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Conservation Laws for Potential Vorticity in a Salty Ocean or Cloudy Atmosphere

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Key Points:

- Important conservation laws of circulation and potential vorticity are known only under idealized assumptions
- Here, conservation laws are described with additional realism, namely salinity in the ocean or moisture and clouds in the atmosphere
- Potential vorticity is not conserved for each fluid parcel but for certain pancake-shaped volumes

Supporting Information:

Supporting Information may be found in the online version of this article.

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Abstract One of the most important conservation laws in atmospheric and oceanic science is conservation of potential vorticity. The original derivation is approximately a century old, in the work of Rossby and Ertel, and it is related to the celebrated circulation theorems of Kelvin and Bjerknes. However, the laws apply to idealized fluids, and extensions to more realistic scenarios have been problematic. Here, these laws are extended to hold with additional fundamental complexities, including salinity in the ocean, or moisture and clouds in the atmosphere. In the absence of these additional complexities, it is known that potential vorticity is conserved following each fluid parcel; here, for a salty ocean or cloudy atmosphere, the general conserved quantity is potential vorticity integrated over certain pancake-shaped volumes. Furthermore, the conservation laws are also related to a symmetry in the Lagrangian, which brings a connection to the symmetry-conservation relationships seen in other areas of physics.

Plain Language Summary A vortex can be seen in the atmosphere or ocean in a variety of settings, including the intense vortices that are familiar from tornadoes and hurricanes. A quantity called vorticity is a measure of the strength of a vortex, or, more generally, of the amount of swirl in a fluid. Intimately connected to vorticity is an important conserved quantity referred to as potential vorticity. This quantity remains unchanged following the trajectory of a fluid parcel. Owing to its conserved nature, potential vorticity is widely used to study a wide range of weather events. However, this century-old conservation result is only valid in an idealized setting: an atmosphere devoid of clouds or an ocean without dissolved salt—i.e., without salinity. Here, the conservation law is extended to hold in a cloudy atmosphere and salty oceans. The generalized conservation statement presented here will aid in the wider applicability of potential vorticity for analyzing and understanding atmospheric and oceanic dynamics.

1. Introduction

In addition to conservation laws of mass, momentum, and energy, other conservation laws arise for fluids, such as conservation of circulation and potential vorticity. The circulation Γ is defined as the integral of the velocity $\vec{u}$ along a closed curve C,

$$\Gamma = \oint_C \vec{u} \cdot d\vec{l}, \tag{1}$$

and the circulation theorems of Kelvin and Bjerknes (Bjerknes, 1898; Thomson, 1867; Thorpe et al., 2003) show that, under certain assumptions, the circulation is conserved if the closed curve $C(t)$ moves with the fluid. See Figure 1 for an illustration. The circulation is also related to the vorticity, $\vec{\omega} = \nabla \times \vec{u}$, since Equation 1 can be rewritten, using Stokes' theorem, as

$$\Gamma = \int_A \vec{\omega} \cdot d\vec{A}, \tag{2}$$

which is an area integral over any surface A whose boundary is the closed curve C, and where $d\vec{A}$ is a surface area element whose vector direction is perpendicular to the surface. Helmholtz (1858) had earlier formulated some related conservation statements in terms of vorticity. An associated quantity is the potential vorticity (PV), which is the dot product of the vorticity vector and the entropy gradient, ∇s, weighted by the fluid density ρ; that is,

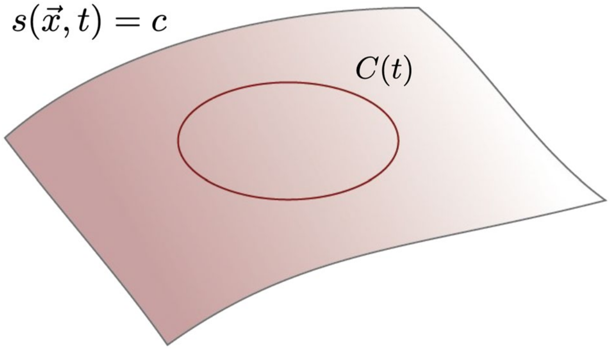

Figure 1. Geometry of the circulation theorem for a dry atmosphere or freshwater ocean. The curve $C(t)$ may be any curve on a surface of constant entropy, $s(\vec{x}, t) = c$.

$$PV = \frac{\vec{\omega} \cdot \nabla s}{\rho}. \tag{3}$$

Ertel (Ertel, 1942) showed that PV is conserved following the trajectory of a fluid parcel, which builds on an earlier result of Rossby (1940) for the shallow water equations.

