A Spatiotemporal Stochastic Model for Tropical Precipitation and Water Vapor Dynamics

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ABSTRACT

A linear stochastic model is presented for the dynamics of water vapor and tropical convection. Despite its linear formulation, the model reproduces a wide variety of observational statistics from disparate perspectives, including (i) a cloud cluster area distribution with an approximate power law; (ii) a power spectrum of spatiotemporal red noise, as in the "background spectrum" of tropical convection; and (iii) a suite of statistics that resemble the statistical physics concepts of critical phenomena and phase transitions. The physical processes of the model are precipitation, evaporation, and turbulent advection–diffusion of water vapor, and they are represented in idealized form as eddy diffusion, damping, and stochastic forcing. Consequently, the form of the model is a damped version of the two-dimensional stochastic heat equation. Exact analytical solutions are available for many statistics, and numerical realizations can be generated for minimal computational cost and for any desired time step. Given the simple form of the model, the results suggest that tropical convection may behave in a relatively simple, random way. Finally, relationships are also drawn with the Ising model, the Edwards–Wilkinson model, the Gaussian free field, and the Schramm–Loewner evolution and its possible connection with cloud cluster statistics. Potential applications of the model include several situations where realistic cloud fields must be generated for minimal cost, such as cloud parameterizations for climate models or radiative transfer models.

1. Introduction

Tropical clouds and precipitation display a remarkable variety of behaviors. This behavior has been quantified statistically using many different measures which have been largely aided by satellite observations in the past several decades. Three examples are (i) cloud size distributions, from individual convective cells to organized convection on mesoscales and synoptic scales (Peters et al. 2009; Wood and Field 2011); (ii) the power spectrum of precipitation and cloudiness, which shows characteristics of spatiotemporal red noise and also coherent propagating wave features (Takayabu 1994; Wheeler and Kiladis 1999); and (iii) statistical physics

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perspectives including critical phenomena (Peters and Neelin 2006; Neelin et al. 2009).

The main aim of this paper is to show that a relatively simple model of water vapor dynamics has behavior very similar to these observational statistics. The new water vapor model proposed in this paper is a discrete version of the stochastic PDE:

$$\frac{\partial q}{\partial t} = b_0 \nabla^2 q - \frac{1}{\tau} (q - q^*) + F + D_* \dot{W}, \qquad (1)$$

where q(x, y, t) is the column water vapor at horizontal location (x, y). The parameters of the model are described in detail in section 2. In short, this model represents the effects of precipitation, evaporation, and turbulent advection-diffusion of water vapor in a compact, linearized form, as described in more detail below.

The motivation for this investigation is our incomplete understanding of tropical convection and moist convection. This incomplete understanding is manifest, for example,

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in the continuing struggle to parameterize tropical convection in climate models (Randall et al. 2003; Lin et al. 2006; Hung et al. 2013).

Several stochastic models have been introduced to provide insight into tropical convection and its spatiotemporal variability. Lin and Neelin (2000, 2002) introduced a stochastic version of the convective parameterization of Betts and Miller (1986), and they investigated it in an intermediate-complexity model and later in a general circulation model (Lin and Neelin 2003). Majda and Khouider (2002) used ideas from nonequilibrium statistical mechanics to model subgridscale convective variability, and Khouider et al. (2003) introduced a coarse-graining procedure to drastically reduce the computational cost of the subgrid-scale model while still retaining its important statistical features. Khouider et al. (2010) introduced a stochastic multicloud model that was later incorporated into the convective parameterization of an idealized model (Frenkel et al. 2012, 2013) and a general circulation model (Deng et al. 2015) and reproduces many realistic features of convectively coupled equatorial waves (CCEWs) and the Madden–Julian oscillation (MJO). Bengtsson et al. (2011, 2013) used a cellular automaton to model deep convective cloud patterns on subgrid scales of a numerical weather prediction model, and they explored its effect on equatorial waves and on the ensemble spread of ensemble forecasts. Thual et al. (2014) introduced a stochastic model for the skeleton of the MJO that produces the intermittent generation and termination of wave trains of MJO events.

Common to almost all the models of the previous paragraph is an aim toward convectively coupled waves and the MJO. In the present paper, the focus is instead on the "background spectrum" of tropical convection, as illustrated in Fig. 2 of Wheeler and Kiladis (1999), which is reproduced here in Fig. 1.

The background spectrum can be viewed from at least two perspectives (and it is also possible that nature is actually a combination of the two). One possible perspective is wave centric: CCEWs and the MJO provide the dominant coherent features of tropical variability, and their signature in a power spectrum can potentially be smeared away from their idealized dispersion curves to create the raw spectrum in Fig. 4 of Takayabu (1994) or Fig. 1 of Wheeler and Kiladis (1999). The smearing is potentially created as a wave's dispersion curve changes in accordance with changes in the background state through, for example, background wind shear or Doppler shifting (Majda and Stechmann 2009a; Han and Khouider 2010; Dias and Kiladis 2014). A second perspective can be formed around the background spectrum itself and the highly stochastic nature of tropical



FIG. 1. The observed background power spectrum of outgoing longwave radiation (OLR) data, a proxy for tropical convection and cloudiness, reproduced from Fig. 2 of Wheeler and Kiladis (1999). This background spectrum was created by several iterations of smoothing the raw spectrum, and it resembles the spectrum of spatiotemporal red noise. Copyright American Meteorological Society; used with permission.

convection: perhaps, in the raw spectrum, the signature of CCEWs is a slight deformation of a fundamental background spectrum of highly stochastic tropical convection.

A strong theoretical basis is currently lacking for the background-centric perspective. This is because the background spectrum is typically obtained in an empirical way (by smoothing the raw spectrum) and because no theoretical explanation exists for a background spectrum. In this paper, based on the model in (1), a simple theoretical justification will be offered.

Another perspective on tropical convection has been provided by analogies with statistical physics paradigms of phase transitions and critical phenomena (Peters and Neelin 2006; Neelin et al. 2009). Stochastic models have been introduced for this dynamical behavior in time (Stechmann and Neelin 2011, 2014; Hottovy and Stechmann 2015). In the present paper, one aim is to build on this work by introducing a model for spatiotemporal variability: the linear model in (1). Given that precipitation is typically modeled as a nonlinear component of water vapor dynamics and given that most classical models for phase transitions are inherently nonlinear (Yeomans 1992; Christensen and Moloney 2005), one might wonder: Can the linear model in (1) really display a similar type of behavior? Precipitation will be introduced as a nonlinear statistic, formulated a posteriori in terms of the underlying linear dynamics of water vapor in (1).



