbia, J. (2020). Modal view of atmospheric variability: Applications of normal-mode function decomposition in weather and climate research. In *Mathematics of Planet Earth (MPE)* (Vol. 8). Springer Nature. <https://doi.org/10.1007/978-3-030-60963-4>
- Žagar, N., Tribbia, J., Anderson, J. L., & Raeder, K. (2009). Uncertainties of estimates of inertia-gravity energy in the atmosphere. Part I: Intercomparison of four analysis datasets. *Monthly Weather Review*, 137(11), 3837–3857. <https://doi.org/10.1175/2009mwr2815.1>
- Zhang, Y., Smith, L. M., & Stechmann, S. N. (2021). Effects of clouds and phase changes on fast-wave averaging: A numerical assessment. *Journal of Fluid Mechanics*, 920, A49. <https://doi.org/10.1017/jfm.2021.427>
- Zhao, N., Manda, A., Guo, X., Kikuchi, K., Nasuno, T., Nakano, M., et al. (2021). A Lagrangian view of moisture transport related to the heavy rainfall of July 2020 in Japan: Importance of the moistening over the subtropical regions. *Geophysical Research Letters*, 48(5), e2020GL091441. <https://doi.org/10.1029/2020gl091441>