PV is a fundamental quantity of interest in geophysical flows (Müller, 1995; Salmon, 1998). It is extensively used in analysis of weather events (Davis & Emanuel, 1991; Lackmann, 2002), the general ocean circulation (Holland et al., 1984; Müller, 1995; Pollard & Regier, 1990; Rhines, 1986), magnetohydrodynamics (MHD) and astrophysical fluid dynamics (Webb & Mace, 2015). The wide-ranging applications of PV stem from its conservation properties and the notion of PV inversion (Hoskins et al., 1985; Smith & Stechmann, 2017; Wetzel et al., 2020).

Despite the importance of PV and the further knowledge gained over approximately a century, the conservation laws of circulation and PV are limited to fluids that are relatively idealized. In particular, these conservation laws apply to an ocean or atmosphere that is composed of a single fluid substance—water in the ocean, or dry air in the atmosphere. In nature, the ocean and atmosphere include other substances which are of fundamental significance. Moisture and clouds in the atmosphere are associated with weather events such as hurricanes, tornadoes, and monsoons. Salinity in the ocean affects the density and helps drive the thermohaline circulation and meridional overturning circulation that move heat from low latitudes to high latitudes (Curry & Mauritzen, 2005; Marshall & Schott, 1999; Peterson et al., 2006; Vallis, 2017). As one route to better understanding and predicting these phenomena, it is desirable to have knowledge of the associated conserved quantities.

The main goal of the present paper is to derive conservation principles for circulation and potential vorticity, for the case of an ocean with salinity or an atmosphere with moisture and clouds. The atmospheric case will be described first, and then the oceanic case will follow in a similar way. Also, circulation will be treated first, and then potential vorticity.

2. Circulation

The starting point is the evolution equation for the circulation Γ, which was defined above in Equation 1. Its evolution equation follows from the equations for conservation of mass and momentum (Bjerknes, 1898; Thorpe et al., 2003; Vallis, 2017) and it takes the form

$$\frac{D\Gamma}{Dt} = \oint_{C(t)} \frac{1}{\rho} \nabla p \cdot d\vec{l}, \tag{4}$$

where p is the pressure, $C(t)$ is a closed curve that moves along with the fluid, and $D/Dt = \partial/\partial t + \vec{u} \cdot \nabla$ is a material derivative—i.e., a derivative along the trajectory of a fluid parcel. See Supporting Information S1 for a derivation. The fluid is assumed to be compressible and inviscid.

Before considering the moist case, it is instructive to consider the dry case, for which it is known that circulation is conserved, under certain assumptions on the curve $C(t)$. To see this, one can rewrite (Equation 4) as

$$\frac{D\Gamma}{Dt} = \oint_{C(t)} \theta \, d\pi, \tag{5}$$

where the Exner function $\pi(p)$ is a function of pressure, and where the potential temperature θ is a function of entropy, as $s = c_p \log \theta + const.$, where c_p, commonly assumed constant for a dry atmosphere, is the specific heat at constant pressure. The details of this derivation are provided in Supporting Information S1. From Equation 5, one can deduce the scenarios when the right-hand side is zero and circulation Γ is conserved. In particular, if θ (and hence also entropy s) is a constant on the closed curve $C(t)$, then the right-hand side of Equation 5 is zero, by the fundamental theorem of calculus for line integrals. Consequently, circulation is conserved,

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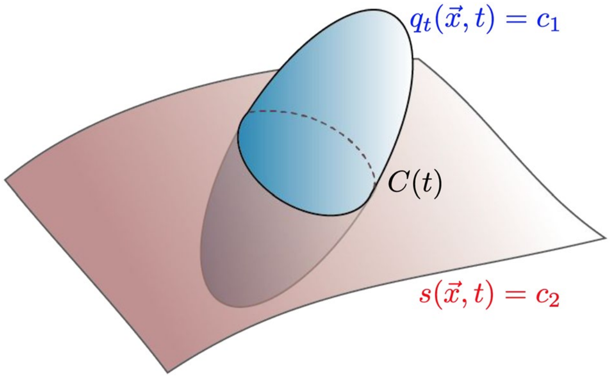