FIG. 2. A diagram of the two-dimensional lattice. Each 5 km × 5 km square is represented by a lattice point, (i, j), which has the column water vapor value of $q_{i,j}(t)$. Each site has nearest-neighbor interactions. Nearest neighbors are defined as one site to the right, left, up, and down.

The outline of the paper is as follows. In section 2, the two-dimensional lattice model [(2)] is introduced, its exact solution given, and how the parameters are chosen is described. In section 3, a variety of statistics is presented relating to water vapor, precipitation, and cloud clusters. In section 4, additional physical processes and extensions of the model are discussed. In section 5, the results of this paper are connected to other models in statistical physics. In section 6, the results are summarized.

2. Model description

a. Equations and physical interpretation

A large swath of the atmosphere is modeled using a twodimensional grid. The domain is a L by L section of the tropics, where L = 5000 km. In this domain an N by N point lattice, with N = 1000, is arranged so that each lattice point lies in the center of a Δx by Δy section of the tropics (see Fig. 2) with $\Delta x = \Delta y = 5$ km. The grid spacing of 5 km is chosen to be roughly the smallest width of individual cells of tropical deep convection. Define $q_{i,j}(t)$ to be the integrated column water vapor (mm) of the (i, j)th column of the atmosphere. The column water vapor dynamics are given by the stochastic differential equation (SDE):

$$\frac{dq_{ij}(t)}{dt} = F + b[q_{i+1,j}(t) + q_{i-1,j}(t) + q_{i,j+1}(t) + q_{i,j-1}(t) - 4q_{i,j}(t)] - \tau^{-1}[q_{i,j}(t) - q^*] + D_* \dot{W}_{i,j}(t),$$
(2)

for i, j = 1, ..., N, and $W_{i,j}(t)$ are independent white noises, denoted formally as the derivative of a Wiener process. The dynamics of $q_{i,j}(t)$ depend on an external force F (mm h⁻¹), spacial interaction constant b (h⁻¹), relaxation time τ (h), relaxation target q^* (mm), and stochastic forcing variance D_*^2 (mm² h⁻¹). For simplicity, periodic boundary conditions are imposed for the above equation. That is, $q_{N+1,j}(t) = q_{1,j}(t)$ for all j =1, 2, ..., N and $q_{i,N+1}(t) = q_{i,1}(t)$ for all i = 1, 2, ..., N.

The model [(2)], or its continuum form [(1)], can be related to atmospheric fluid dynamics in the following way. The water vapor mass concentration evolves according to

$$\frac{\partial q}{\partial t} + (uq)_x + (vq)_y + (wq)_z = S, \qquad (3)$$

where u, v, and w are the velocity components and S represents any source or sink such as precipitation. Next, q is decomposed as $q = \overline{q} + q'$ into a large-scale component \overline{q} and a small-scale component q'. The large-scale component \overline{q} represents a vertical average of q over the depth of the atmosphere [to give the column water vapor (CWV)] and a horizontal average over a scale similar to the lattice grid spacing Δx . (One could argue that the vertical integral should be taken in the free troposphere; i.e., above the atmospheric boundary layer.) The dynamics of \overline{q} is then found from (3) to be

$$\frac{\partial \overline{q}}{\partial t} = \overline{S} - \left[\left(\overline{u} \, \overline{q} \right)_x + \left(\overline{v} \, \overline{q} \right)_y \right] - \left[\left(\overline{u'q'} \right)_x + \left(\overline{v'q'} \right)_y \right]. \tag{4}$$

The relationship with (1) and (2) can then be seen after two common simplifying assumptions for turbulent flows: (i) the small-scale flux convergence, $-(\overline{u'q'})_x - (\overline{v'q'})_y$, is modeled as eddy diffusion, $b_0 \nabla^2 q$, and (ii) the nonlinear turbulent effects of $-(\overline{u} \overline{q})_x - (\overline{v} \overline{q})_y$ are modeled with additional turbulent damping, $-\tau^{-1}q$, and stochastic forcing, $D_* \dot{W}$ (DelSole 2004; Majda and Grote 2007, 2009). The term \overline{S} in (4) includes the effects of water vapor sources such as precipitation and evaporation, which are represented in (1) and (2) in idealized form as a net mean constant forcing F and a partial contribution to the stochastic forcing $D_* \dot{W}$. With this connection to atmospheric dynamics, the terms of the model can be identified with physical processes of precipitation, evaporation, and turbulent advection-diffusion.

The precise relationships with precipitation and water are somewhat open to interpretation in such a simple model. For example, q could potentially represent total water (sum of vapor and cloud condensate) or water vapor alone. Corresponding to each of these possibilities is an interpretation of q^* as a saturation value (if q is total water) or a threshold for the onset of convection (if q is water vapor). Here q will be described as water vapor to be consistent with the interpretation in the earlier work of Peters and Neelin (2006) and Neelin et al. (2009). As another example, the terms F, $\tau^{-1}(q-q^*)$, and $D_*\dot{W}$



FIG. 3. A realization of the model for CWV and the cloud indicator (σ). (a),(b) The model with each lattice point denoting a 5 km × 5 km area. (c),(d) The model coarsened, as described in section 2b, to a grid spacing of 25 km × 25 km.

represent a combination of effects from precipitation, evaporation, and turbulent advection–diffusion. In the spirit of simplicity, precipitation will be identified in the model as follows.

Cloudiness and convection are indicated in cell (i, j) when $q_{i,j}(t) > q^*$. We define the threshold in the model to be $q^* = 65$ mm, which is close to the critical value found in observational studies (e.g., Peters and Neelin 2006; Neelin et al. 2009). Note that q^* is also defined as the relaxation

target. It will be referred to as a threshold throughout the rest of the paper. Define the cloud indicator variable

$$\sigma_{i,j}(t) = \mathcal{H}(q_{i,j}(t) - q^*), \tag{5}$$

where \mathcal{H} is the Heaviside function, equaling one if the (i, j)th column is undergoing strong convection and equaling zero if it is not. The connection to cloud cover is seen in Fig. 3, which shows a snapshot of column water



FIG. 4. Comparison of observational and model precipitation. (a) Precipitation data from TRMM (from http:// trmm.gsfc.nasa.gov/). (b) A realization of the model for the precipitation $(|F| + q_{ij}/\tau)\sigma_{ij}$, which is coarsened to a 25 km × 25 km grid spacing.

vapor (Fig. 3a) next to the values of $\sigma_{i,j}$ (Fig. 3c). A variety of large and small clusters can be identified in both $q_{i,j}$ and $\sigma_{i,j}$, and their statistics will be examined in more detail below.

