Supporting Information for "Conservation laws for potential vorticity in a salty ocean or cloudy atmosphere"

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Contents of this file

1. Text S1 to S4

Introduction Supplementary information contains derivations of the evolution equations for circulation/potential vorticity and the dry circulation theorem for the purpose of completeness. Also provided is an alternate derivation of the moist potential vorticity conservation theorem.

Text S1. Evolution of circulation

The definition of the circulation Γ is the integral of the velocity $\vec{u}$ along a closed curve C:

$$\Gamma = \oint_C \vec{u} \cdot d\vec{x} = \oint_C \vec{u} \cdot \frac{\partial \vec{x}}{\partial \sigma} d\sigma, \tag{1}$$

where the curve has been parametrized with parameter σ.

The evolution equation for the circulation $\Gamma(t)$ can be derived as follows, for a curve $C(t)$ that moves with the fluid. The starting point is the momentum equation for an inviscid fluid with a conservative body force acting on it:

$$\frac{D\vec{u}}{Dt} = -\rho^{-1}\nabla p + \nabla\phi, \tag{2}$$

where $D/Dt = \partial/\partial t + \vec{u} \cdot \nabla$ is the material derivative, ρ is the density of the fluid, p is the pressure and ϕ is the force potential. Now, by taking a material derivative of (1), we have

$$\frac{D\Gamma}{Dt} = \oint \frac{D\vec{u}}{Dt} \cdot d\vec{x} + \oint \vec{u} \cdot \frac{D}{Dt}\left(\frac{\partial \vec{x}}{\partial \sigma}\right) d\sigma = \oint \frac{D\vec{u}}{Dt} \cdot d\vec{x} + \oint \vec{u} \cdot \frac{\partial \vec{u}}{\partial \sigma} d\sigma. \tag{3}$$

The last term on the right hand side is an exact differential $d\left(\frac{1}{2}\vec{u} \cdot \vec{u}\right)$ and consequently goes to zero when integrated along a closed curve. The momentum equation can then be used to replace $\frac{D\vec{u}}{Dt}$ and arrive at

$$\frac{D\Gamma}{Dt} = \oint \frac{D\vec{u}}{Dt} \cdot d\vec{x} = -\oint \rho^{-1}\nabla p \cdot d\vec{x} + \oint \nabla\phi \cdot d\vec{x}. \tag{4}$$

The second integral can be written as the integral of an exact differential $d\phi$ and therefore is zero. From this, we obtain our final result,

$$\frac{D\Gamma}{Dt} = -\oint \rho^{-1}\nabla p \cdot d\vec{x}, \tag{5}$$

the evolution equation for circulation Γ.

One can also easily incorporate the effects of rotation. In this case, the momentum evolution equation becomes

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} = -\rho^{-1}\nabla p + \nabla \phi, \tag{6}$$

which includes the Coriolis term, $2\vec{\Omega} \times \vec{u}$, where $\vec{\Omega}$ is the rotation vector. Following a similar derivation as above, one can arrive at the same evolution equation for circulation Γ as in (5), except for a modified definition of the circulation,

$$\Gamma = \oint_C (\vec{u} + \vec{\Omega} \times \vec{x}) \cdot d\vec{x}, \tag{7}$$

which includes a term due to rotation.

Text S2. Circulation theorem (dry atmosphere)

We review a derivation of the circulation theorem for a dry atmosphere (i.e., for an atmosphere with no moisture and no clouds). This dry derivation is useful for illustrating that it is difficult to see how a similar derivation could be used for a moist atmosphere with clouds and phase changes.