Figure 2. Geometry of the circulation theorem for an atmosphere with clouds or an ocean with salinity. The curve $C(t)$ is a curve of constant entropy s and total water specific humidity q_t.

$$\frac{D\Gamma}{Dt} = 0, \tag{6}$$

if the closed curve $C(t)$ lies on a surface of constant entropy. See Figure 1 for an illustration. In order for the material curve $C(t)$ to remain on a surface of constant entropy for all times, the fluid is assumed to be adiabatic, so that $Ds/Dt = 0$. Finally, from Equation 6, the circulation Γ is then said to be a *material invariant*, since its material derivative is zero.

In the moist case, one could try to repeat the derivation from the dry case to arrive at a circulation theorem. However, this line of reasoning fails for a moist system, since the integrand that would arise in Equation 5 would no longer be a material invariant, due to phase changes and cloud latent heating.

We now show that the challenges of the moist case can actually be overcome by referring to fundamental principles of moist thermodynamics. Two derivations will be shown: a derivation that directly refers to enthalpy, and a derivation that does not. In terms of the enthalpy $h(s, p, q_t)$, the fundamental thermodynamic relation (Landau & Lifshitz, 1980, 1987; Pauluis, 2008) states that

$$dh = T\,ds + \rho^{-1}\,dp + \mu\,dq_t, \tag{7}$$

which can be viewed as the differential of the function $h(s, p, q_t)$ as

$$dh = \left(\frac{\partial h}{\partial s}\right)_{p,q_t} ds + \left(\frac{\partial h}{\partial p}\right)_{s,q_t} dp + \left(\frac{\partial h}{\partial q_t}\right)_{s,p} dq_t, \tag{8}$$

where h is enthalpy, T is temperature, q_t is the total water specific humidity, and $\mu = \mu_v - \mu_d$ is the difference of the chemical potentials associated with water vapor and dry air, respectively. Note that the same symbols (such as p, ρ, s, etc.) will be used for discussing both the dry and moist cases, although the physical interpretations are different (Ooyama, 2001; Pauluis, 2008) and should be clear here from context. As a first important observation, notice that Equation 7 includes the term $\rho^{-1}\,dp$, which is also the sole term that appears in the evolution equation for circulation in Equation 4. Therefore, inserting the fundamental relation (Equation 7) into the evolution equation for circulation in Equation 4, we have

$$\frac{D\Gamma}{Dt} = -\oint_{C(t)} T\,ds - \oint_{C(t)} \mu\,dq_t, \tag{9}$$

where it was used that $\oint_{C(t)} dh = 0$ for a closed curve. The aim now is to identify scenarios when the right-hand side is zero. In this direction, note that the line integrals are zero if the curve $C(t)$ lies on a surface of constant entropy s *and* a surface of constant q_t. Furthermore, in order for the material curve to have constant s and q_t for all times, it is assumed that $Ds/Dt = 0$ and $Dq_t/Dt = 0$, so that s and q_t are both material invariants, if reversible phase changes are assumed between water vapor and liquid water, and precipitation is absent. Therefore, the right hand side of Equation 9 vanishes and we have

$$\frac{D\Gamma}{Dt} = 0, \tag{10}$$

if $C(t)$ is a closed curve of constant s and q_t. See Figure 2 for an illustration. This is the moist, cloudy analog of the circulation theorems of Kelvin and Bjerknes.