A precipitation rate $r_{i,j}$ is assigned to a column if $\sigma_{i,j} = 1$. The precipitation rate used in previous models varies greatly. In Betts and Miller (1986), the precipitation rate is proportional to the column water vapor: $(q_{i,j} - q^*)\sigma_{i,j}/\tau$. In Stechmann and Neelin (2011, 2014) and Hottovy and Stechmann (2015), the precipitation rate is constant: $|F|\sigma_{i,j}$, where *F* is the constant source from (2). In addition, one could argue that the *F* term should be incorporated into a Betts–Miller-like rain rate as

$$\mathbf{r}_{ij} = \left[|F| + \frac{(q_{ij} - q^*)}{\tau} \right] \sigma_{ij}.$$
 (6)

All three of these precipitation definitions will be compared below.

A coarsened grid is used to compare the model to observational data. The motivation for the coarsening is the relatively large footprint of satellite observations such as the TRMM Microwave Imager (Huffman et al. 2007). Specifically, the satellite footprint represents an area of roughly $25 \text{ km} \times 25 \text{ km}$ or $50 \text{ km} \times 50 \text{ km}$ (Neelin et al. 2009), whereas each column in the model accounts for a $5 \text{ km} \times 5 \text{ km}$ area of vertically integrated atmosphere. To compare the model to observational data, a sum is taken over a subblock of the lattice. For example, to compare column water vapor statistics, the entire

lattice is divided into M by M blocks, where M = 5 (see Fig. 2). This results in a two-dimensional lattice that is N/M by N/M points with N/M = 200. Thus each point represents a 25 km \times 25 km area of the atmosphere. The column water vapor in the $M_{i,j}$ th column of this coarsened lattice is

$$q_{M_{i,j}} = \frac{1}{M^2} \sum_{i',j' \in M_{i,j}} q_{i',j'}.$$
(7)

This procedure is repeated for the cloud indicator (σ_{ij}) and precipitation (r_{ij}) statistics. The effect of the coarsening is most notable in the conditional cloud indicator statistic [given by (A15)]. Given a single column value q_{ij} , $E(\sigma_{ij} | q_{ij})$ is a discontinuous Heaviside function centered at q^* ; on the other hand, for the coarsened grid, $E(\sigma_{M_{ij}} | q_{M_{ij}})$ is a smooth function that rapidly increases to one after q^* . The results of coarsening the column water vapor (Fig. 3b) and cloud indicator (Fig. 3d) are shown in Fig. 3.

To further evaluate the realism of the coarsened spatial variability, the corresponding precipitation from (6) is compared in Fig. 4, after coarsening, against a TRMM Microwave Imager sample data [Huffman et al. (2007) and http://trmm.gsfc.nasa.gov/affinity/affinity_3hrly_rain. html]. Since the model is currently set up to represent tropical deep convection, TRMM data from the Indian and western Pacific oceans is shown for comparison. The model domain of $5000 \text{ km} \times 5000 \text{ km}$ is chosen to roughly represent the same area as this TRMM data snapshot. To the eye, the spatial variability compares favorably, with the understanding that one can only make statistical

comparisons, and with the caveat that the relatively cold waters of the southern Indian Ocean do not favor tropical deep convection at this time, whereas convection is equally probable at any location in the stochastic model domain. Detailed statistical comparisons will be presented below in section 3.

b. Analytic stationary pdf

In this section the stationary pdf is given in both physical and Fourier spaces.

In physical space, the stationary pdf of the system is

$$\rho(\mathbf{q}) = Z^{-1} \exp\left\{ D_{*}^{-2} \left[F \sum_{i,j=0}^{N-1} q_{i,j} - \frac{1}{2\tau} \sum_{i,j=0}^{N-1} (q_{i,j} - q^{*})^{2} - b \sum_{\langle (i,j), (\tilde{i}, \tilde{j}) \rangle} (q_{i,j} - q_{\tilde{i}, \tilde{j}})^{2} \right] \right\},$$
(8)

where **q** is a vector of the $q_{i,j}$ values, $\langle \cdot, \cdot \rangle$ denotes nearest neighbors, and Z is a normalizing constant. (See the appendix for a derivation.) Notice that this resembles the continuum formula

$$\rho(q) \approx Z^{-1} \exp\left\{ D_{*}^{-2} \left[F \iint q(x, y) \, dx \, dy - \frac{1}{2\tau} \iint |q(x, y) - q^{*}|^{2} \, dx \, dy - b_{0} \iint |\nabla q(x, y)|^{2} \, dx \, dy \right] \right\},$$
(9)

which indicates that (i) spatial gradients $|\nabla q|^2$ are penalized via b_0 , (ii) anomalies $|q - q^*|$ are penalized via τ^{-1} , and (iii) positive anomalies, q > 0, are penalized if F < 0 (and vice versa for F > 0). In other words, even though $q_{i,j}(t)$ is random, the parameters τ and b act to promote some degree of temporal and spatial correlation.

Mathematically, (8) takes the form of a multivariate Gaussian distribution. The mean is given by

$$E[q(t)] = \tau F + q^*.$$
 (10)

The covariance can be obtained in either physical space or Fourier space.