The derivation starts from (5), which can be rewritten as

$$\frac{D\Gamma}{Dt} = -\oint \rho^{-1}\, dp. \tag{8}$$

The idea now is to rewrite the integrand in a way that will expose an exact differential. To do this, define the potential temperature, θ, as

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}. \tag{9}$$

where p_0 is a constant reference pressure, c_p is the specific heat at constant pressure, and R is the gas constant from the ideal gas law, $p = \rho R T$. Now introduce the Exner function,

$\pi(p)$, defined as

$$\pi = c_p \left(\frac{p}{p_0}\right)^{\frac{R}{c_p}}, \tag{10}$$

and use the ideal gas law, $p = \rho RT$, to see that

$$\oint \rho^{-1} dp = \oint \theta d\pi. \tag{11}$$

Therefore, (8) can be rewritten as

$$\frac{D\Gamma}{Dt} = -\oint \theta d\pi. \tag{12}$$

Note that $\theta d\pi$ is not, in general, an exact differential. However, by restricting the material circuit to a surface of constant potential temperature θ, it is an exact differential and we have

$$\frac{D\Gamma}{Dt} = 0, \tag{13}$$

where Γ is the circulation of material curves on a level surface of constant potential temperature, or on a level surface of constant entropy, since $s = c_p \log \theta + const.$ for a dry atmosphere, where c_p, commonly assumed constant for a dry atmosphere, is the specific heat at constant pressure. In order for the material curve $C(t)$ to remain on a surface of constant entropy for all times, the fluid is assumed to be adiabatic, so that $Ds/Dt = 0$.

Text S3. Evolution of PV

A derivation of the evolution equation for PV is now presented. The derivation follows common derivations (Schubert et al., 2001; Vallis, 2017) and is presented here for completeness and to keep the manuscript self-contained.

The momentum equation in (2) can be rewritten using $\vec{u}\cdot\nabla\vec{u} = \frac{1}{2}\nabla(\vec{u}\cdot\vec{u}) + \vec{\omega}\times\vec{u}$ to arrive at

$$\frac{\partial \vec{u}}{\partial t} + \vec{\omega}\times\vec{u} + \nabla\left(\frac{1}{2}\vec{u}\cdot\vec{u} - \phi\right) + \rho^{-1}\nabla p = 0, \tag{14}$$

where $\vec{\omega} = \nabla\times\vec{u}$ is the vorticity. One can then obtain the vorticity equation by taking the curl of the momentum equation to arrive at

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla\times(\vec{\omega}\times\vec{u}) = \nabla\rho^{-1}\times\nabla p. \tag{15}$$

From the vorticity equation, the evolution equation for $PV_\psi = \frac{1}{\rho}\vec{\omega}\cdot\nabla\psi$, for any fluid dynamic variable ψ, can be obtained by first taking a dot product with $\nabla\psi$:

$$\frac{\partial \vec{\omega}\cdot\nabla\psi}{\partial t} + \nabla\psi\cdot(\nabla\times(\vec{\omega}\times\vec{u})) = \nabla\psi\cdot\nabla\rho^{-1}\times\nabla p + \vec{\omega}\cdot\nabla\frac{\partial\psi}{\partial t}. \tag{16}$$

The equation can be further simplified by noting that

$$\nabla\psi\cdot(\nabla\times(\vec{\omega}\times\vec{u})) = \nabla\cdot(\nabla\psi\times(\vec{u}\times\vec{\omega})) - (\vec{u}\times\vec{\omega})\cdot(\nabla\times\nabla\psi) = \nabla\cdot(\nabla\psi\times(\vec{u}\times\vec{\omega})), \tag{17}$$

where the second term goes to zero since $\nabla\times\nabla A = 0$ for all A. Furthermore, using $\nabla\psi\times(\vec{u}\times\vec{\omega}) = \vec{u}(\vec{\omega}\cdot\nabla\psi) - \vec{\omega}(\vec{u}\cdot\nabla\psi)$, (16) reduces to

$$\nabla\psi\cdot\frac{\partial\vec{\omega}}{\partial t} + \nabla\cdot(\vec{u}(\vec{\omega}\cdot\nabla\psi) - \vec{\omega}(\vec{u}\cdot\nabla\psi)) = \nabla\psi\cdot\nabla\rho^{-1}\times\nabla p, \tag{18}$$

or

$$\nabla\psi\cdot\frac{\partial\vec{\omega}}{\partial t} + \vec{u}\cdot\nabla(\vec{\omega}\cdot\nabla\psi) + (\nabla\cdot\vec{u})(\vec{\omega}\cdot\nabla\psi) - (\nabla\cdot\vec{\omega})(\vec{u}\cdot\nabla\psi) - \vec{\omega}\cdot\nabla(\vec{u}\cdot\nabla\psi) = \nabla\psi\cdot\nabla\rho^{-1}\times\nabla p. \tag{19}$$