As an alternative derivation of Equation 10 that does not directly refer to enthalpy, return again to the starting point of Equation 4, where the integrand is ρ^{-1}. Now use a fundamental property of moist thermodynamics: any moist thermodynamic variable can be written as a function of s, q_t and p (Pauluis, 2008). Hence, $\rho^{-1}\,dp$ can be written as $\rho^{-1}(s, q_t, p)\,dp$, which, on a circuit $C(t)$ of constant entropy s and total water mixing ratio q_t, is a function of pressure p alone. Consequently, (Equation 4) can furthermore be simplified to

$$\frac{D\Gamma}{Dt} = \oint_{C(t)} df, \tag{11}$$

where $f(p)$ is an antiderivative that satisfies $df = (df/dp)\,dp = \rho^{-1}(s, q_t, p)\,dp$. By Equations 7 and 8, since $(dh/dp)_{s,q_t} = \rho^{-1}$, the function f can be identified as the enthalpy h, which is insightful but not essential for this derivation. Finally, since $C(t)$ is a closed curve, it follows from Equation 11 that

$$\frac{D\Gamma}{Dt} = 0, \tag{12}$$

if $C(t)$ is a closed curve of constant s and q_t.

As a comparison of the dry and moist cases, notice that both apply to material curves $C(t)$ on surfaces of constant entropy s. However, while *any* such curve is valid for a dry atmosphere, the curve must also have constant q_t for a moist, cloudy atmosphere.

3. Potential Vorticity

Now consider the second quantity of interest here, defined above in Equation 3: potential vorticity. Its evolution equation follows from the equations for conservation of mass, momentum, and entropy (Müller, 1995; Schubert et al., 2001) and it takes the form

$$\rho \frac{D}{Dt}\left(\frac{\vec{\omega}\cdot\nabla s}{\rho}\right) = \nabla s \cdot \nabla \times \left(\frac{1}{\rho}\nabla p\right) \tag{13}$$

see Supporting Information S1 for a derivation. On the right-hand side, the term $\nabla \times (\rho^{-1}\nabla p)$ arises from the curl of the pressure gradient in the statement of conservation of momentum.

For a dry atmosphere, PV is conserved for each fluid parcel, and it is helpful to review this case as a prelude to the moist case. To see how dry PV is conserved, note that the right-hand side of Equation 13 can be written as

$$\nabla s \cdot \nabla \times \left(\frac{1}{\rho}\nabla p\right) = -\frac{1}{\rho^2}\nabla s \cdot \nabla p \times \nabla \rho, \tag{14}$$

which is a scalar triple product. Then one can use a fundamental property of thermodynamics for a dry atmosphere: the entropy can be expressed as a function of pressure and density, so that $s = s(p, \rho)$. Consequently, we have

$$\nabla s = \frac{\partial s}{\partial p}\nabla p + \frac{\partial s}{\partial \rho}\nabla \rho \tag{15}$$

and inserting this into (Equation 14) shows that the right-hand side of Equation 13 is zero, and the PV is conserved for each fluid parcel:

$$\frac{D}{Dt}\left(\frac{\vec{\omega}\cdot\nabla s}{\rho}\right) = 0. \tag{16}$$

In essence, this proof works smoothly because the entropy s is a function of pressure p and density ρ, for a dry atmosphere. However, for a moist atmosphere, due to the effects of phase changes and cloud latent heating, it is not true that $s = s(p, \rho)$, and it is therefore difficult to see how to extend these proof ideas to apply to the moist case.

For a moist atmosphere with clouds, we now show that the difficulty of PV conservation can be resolved by appealing to fundamentals of moist thermodynamics. In particular, use the gradient form of the fundamental thermodynamic relation in Equation 7, and insert into (Equation 13) for the $\rho^{-1}\nabla p$ term. Then, after using vector calculus identities, (Equation 13) becomes

$$\rho \frac{D}{Dt}\left(\frac{\vec{\omega}\cdot\nabla s}{\rho}\right) = -\nabla s \cdot \nabla \times (\mu \nabla q_t). \tag{17}$$

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The benefit of having replaced $\rho^{-1}\nabla p$ with $\mu \nabla q_t$ will be seen below and arises from the fact that the right-hand side of Equation 17 now involves two quantities, s and q_t, which are both conserved for each fluid parcel, for the case of reversible phase changes between water vapor and cloud liquid water, with no precipitation.