In Fourier space, the system [(2)] decouples for each Fourier mode, which greatly simplifies the formulas for the stationary pdf and covariance. Define the two-dimensional discrete Fourier transform of $q_{i,j}(t)$ as

$$Q_{k,\ell}(t) = \frac{1}{N^2} \sum_{i=0}^{N-1} \left[\sum_{j=0}^{N-1} e^{-2\pi i \Delta x (ki+\ell j)/L} q_{i,j}(t) \right], \quad (11)$$

where \hat{i} is the usual definition of the imaginary number. The inverse formula for the discrete Fourier transform is

$$q_{i,j}(t) = \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} e^{2\pi i \Delta x (ki+\ell j)/L} Q_{k,\ell}(t).$$
(12)

Equation (2) is diagonalized in the Fourier domain and thus

$$\frac{dQ_{k,\ell}(t)}{dt} = -c_{k,\ell}Q_{k,\ell}(t) + D_*\hat{W}_{k,\ell}(t), \qquad (13)$$

where an additional term $(F + \tau^{-1}q^*)$ must be added to the right-hand side when k = 0, $\ell = 0$, and where $c_{k,\ell}$ is defined as

$$c_{k,\ell} = b \left[4 + (b\tau)^{-1} - 2\cos\left(\frac{2\pi k\Delta x}{L}\right) - 2\cos\left(\frac{2\pi\ell\Delta y}{L}\right) \right].$$
(14)

The terms $\hat{W}_{k,\ell}(t)$ in (13) are obtained by the discrete Fourier transform of $\dot{W}_{i,j}$ in a manner similar to (11). Thus, $\hat{W}_{k,\ell}(t)$ are independent Gaussian random variables, except that $\hat{W}_{k,\ell}$ and $\hat{W}_{N-k,N-\ell}$ are complex conjugates to ensure that $q_{i,j}$ is real valued. Equation (13) defines a complex-valued Ornstein–Uhlenbeck process. Since $c_{k,\ell} > 0$, the process has a (Gaussian) stationary distribution with mean

$$E(Q_{k\ell}) = 0 \tag{15}$$

and variance

$$E(|Q_{k,\ell}|^2) = \frac{D_*^2}{2c_{k,\ell}}.$$
 (16)

For $k = \ell = 0$ the mean is instead $(F\tau + q^*)$. Furthermore, the spatial and temporal covariance can be computed. More details are described below.

A sample realization of **q** from the stationary state was shown in Fig. 3a. Note that such realizations can be generated very computationally efficiently; an independent Gaussian random variable is drawn for each Fourier mode according to (15) and (16), and then an inverse Fourier transform yields $q_{i,j}$.

c. Semianalytic evolution in time

In this section, a formula to compute realizations of the model is derived. Realizations of the model are used in this study in two ways: (i) to visualize individual realizations and (ii) for computing some complicated statistics that cannot be found analytically. This method is called semianalytic because the solution



FIG. 5. A Hovmöller plot of (a) the column water vapor, (b) cloud indicator, and (c) precipitation. To emphasize the large-scale features, the data are averaged over a 1000-km strip in the y direction, and a boxcar filter is applied over wavenumbers -10 to 10.

for the realization $q_{i,j}(t)$ is derived analytically (meaning there is no numerical integration error), but random variables must be drawn, which introduces sampling error.

The model is solved in Fourier space by integrating exactly. The solution of SDE (13) in the Fourier domain is

$$Q_{k,\ell}(t) = e^{-c_{k,\ell}(t-t_0)}Q_{k,\ell}(t_0) + D_* \int_{t_0}^t e^{-c_{k,\ell}(t-s)} d\hat{W}_{k,\ell}(s) + \frac{[F + (q^*/\tau)]}{c_{0,0}} [1 - e^{-c_{0,0}(t-t_0)}]\delta_{0k}\delta_{0\ell},$$
(17)

where $\delta_{i,j}$ is the Kronecker delta function. Note that the integral in (17) is a Gaussian random variable at each time *t* with mean zero and variance

$$\operatorname{Var}\left[D_{*}\int_{t_{0}}^{t}e^{-c_{k,\ell}(t-s)}d\hat{W}_{k,\ell}(s)\right] = \frac{D_{*}^{2}}{2c_{k,\ell}}\left[1-e^{-2c_{k,\ell}(t-t_{0})}\right].$$
(18)

Using this formula, $Q_{k,\ell}(t)$ is computed efficiently in the following way. For any desired time t, draw an independent Gaussian random variable for each Fourier mode according to (18) and add it to the contribution from $Q_{k,\ell}(t_0)$ in (17), and then the inverse fast Fourier transform recovers $q_{i,j}(t)$. This procedure can be repeated to advance forward in time with samples at any desired times.

A sample of the time evolution is shown in Fig. 5. The initial condition, at time t = 0, was drawn from the stationary pdf [(15) and (16)], and a realization of the time evolution was sampled one time every 12 h, using the method described in (17) and (18) with $t_0 = 12i$ and





t = 12(i + 1) for i = 1, 2, 3, ... To compactly illustrate the spatiotemporal evolution, $q_{i,j}(t)$ is averaged over a band of latitudes from y = 2000 to y = 3000 km, similar to averaging from 5°S to 5°N as in the standard method for creating Hovmöller diagrams. Figure 5 shows plots of water vapor (Fig. 5a), cloud indicator (Fig. 5b), and precipitation (Fig. 5c), all low-pass filtered spatially to include Fourier wavelengths of 500 km and longer (which turns the cloud indicator variable into a cloud area fraction). There is evidence for coherent traveling structures especially in the CWV.

For example, high water vapor values seem to move to the west from x = 3000 to x = 1800 km over 150–160 days. This indicates that propagating structures can arise even from (spatiotemporal) red noise, with no waves or advection explicitly included in the model.

d. Model calibration

The constants of (2) are chosen to match climatological mean and variance data. The observational data of background power spectral density of OLR [see Fig. 2 of Wheeler and Kiladis (1999)] is used to determine parameters τ and b. The τ is chosen to give a similar decay in power over the k = 0 mode frequencies, and b is chosen to give similar decay in the $k \neq 0$ directions (see Fig. 6). Next, the observational column water vapor probability density function is used to determine D^* and F [see Fig. 3 in Peters and Neelin (2006)] by using the standard deviation and mean, respectively. The numerical values for each parameter can be found in Table 1.

3. Statistics of water vapor, precipitation, and cloud clusters

In this section water vapor statistics are computed for (2). Explicit formulas are given for statistics that are exactly solvable. In some cases there is no formula (e.g., CWV variance). In these cases numerical simulations are used instead.

In section 3a the power spectral density is computed. In section 3b statistics that correspond to critical phenomena are given, including conditional precipitation and the spatial covariance of CWV. In section 3c the cloud cluster area distribution is numerically computed.

a. Power spectral density

For the model, the power spectral density is

$$P_{k,\ell}(\omega) = \frac{D_*^2}{2(\omega^2 + c_{k,\ell}^2)} + \frac{F^2}{c_{0,0}^2} \delta(\omega) \delta_{k0} \delta_{\ell 0}.$$
 (19)

Parameter variable	Description	Numerical value	Units
L	System size in one direction	5000	km
N	Number of lattice points in one direction	1000	_
M	Coarsening block size in one direction	5	_
Δx	Size of one grid cell in x direction	5	km
Δy	Size of one grid cell in y direction	5	km
F	Deterministic forcing	-0.125	$\mathrm{mm}\mathrm{h}^{-1}$
au	Dampening coefficient	96	h
b_0	Diffusion coefficient	750	$\mathrm{km}^{2}\mathrm{h}^{-1}$
$b = b_0 / \Delta x^2$	Interaction coefficient	30	$1 \mathrm{h}^{-1}$
D_*	Magnitude of stochastic forcing	50	$\mathrm{mm}\mathrm{h}^{-1/2}$
q^*	Threshold value for switch to convection	65	mm

TABLE 1. A table of the parameters used in the model (2).