Using the definition of the material derivative D/Dt, we get,

$$\nabla\psi\cdot\frac{\partial\vec{\omega}}{\partial t} + \vec{u}\cdot\nabla(\vec{\omega}\cdot\nabla\psi) + (\nabla\cdot\vec{u})(\vec{\omega}\cdot\nabla\psi) - \vec{\omega}\cdot\nabla\left(\frac{D\psi}{Dt} - \frac{\partial\psi}{\partial t}\right) = \nabla\psi\cdot\nabla\rho^{-1}\times\nabla p. \tag{20}$$

Now, be rearranging terms, we have,

$$\nabla\psi \cdot \frac{\partial \vec{\omega}}{\partial t} + \vec{\omega} \cdot \frac{\partial \nabla\psi}{\partial t} + \vec{u} \cdot \nabla(\vec{\omega} \cdot \nabla\psi) + (\nabla \cdot \vec{u})(\vec{\omega} \cdot \nabla\psi) - \vec{\omega} \cdot \nabla \left(\frac{D\psi}{Dt}\right) = \nabla\psi \cdot \nabla\rho^{-1} \times \nabla p. \tag{21}$$

Simplifying further using the definition of D/Dt and product rule,

$$\frac{D}{Dt}(\vec{\omega} \cdot \nabla\psi) + (\nabla \cdot \vec{u})(\vec{\omega} \cdot \nabla\psi) = \nabla\psi \cdot \nabla\rho^{-1} \times \nabla p + \vec{\omega} \cdot \nabla \left(\frac{D\psi}{Dt}\right), \tag{22}$$

or

$$\rho \frac{D}{Dt}\left(\frac{1}{\rho}\vec{\omega} \cdot \nabla\psi\right) - \rho\,(\vec{\omega} \cdot \nabla\psi)\,\frac{D}{Dt}\left(\frac{1}{\rho}\right) + (\nabla \cdot \vec{u})(\vec{\omega} \cdot \nabla\psi) = \nabla\psi \cdot \nabla\rho^{-1} \times \nabla p + \vec{\omega} \cdot \nabla \left(\frac{D\psi}{Dt}\right), \tag{23}$$

or

$$\rho \frac{D}{Dt}\left(\frac{1}{\rho}\vec{\omega} \cdot \nabla\psi\right) + \frac{1}{\rho}(\vec{\omega} \cdot \nabla\psi)\,\frac{D\rho}{Dt} + (\nabla \cdot \vec{u})(\vec{\omega} \cdot \nabla\psi) = \nabla\psi \cdot \nabla\rho^{-1} \times \nabla p + \vec{\omega} \cdot \nabla \left(\frac{D\psi}{Dt}\right). \tag{24}$$

At this point, we can use the evolution equation for density, ρ given by $\frac{D\rho}{Dt} + \rho\nabla \cdot \vec{u} = 0$ to arrive at the evolution equation for PV_ψ,

$$\rho \frac{D}{Dt}\left(\frac{1}{\rho}\vec{\omega} \cdot \nabla\psi\right) = \nabla\psi \cdot \nabla\rho^{-1} \times \nabla p + \vec{\omega} \cdot \nabla \left(\frac{D\psi}{Dt}\right), \tag{25}$$

where ψ can be any scalar and is commonly chosen to be the entropy s.

One can also easily incorporate the effects of rotation. In this case, the momentum evolution equation becomes

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} = -\rho^{-1}\nabla p + \nabla\phi, \tag{26}$$

which includes the Coriolis term, $2\vec{\Omega} \times \vec{u}$, where $\vec{\Omega}$ is the rotation vector. Following a similar derivation as above, one can arrive at the same evolution equation for PV_ψ as in

(25), except for a modified definition of the potential vorticity,

$$PV_\psi = \frac{1}{\rho}\vec{\omega}_a \cdot \nabla\psi, \tag{27}$$

which is the same as the original PV_ψ definition except with vorticity, $\vec{\omega}$, replaced by the absolute vorticity $\vec{\omega}_a = \vec{\omega} + 2\vec{\Omega}$.