To proceed with finding the conservation law for moist PV, we abandon the idea that moist PV can be conserved for each fluid parcel. In other words, we abandon hope of showing that the right-hand side of Equation 17 is zero. Instead, consider the integral of moist PV over certain volumes. The evolution equation of volume-integrated PV is

$$\frac{D}{Dt}\iiint_{V(t)} \frac{\vec{\omega}\cdot\nabla s}{\rho}\,\rho\, dV = \iiint_{V(t)} \rho \frac{D}{Dt}\left(\frac{\vec{\omega}\cdot\nabla s}{\rho}\right) dV \tag{18}$$

$$= -\iiint_{V(t)} \nabla\cdot(\mu\nabla q_t \times \nabla s)\, dV \tag{19}$$

$$= -\iint_{A(t)} \mu\nabla q_t \times \nabla s \cdot d\vec{A}, \tag{20}$$

where the volume integral in Equation 18 is mass-weighted using density ρ, and where the volume $V(t)$ is a material volume that moves with the fluid but otherwise remains to be specified. See Supporting Information S1 for details of the derivation. In brief, (Equation 18) uses the transport theorem, (Equation 19) uses (Equation 17) and some vector calculus identities, and (Equation 20) follows from the divergence theorem, where $A(t)$ is the surface that encloses the material volume $V(t)$.

Now it is time to identify the special volumes for which (Equation 20) is zero and integrated moist PV is conserved. The intuition is similar to the case of the moist circulation theorem in Equation 9 where the appearance of the line elements ds and dq_t suggested the choice of curves of constant s and q_t. Here, it is a surface integral that arises in Equation 20, and in order for the integral to vanish, the surface area element $d\vec{A}$ must be perpendicular to ∇s or ∇q_t. Hence, we have

$$\frac{D}{Dt}\iiint_{V(t)} \frac{\vec{\omega}\cdot\nabla s}{\rho}\,\rho\, dV = 0, \tag{21}$$

if the volume is defined to be enclosed by surfaces of constant s and surfaces of constant q_t. An alternative derivation, which does not explicitly use the fundamental thermodynamic relation, is shown in Supporting Information S1. Either derivation leads to the same result, (Equation 21), the conservation law for PV for a cloudy atmosphere, which applies to volume-integrated PV.

More generally, the surface could be composed piecewise, based on level surfaces of general functions $F(s, q_t)$, since any such level surface is perpendicular to $\nabla s \times \nabla q_t$. Nevertheless, it is convenient and simple to choose surfaces of constant s or constant q_t, as illustrated in Figure 3. The simplest such volume is shaped like a thin cylinder, or pill box, or pancake.

As perhaps the most special case, notice that the volume $V(t)$ in Figure 3 will shrink to a point at a maximum or minimum value of q_t on a constant-entropy surface. As a result, at minima or maxima of q_t, the PV is conserved for the individual parcel.

As another generalization, notice that the PV could be defined as $\rho^{-1}\vec{\omega}\cdot\nabla q_t$ with q_t in the role that is normally played by entropy s. A similar conservation law also follows for such a q_t-based PV. Moreover, the PV could be defined as $\rho^{-1}\vec{\omega}\cdot\nabla F(s, q_t)$ for any function F of s and q_t. In that case, the proof proceeds in the same way by writing the fundamental thermodynamic relation in a form that replaces s and q_t by $F(s, q_t)$ and $G(s, q_t)$, where G is arbitrary but functionally independent of F.

While the conservation laws above were derived under moist adiabatic conditions and without frictional forces, some other principles hold even in the presence of frictional forces and diabatic heating (such as radiative heating or cooling). In particular, a version of the impermeability theorem (Haynes & McIntyre, 1987, 1990) holds in the present setting: there is no net flux of $\vec{\omega}\cdot\nabla s$ across a surface of constant moist entropy s (i.e., a moist isentropic surface), and there is no net flux of $\vec{\omega}\cdot\nabla q_t$ across a surface of constant total water specific humidity q_t. The proofs follow in the same way as the classic impermeability theorem (Haynes & McIntyre, 1987).