FIG. 7. The PDF of CWV $(q_{M_{i,j}})$ (dashed line) and the conditional mean precipitation (solid line).

(See the appendix for an outline of the derivation.) To compare this equation with observations, we consider the power $P_{k,0}(\omega)$ in the $\ell = 0$ mode as an analog of averaging over a range of latitudes (e.g., 15°S to 15°N). An approximate, but more revealing, form of this power spectral density (PSD) can be obtained by considering $c_{k,0}$ in more detail. First, a Taylor expansion of $c_{k,0}^2$, about k = 0, is taken using the definition of $c_{k,\ell}$ from (14). Assuming a diffusive scaling of $b = b_0/\Delta x^2$ leads to

$$\lim_{\Delta x \to 0} c_{k,0}^2 = \tau^{-2} + \frac{4b_0 \pi^2 k^2}{L^2} (1 - \tau^{-1}).$$
 (20)

Thus the power spectrum, in the limit of large N, is

$$P_{k,0}(\omega) \approx \frac{D_*^2 + F^2 \delta(\omega) \delta_{k0}}{2\left(\omega^2 + \tau^{-2} + \frac{8b_0 \tau^{-1} \pi^2}{L^2} k^2\right)}.$$
 (21)

Thus the column water vapor has a power spectral density function similar to red noise power spectrum in both space and time, as indicated by the k^2 and ω^2 in the denominator.

In Fig. 6, the PSD for zonal wavenumbers -15 to 15 is plotted for the model averaged in the *y* direction. The decay across wavenumbers is similar to the observed background spectrum as shown in Fig. 2 of Wheeler and Kiladis (1999). The PSD of the precipitation $r_{i,j}$ and cloud indicator $\sigma_{i,j}$ can be computed numerically (not shown) and look similar to Fig. 6.

b. Phase transition and criticality

In this subsection, the stochastic model is compared with a recently proposed paradigm for tropical convection. Drawing on ideas from statistical physics, observational analyses have shown that tropical convection has similar characteristics to phase transitions and selforganized criticality (Peters and Neelin 2006; Neelin et al. 2009). What is the simplest model with this type of atmospheric behavior?



FIG. 8. A plot of the conditional mean precipitation with three interpretations of precipitation. The dashed line is precipitation as a constant rain rate |F|, as in Stechmann and Neelin (2011, 2014) and Hottovy and Stechmann (2015). The dashed–dotted line is precipitation proportional to CWV, as in Betts and Miller (1986). The solid line is precipitation as a combination of the two previous values. The gray lines are the best-fit power laws with exponents $\beta = 0.62$ (solid) and $\beta = 0.23$ (dashed).

The most basic principles of the phase transition and self-organized criticality are illustrated in Fig. 7. The mean precipitation, conditioned on each CWV value, displays a rapid pickup as CWV increases beyond the "critical" value $q^* = 65 \text{ mm}$ [see appendix for the analytical formulas, specifically (A15)]. This is reminiscent of a continuous phase transition (Baxter 1989; Yeomans 1992; Christensen and Moloney 2005) and similar to the observational analysis of tropical convection by Peters and Neelin (2006) (see their Figs. 1 and 3). Also shown in Fig. 7 is the pdf of CWV. In this model, the pdf is a Gaussian with mean

$$E(q_{M_{i,j}}) = q^* + F\tau = 53 \,\mathrm{mm}$$
 (22)

and standard deviation 4.70 mm. The bulk of the pdf is positioned just below the "critical value" $q^* = 65$ mm, similar to Fig. 3 of Peters and Neelin (2006) and to many systems displaying self-organized criticality.¹

To further explore the conditional mean precipitation, three different variations of precipitation are plotted in Fig. 8. The dashed line is a fixed rain rate of F, as in Stechmann and Neelin (2011, 2014) and Hottovy and Stechmann (2015). The dashed line is a

¹ In a sense this may not seem "self-organized" since the mean of the pdf can be determined in terms of the model parameters. However, it is self-organized in the sense that the "tuning parameters" here, the CWV, is allowed to vary through its own internal dynamics, in contrast to classical phase transitions where an external tuning parameter such as temperature is fixed in a laboratory experiment.



FIG. 9. (top) Numerically computed precipitation variance and (bottom) the mean precipitation, conditioned on column water vapor. Error bars are shown in gray. The number of model samples was chosen to be comparable to the number of observational data samples available in earlier studies.

rain rate proportional to the CWV as in Betts and Miller (1986). The solid line is a combination of both. Also in Fig. 8, the conditional mean precipitation is plotted with the best-fit line of a power law (gray dashed lines). That is, q^c , α , and β are chosen to fit the function $\alpha(q-q^c)^{\beta}$, for $q > q^* \ge q^c$. The critical value q^{c} for the precipitation rate $[|F| + (q_{ij} - q^{*})/\tau]\sigma_{ij}$ is strictly less than the threshold value of q^* . This is analogous to what is seen in observations [see Neelin et al. (2009), their Fig. 3], where q^* would be identified with the saturation value. On the other hand, the Betts-Miller-like precipitation rate $(q_{ij} - q^*)\sigma_{ij}/\tau$ bears little resemblance to a power law. Finally, the third potential precipitation definition, $|F|\sigma_{i,j}$, is shown with a black dashed-dotted line and also appears to follow a power law (gray dashed line). In fact, its power-law exponent of 0.23 is close to the values of 0.215 (Peters and Neelin 2006) and 0.265 (Peters et al. 2009) estimated from observational data.