Text S4. Local volume-integrated moist PV conservation: Alternative derivation

Here is an alternative direct proof for moist PV conservation, without explicitly referring to enthalpy. Now take $\psi = s$ and integrate over a distorted cylinder with base and lid given by $s = C_1$ and $s = C_2$, respectively, and sides given by $q_t = q_t(s)$. The first steps in evaluating the integral are

$$\frac{D}{Dt}\int_{V_m} dV \; (\vec{\omega}\cdot\nabla s) = \frac{D}{Dt}\int_{V_a} dV_a J\,(\vec{\omega}\cdot\nabla s) \tag{28}$$

$$= \int_{V_a} dV_a \,\frac{\partial}{\partial t}\left(\frac{\vec{\omega}\cdot\nabla s}{\rho}\right) \tag{29}$$

$$= \int_{V_a} dV_a J\,\rho\,\frac{\partial}{\partial t}\left(\frac{\vec{\omega}\cdot\nabla s}{\rho}\right) \tag{30}$$

$$= \int_{V_m} dV\,\rho\,\frac{D}{Dt}\left(\frac{\vec{\omega}\cdot\nabla s}{\rho}\right) \tag{31}$$

The second line results from knowing $J = \frac{\partial(\vec{x})}{\partial(\vec{a})} = \frac{1}{\rho}$ by appropriately choosing particle label $\vec{a}$ as done by Salmon (1998). Now, using the divergence theorem, since $\nabla s\cdot\nabla\times\left(\frac{1}{\rho}\nabla p\right) = \nabla\cdot\left(s\nabla\times\left(\frac{1}{\rho}\nabla p\right)\right)$, we have

$$\frac{D}{Dt}\int_{V_m} dV \; (\vec{\omega}\cdot\nabla s) = \int_{V_m} dV\;\nabla\cdot\left(s\nabla\times\left(\frac{1}{\rho}\nabla p\right)\right) \tag{32}$$

$$= \oint_{S_m} d\vec{S} \cdot s\nabla \times \left(\frac{1}{\rho}\nabla p\right) = \oint_{S_m} d\vec{S} \cdot \left(\nabla \times \left(\frac{1}{\rho}s\nabla p\right) - \left(\frac{1}{\rho}\nabla s \times \nabla p\right)\right) \tag{33}$$

$$= -\oint_{S_m} d\vec{S} \cdot \left(\frac{1}{\rho}\nabla s \times \nabla p\right) \tag{34}$$

On the base and lid, $d\vec{S} \parallel \nabla s$ i.e., the normal to the surface is perpendicular to ∇s. If $q_t = q_t(s)$ on the sides, then

$$-\int\int_{S_{sides}} d\vec{S} \cdot \left(\frac{1}{\rho}\nabla s \times \nabla p\right) = -\int\int_{S_{sides}} d\vec{S} \cdot (f(p,s)\nabla s \times \nabla p) = -\int\int_{S_{sides}} d\vec{S} \cdot \nabla \times (g(p,s)\nabla p)$$

where $\partial g(p,s)/\partial s = f(p,s)$. Now using Stokes' theorem, we have,

$$-\int\int_{S_{sides}} d\vec{S} \cdot \left(\frac{1}{\rho}\nabla s \times \nabla p\right) = -\int_{C_1} d\vec{x} \cdot (g(p,s)\nabla p) + \int_{C_2} d\vec{x} \cdot (g(p,s)\nabla p) \tag{35}$$

$$-\int\int_{S_{sides}} d\vec{S} \cdot \left(\frac{1}{\rho}\nabla s \times \nabla p\right) = \int_C d\vec{x} \cdot (g(p)\nabla p) \tag{36}$$

$$= \int_C d\vec{x} \cdot \nabla G(p) = \int_C dG(p) = 0 \tag{37}$$

References

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