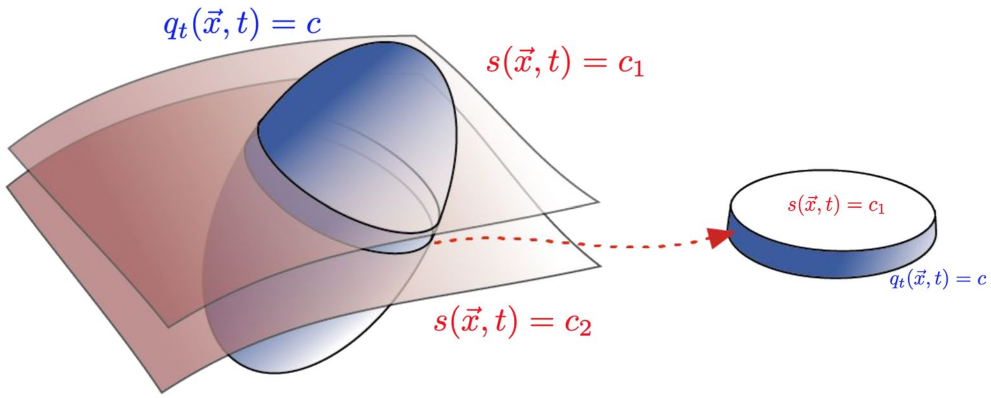

Figure 3. Geometry of the potential vorticity (PV) conservation law for an atmosphere with clouds or an ocean with salinity. The conserved quantity is not PV of individual parcels, but PV integrated over certain cylinder-like volumes.

For an ocean with salinity, the conservation laws for circulation and PV follow in the same way. Simply replace specific humidity q_t by salinity S, and to avoid confusion, change the notation of the entropy s to instead be η, as is common in oceanography. Table 1 summarizes the conservation laws for both the cloudy atmosphere and salty ocean.

4. Symmetry–Conservation Relationship

A symmetry–conservation relationship is also associated with these conservation laws of circulation and PV. In particular, it is a particle-relabeling symmetry, which, as the name suggests, involves particles whose positions are $\vec{x}(\vec{a}, t)$ or the relabeled positions $\vec{x}(\vec{a}\,', t)$, where the original label $\vec{a}$ was replaced by the relabeling $\vec{a}\,' = \vec{a}\,'(\vec{a}, t)$. In a discrete setting, the relabeling of $\vec{x}_n(t)$ to $\vec{x}_{n'}(t)$ would simply involve a permutation of the labels, $n = 1, 2, 3, \ldots, N$, but there is no guarantee of an associated conservation law for such a discrete symmetry. For the continuum setting of a cloudy atmosphere, the symmetry applies to the Lagrangian

$$\mathcal{L}(t) = \iiint \rho\left[\frac{1}{2}|\vec{u}|^2 - E(\rho, s, q_t) - \Phi(\vec{x})\right] dV_a, \tag{22}$$

where $E(\rho, s, q_t)$ is the internal energy, $\Phi(\vec{x})$ is the gravitational potential, and the integral is over the particle labels $\vec{a}$, as the continuum version of a sum over the particles.

A key observation is that only some special particle relabeling will leave the Lagrangian unchanged. For instance, for a dry atmosphere, the internal energy $E(\rho, s)$ is a function of density and entropy, and it is well-known that $E(\rho, s)$ and the Lagrangian remain unchanged for certain particle relabeling on surfaces of constant entropy s (Salmon, 1998); and it follows that the Kelvin–Bjerknes circulation theorem holds for curves $C(t)$ that lie on surfaces of constant entropy, as mentioned in Equation 6 and Figure 1 above. Now, for a moist atmosphere, one can see the natural extension: the particle relabeling should keep both s and q_t unchanged in order to keep $E(\rho, s, q_t)$ and the Lagrangian unchanged; consequently, the moist circulation theorem in Equation 10 and Figure 2 holds

Table 1
Summary of Conservation Laws Conservation Laws for an Atmosphere With Clouds and an Ocean With Salinity

	Cloudy atmosphere	Salty ocean
Circulation Theorem	$\frac{D}{Dt}\oint_{C(t)} \vec{u}\cdot d\vec{l} = 0$	$\frac{D}{Dt}\oint_{C(t)} \vec{u}\cdot d\vec{l} = 0$
Curve $C(t)$	Constant s and q_t	Constant η and S
PV Conservation Law	$\frac{D}{Dt}\int\int\int_{V(t)} \frac{\vec{\omega}\cdot\nabla s}{\rho}\,\rho\, dV = 0$	$\frac{D}{Dt}\int\int\int_{V(t)} \frac{\vec{\omega}\cdot\nabla \eta}{\rho}\,\rho\, dV = 0$
Surface of Volume $V(t)$	Constant s or q_t	Constant η or S