In Fig. 9 the variance of CWV for $q_{M_{i,j}}$ (top panel) is plotted with error bars. This plot is numerically computed. An analytical formula does not exist because the expectation of the product $\sigma_{i,j}\sigma_{i',j'}$ can only be explicitly calculated in the simplest of cases (e.g., F = 0). Therefore, independent realizations of the model are used and the variance of one subblock is computed. In the bottom



FIG. 10. Variance for block sizes R = 1 (M = 5), R = 2 (M = 10), and R = 4 (M = 40). (top) The decrease in variance as block size increases. (bottom) The variance multiplied by a scaling factor of $R^{0.6}$ to collapse all the curves.

panel, the conditional mean is plotted based on the same numerical samples. Notice that the large CWV values (q > 70) are not well sampled but are sampled similarly to the observational data analysis of Peters and Neelin (2006) (see their Fig. 2). [We have run other tests (not shown) with many more samples in order to obtain more accurate estimates of the variance, and the result is essentially the same.] The main feature is a peak in the variance near the critical point $q^c < q^*$, similar to Fig. 1 of Peters and Neelin (2006) and to many examples of phase transitions (Baxter 1989; Yeomans 1992; Christensen and Moloney 2005).

A further characteristic of criticality is a nontrivial scaling of variance on different length scales. Specifically, in Fig. 10 the variance is plotted when averaging over block sizes of M = 5, 10, and 20. This corresponds to coarsening factors of R = 1, 2, and 4, respectively, relative to the standard coarsening M = 5. These values of M are similar to the ones used in Peters and Neelin (2006), their Fig. 2. As the block size increases, the variance decreases because of regression toward the mean (top panel). For independent random variables, the central limit theorem states that the variance of the average of a block size scales as R^{-2} . For the stochastic model here, in the bottom panel, the variance scales as $R^{-0.6}$. The power law of 0.6 < 2 means that



FIG. 11. Spatial correlation of CWV at two locations as a function of the separation in x. (top) Linear–linear axis scaling. (bottom) Log–log axis scaling.

there are long correlations. For a system with short correlations [i.e., here, correlations that exponentially decay before 100 km (M = 20)], the exponent will depend on R and increase to 2 as R increases. However, for a system exhibiting criticality, the exponent will be a fixed constant, independent of R. This means that, at least over the limited coarsened block scales considered here, the system has approximately scale-free correlations.

To further explore the correlations in the steady state between $q_{i,j}$ and sites $q_{i',j'}$, in Fig. 11 the spatial correlation of a single site is plotted as a function of the distance in x (i.e., i - i'). The distance is varied only in x because the correlation function is homogeneous and isotropic. In the top panel, the correlation decays quickly over the first 1000 km. In the bottom panel, the correlation is plotted on a log-log axis. In this panel, there is a period of power-law decay over distances from 5 to 500 km. This power law (dashed line) has slope -0.454(using linear least squares over this period). Then at roughly 500-1000 km, there is a cutoff where exponential decay begins. The correlation is compared to the power law (dashed line) to emphasize the long spatial correlations of the system. The fit to the power law can be enhanced by varying the parameters of the system as described in section 4a. This power law with an exponential cutoff behavior is typical of systems with criticality. In a system with criticality, the exponential cutoff



FIG. 12. A log–log plot of the PDF of cloud cluster area with error bars.

increases to arbitrarily long scales for certain parameter regimes (Yeomans 1992).

c. Cloud cluster

A cloud cluster in the model is a group of adjacent sites such that $\sigma_{i,j} = 1$ for (i, j) in the group. Recall that adjacent sites are nearest neighbors—that is, one lattice site away in either direction, but not both (not diagonal). This leads to the definition of a cluster as a connected graph of sites with $q_{i,j} > q^*$, where edges connect nearest neighbor sites. The cluster area is the number of sites in a cluster (times the 5 km × 5 km area of one site). To count the number of clusters in a realization, we implement the buffer algorithm [see Newman and Barkema (1999), their section 13.2.5].

In Fig. 12 the results of counting the cloud clusters are plotted. A PDF is created by normalizing each point by the total number of clusters counted. Error bars are included. There is a power-law trend over at least four decades in cloud area. The approximate power law is -1.7254. A similar power law is seen in observational data in Peters et al. (2009), their Fig. 8, and in Wood and Field (2011), their Fig. 4.

The robustness of the power law to changes in parameter τ is shown in Fig. 13 (the mean $F\tau$ is held constant). A vast range of values of τ is shown, from $\tau = 1$ to $\tau = 10^4$, where the standard value of the present paper falls in the middle of the range at $\tau = 96$. The larger values of τ promote a more extensive power-law range, and the cloud size distribution appears to be converging to a fixed power law of approximately -1.19 as τ increases. These plots provide additional confidence that the model's cloud size distribution follows a power law.



FIG. 13. As in Fig. 12, but for several values of τ .

4. Discussion

a. Parameter sensitivity

We have explored a variety of alternative parameter choices, and here we describe some broad aspects of parameter sensitivity for each parameter. Many aspects can be inferred from the formulas for the exact statistics. First, D_* controls the randomness, and it impacts the standard deviation of the CWV pdf (Fig. 7) and hence also the cloud fraction. However, it has no impact on the shape of the power spectrum (Fig. 6) because it impacts each Fourier mode in the same way in (19). Second, F impacts the mean of the CWV pdf in (22) and Fig. 7, and hence it also impacts the cloud fraction. Third, b impacts the spatial scale of CWV variations and hence also cloud clusters. In the power spectrum, it is b that causes the decrease in power as |k| increases. Finally, τ seems to have an interesting effect on the cloud cluster area distribution (Fig. 12). Specifically, in additional cases (shown in Fig. 13), large τ seems to promote a more robust power law and simultaneously promote the power law in spatial correlation function (Fig. 11). This is perhaps consistent with the connection between the $\tau \rightarrow \infty$ limit and some existing statistical physics models, which is described in section 5.

b. Thermodynamics and vertical structure

The thermodynamics of clouds is treated here in highly simplified form, based only on water vapor (or total water, depending on one's interpretation of the model variable q), and based only on its vertical average.

This formulation neglects the potential effects of the vertical structure of water vapor (Holloway and Neelin 2009) and effects of temperature variations, which have been explored from the statistical physics perspective by Neelin et al. (2009). Furthermore, associated with the column-averaged formulation is a tacit focus on deep convection without explicit consideration of other cloud types with more detailed vertical structure, such as congestus and stratiform clouds (Houze 1989; Johnson et al. 1999). The effect of congestus and stratiform clouds has been incorporated into some modeling frameworks (Moncrieff 1992; Mapes 2000; Khouider and Majda 2006) and it would be interesting to combine these approaches with the effects of the present model.

c. Nonlinear precipitation

The precipitation rate parameterization here is highly idealized and is simply a linear function of CWV multiplied by the cloud indicator variable. Furthermore, the same linear dependence of $\tau^{-1}q$ is also used to represent evaporation with the same time scale τ . As a result, the value of τ is intermediate between the short and long time scales that would normally be used (Neelin and Zeng 2000; Khouider and Majda 2006; Craig and Mack 2013) for precipitation and evaporation, respectively, and the model precipitation rates are somewhat reduced accordingly. In light of this, it is perhaps somewhat surprising that the model still displays features that resemble a continuous phase transition and power-law scaling of cloud cluster size distribution. Nonlinear precipitation has been investigated, for example, by Craig and Mack (2013), in a model without stochastic forcing, and the model produces phase separation with precipitation confined to a small number of circular moist regions. In the future, it would be interesting to explore the effects of the simple nonlinear precipitation formulation of Stechmann and Neelin (2011, 2014).

d. Entrainment

Entrainment and detrainment effects were included here only in a simplified linear form. More specifically, they are inherent to some degree in the eddy diffusion, which promotes an exchange of water vapor between neighboring cells. However, this exchange occurs here between all neighboring grid cells, not just at cloud boundaries. Nonlinearity could be introduced to isolate water vapor exchange at cloud boundaries; for example, a nonlinear term of the form $(\sigma_{i,j} + \sigma_{i+1,j} - 2\sigma_{i,j}\sigma_{i+1,j})(q_{i,j} - q_{i+1,j})$ would be active only if cells (i, j) and (i + 1, j) represent a cloud boundary with $\sigma_{i,j} = 1$ and $\sigma_{i+1,j} = 0$ or vice versa. It would be interesting to explore such nonlinear aspects of entrainment and detrainment. However, nonlinearity would cause exact solutions to be very challenging if not impossible, and it would cause even numerical solutions to be much more computationally expensive, since the semianalytic numerical method of section 2c would no longer be applicable.

e. Turbulent advection-diffusion

The representation of turbulent advection-diffusion here is in a highly idealized form as eddy diffusion, stochastic forcing, and damping. More sophisticated representations could also be used to provide additional realism, as described, for example, by Majda and Kramer (1999) and Majda and Gershgorin (2013) and references therein. For instance, these more sophisticated models can reproduce the fat-tailed pdfs that are commonly seen in situations of turbulent advection-diffusion with a background mean gradient, including the case of water vapor pdfs with exponential tails (Neelin et al. 2010). With water vapor, an added complication are the source terms due to precipitation and evaporation, and Stechmann and Neelin (2011) suggested that the source terms may play a key role in accounting for the exponential tails analyzed by Neelin et al. (2009). It would be interesting to possibly extend the model used here to explore the contributions of source terms versus turbulent advection-diffusion in accounting for the exponential tails in water vapor pdfs.

5. Connections with existing mathematical and statistical physics models

In this section, several connections are made with existing models in the mathematical and physics literature.

a. Ising model

To make a connection to the Ising model, notice that (8) can be written as

$$\rho(\mathbf{q}) = Z_q^{-1} \exp\left\{ \left(D_*\right)^{-2} \left[2b \sum_{\langle (i,j), \langle i',j' \rangle \rangle} q_{i,j} q_{i',j'} + F \sum_{i,j=0}^{N-1} q_{i,j} - \frac{1}{2\tau} \sum_{i,j=0}^{N-1} (q_{i,j} - q^*)^2 - 5b \sum_{i,j=0}^{N-1} q_{i,j}^2 \right] \right\}.$$
(23)

The above pdf is analogous to the classical Ising model's pdf. In the spin-1/2 Ising model, the distribution is characterized by the density

$$\rho(s_{i,j}) = Z_s^{-1} \exp\left\{ (k_B T)^{-1} \left[J \sum_{\langle (i,j), (i',j') \rangle} s_{i,j} s_{i',j'} - H \sum_{i,j=0}^N s_{i,j} \right] \right\},$$
(24)

where $s_{i,j} \in \{-1, 1\}$ is the spin, *J* is the exchange energy, *H* is an external field, k_B is the Boltzmann constant, and *T* is the temperature. Comparing the density in (23) with (24), D_* is analogous with k_BT , *b* to *J*, and *F* to *H*. (Additionally, there is a term proportional to $1/\tau$ and a term proportional to *b* that are not in the Ising model). This is a useful analogy that provides insight into water vapor model parameters because the Ising model is a wellknown system and its parameter sensitivity has been extensively studied (Christensen and Moloney 2005).

There are key differences between the model of column water vapor and the Ising model. The Ising model has discrete random variables $s_{i,j}$, which take only the values -1 or 1. The water vapor at a site $q_{i,j}$ can be any real number. This changes the interpretation of the partition functions (Z_q and Z_s) from a sum (in the Ising case) to an integral. Furthermore, the Ising model has a phase transition at a critical temperature, T > 0, for dimensions two and greater. The column water vapor equations are linear and thus there are not any phase transitions with respect to the $q_{i,j}$ variables themselves. However, the inclusion of a nonlinear threshold function $\sigma_{i,j}$ adds a complexity that leads to phase transition-like statistics.

b. Stochastic partial differential equations and the Edwards–Wilkinson model

The model [(2)] is a discretized form of the stochastic heat equation with damping in two dimensions (Hairer 2009). In one dimension the solution has finite variance; however, in two dimensions, the continuum solution has infinite variance. Thus, taking a limit as Δx , $\Delta y \rightarrow 0$ of (2) does not lead to a "nice" limit. To obtain a limit, the white noise (uncorrelated in space) must be replaced by a spatially correlated noise—for example, the convolution $R * \dot{W}$ of a smooth function R(x, y) with white noise $\dot{W}(x, y, t)$.