Note. Commonly used notation for a cloudy atmosphere is s and q_t for the entropy and total water specific humidity, respectively, and for a salty ocean is η and S for the entropy and salinity, respectively. Rotation and the Coriolis force could also be incorporated with modifications described in Equations 23 and 24.

for curves $C(t)$ of constant s and q_t. These symmetry–conservation ideas have been derived in detail for a moist Boussinesq setting (Kooloth et al., 2022) and here provide motivation and further understanding of the moist conservation laws of circulation and PV. For instance, because the particle relabeling is constrained to curves $C(t)$ of both constant s and q_t, it follows that moist PV is not conserved for individual parcels.

5. Concluding Discussion

Rotation and the Coriolis term could also be incorporated here, in which case the circulation theorem becomes

$$\frac{D}{Dt}\oint_{C(t)}\left(\vec{u}+\vec{\Omega}\times\vec{x}\right)\cdot d\vec{l}=0, \tag{23}$$

where Ω is the angular rotation vector and $\vec{x}$ is the position vector. The extension to include rotation follows standard procedures as in cases without clouds or salinity (Cotter & Holm, 2014; Salmon, 1998; Vallis, 2017). Furthermore, the PV conservation laws have the same form as in Table 1 except with $\omega = \nabla \times \vec{u}$ replaced by absolute vorticity,

$$\vec{\omega}_a = \nabla\times\vec{u}+2\vec{\Omega}, \tag{24}$$

which is the sum of the relative vorticity and the angular rotation vector.

Many applications are possible for the conservation laws presented here. One common use of PV conservation is to diagnose non-conservation, which is an indication of additional physical processes (Haynes & McIntyre, 1987). For example, PV diagnostics can help in understanding the dynamics of extratropical cyclones as well as hurricanes (Davis & Emanuel, 1991; Hausman et al., 2006; Lackmann, 2002; Martinez et al., 2019; Shapiro & Franklin, 1995). The PV conservation laws here could be used for similar purposes, with certain modifications. For instance, cloud latent heating is considered one of the sources of non-conservation for dry PV, but is incorporated into the conservation law here for moist PV.

Besides PV conservation, other important PV concepts are PV inversion and balance. While the concepts and properties of PV are well-established for a dry atmosphere, these ideas are still developing in the case of a moist atmosphere with clouds, phase change and latent heating. In particular, in the dry setting, the dry PV is materially conserved, encodes the balanced component of the flow, and is associated with an invertibility principle. The latter means that, given the dry PV, one can find the balanced components of velocity and temperature by inverting a linear elliptic equation between PV and a streamfunction. For a moist atmosphere with clouds, phase changes and latent heating, several definitions of moist PV have been proposed in analogy with dry PV, but none are simultaneously materially conserved and possessing an invertibility principle. For example, a moist PV defined in terms of the virtual potential temperature (PV_v) can be inverted to find certain velocity and temperature fields (Schubert et al., 2001), but it is coupled to gravity waves and therefore is not a purely balanced quantity (Wetzel et al., 2020), so the velocity and temperature obtained from PV_v inversion do not represent the balanced part of a moist atmosphere with phase changes. The moist PV based on equivalent potential temperature (PV_e) is balanced, but by itself does not possess an invertibility principle in the presence of phase changes (Cao & Cho, 1995; Schubert et al., 2001). On the other hand, taking advantage of one or more additional balanced quantities M involving water, extended PV_e–M inversion can indeed be used to solve for the balanced moist flow (Smith & Stechmann, 2017; Wetzel et al., 2019, 2020). As a general principle, for a moist system, a single moist PV variable is not sufficient information to find the balanced flow components, and additional moisture variables (M) need to be retained (Smith & Stechmann, 2017). In summary, balanced theory based on PV_e is a good candidate for close analogy to the dry theory, by generalizing to PV_e–M inversion and conservation of patch-integrated PV_e as shown here.

Another application of PV conservation is in numerical models of weather and climate. It is often desirable to design such numerical methods to accurately satisfy the PV conservation law (Taylor & Fournier, 2010; Thuburn, 2008). The conservation laws here provide additional targets for numerical conservation, now incorporating additional physical processes of salinity and cloud latent heating.

Data Availability Statement

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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