The one-dimensional Edwards–Wilkinson (EW) model is a model of the random growth of a surface (Edwards and Wilkinson 1982; Yu et al. 1994; Barabási and Stanley 1995; Antal and Rácz 1996). The EW model has the form

$$\frac{\partial q(x,t)}{\partial t} = b_0 \frac{\partial^2 q}{\partial x^2} + \dot{W}(x,t).$$
(25)

Comparing this equation to the continuum model [(1)], the EW model arises by taking the limit as $\tau \rightarrow \infty$. Because of the absence of a relaxation term, on average, the solution to (25) grows unbounded as time *t* tends to infinity. This unbounded growth can be characterized in terms of power laws and can be viewed as an example of nonequilibrium self-organized criticality (Pruessner 2012). Equation (1), unlike the model in (25), has an equilibrium distribution because of the addition of the relaxation term $\tau^{-1}(q-q^*)$.

c. Self-organized criticality

Peters and Neelin (2006) showed that tropical precipitation and water vapor can be viewed as self-organized criticality (SOC) (Bak et al. 1987, 1988; Christensen and Moloney 2005) in the sense that a critical value of water vapor marks a continuous phase transition, and the system is naturally attracted to the critical point (rather than externally tuned to lie near the critical point as in a laboratory setting or in the classical Ising model). This provides a new conceptual viewpoint of tropical convection, and in a sense it extends the ideas of convective "quasi equilibrium" that have been widely in use for decades (Arakawa and Schubert 1974; Emanuel et al. 1994; Neelin et al. 2008). Such a new conceptual viewpoint may potentially provide practical advances in, for instance, the parameterization of convection in climate models, some of which are based on convective quasi-equilibrium ideas.

In the present paper, further possible connections are drawn between tropical convection and ideas from statistical physics such as self-organized criticality. Most importantly, a simplified model for water vapor dynamics was introduced in (1), its relationship with atmospheric fluid dynamics was described in (3) and (4), and its statistics were shown to conform in many ways to the paradigm of self-organized criticality. These results provide some insight into the underlying water vapor and precipitation mechanisms. Furthermore, another possible connection is that the dynamics in (1) has some similarities to the model of Edwards and Wilkinson (1982) (a model of critical phenomena in interface growth) and, in turn, a quenched version of the Edwards and Wilkinson (1982) model has been shown by Pruessner (2003) to be related to the Oslo rice pile model, which is a well-known model of SOC.

Several caveats should also be noted in the possible connection with SOC. First, the observational data analyzed by Peters and Neelin (2006) has notable uncertainties. While some statistics have shown agreement when data were retrieved from different sources [microwave and radar measurements; see Fig. 1 of Peters et al. (2009)], it has also been suggested that measurement uncertainties may contribute to the appearance of a continuous phase transition (Gilmore 2015). Second, while the present paper's model displays statistics similar to SOC, some of the aspects of criticality are not established rigorously here. For example, the power laws suggested in Figs. 11 and 8 are seen over relatively small ranges of scales [although a more substantial power law in Fig. 11 could be seen (not shown) for larger values of τ , similar to the cloud cluster statistic in Fig. 13 when τ is increased]. In any case, the paradigms of phase transitions and self-organized criticality are useful as organizing principles for the vast range of statistics of tropical convection.

d. Gaussian free fields

The Gaussian free field (GFF) is a statistical mechanics model of random surfaces (Schramm and Sheffield 2009, 2013). The discrete GFF takes the same form as the stationary pdf [(8)] if F = 0 and in the limit as $\tau \rightarrow \infty$:

$$\rho(\mathbf{q}) = Z^{-1} \exp\left\{ D_*^{-2} \left[-b \sum_{\langle (i,j), (\bar{i},\bar{j}) \rangle} (q_{i,j} - q_{\bar{i},\bar{j}})^2 \right] \right\}.$$
 (26)

One interesting connection between GFFs and our model is the statistics of cloud clusters as seen in Fig. 12. A cluster of clouds is analogous to the interior of a contour loop for the value $q = q^*$ in the GFF case. These are not yet well understood. However, there are results for contour lines of the GFF connecting two boundaries. The statistics of these lines are governed by the Schramm–Loewner evolution (SLE). It would be interesting to further explore this connection.

6. Conclusions

A linear stochastic model was presented for tropical convection, clouds, and water vapor dynamics. This simple model displays a variety of characteristics that conform to a statistical physics perspective of tropical convectionfor example, a fractal, scale-free distribution of cloud cluster sizes and statistics resembling a phase transition and criticality. These features are in line with the ideas of self-organized criticality. While the underlying model itself is linear, nonlinear statistics were analyzed involving the cloud indicator $\sigma_{i,j} = \mathcal{H}(q_{i,j} - q^*)$. In addition, toward a different perspective, the model also displays a "red noise" power spectrum in both space and time, and hence it can be viewed as a model of the "background spectrum" of tropical convection [as illustrated in Fig. 2 of Wheeler and Kiladis (1999)]. Interestingly, these disparate perspectives on clouds and convection are unified by this model.

The physical processes incorporated into the model are precipitation, evaporation, and turbulent advectiondiffusion of water vapor. These processes are modeled here in a highly idealized form. Nevertheless, it is reassuring that the model equations are related to actual water vapor dynamics, rather than using a cloud indicator variable or rainfall variable as the stochastic dynamical variable in the model. Given the relation to actual water vapor dynamics, the results here suggest

Connections were drawn with some existing models of applied math and physics, including the Ising model, Edwards-Wilkinson model, and the Gaussian free field. The Ising model parameters are directly analogous to the water vapor model parameters, which facilitates their interpretation and the model's behavior. However, a key distinction is that the Ising model is based on a discrete variable s (which can take only two values: -1 or 1), whereas the water vapor q is a continuum variable (which can take the value of any real number). The Edwards-Wilkinson model arises from the water vapor model when damping is neglected and is usually studied in one spatial dimension. The Gaussian free field arises from the water vapor model in a special limiting case of the parameter values. This connection may offer further understanding of cloud cluster statistics, since cloud cluster boundaries are contour lines where the water vapor satisfies $q(x, y) = q^*$, and the statistics of Gaussian free field contour lines have been shown to be governed by the Schramm-Loewner evolution.

Wave dynamics were not explicitly included in the present study for the sake of simplicity. As a result, the model does not produce convectively coupled equatorial waves (Kiladis et al. 2009) or the Madden–Julian oscillation (Zhang 2005) in its present formulation. We are currently investigating the additional effects of waves and plan to present the results elsewhere in the near future. Furthermore, it would be interesting to incorporate the present model's features into other models of tropical dynamics (Frenkel et al. 2012, 2013; Khouider 2014; Majda and Stechmann 2009b; Thual et al. 2014).

Potential applications of the model include several situations where realistic cloud fields must be generated for minimal cost. For example, in some convective parameterizations in climate models, improvements in tropical variability have been achieved by incorporating the effects of subgrid-scale variability in time and/or space (Randall et al. 2003; Deng et al. 2015), and in radiative transfer models, stochastically generated cloud fields are sometimes utilized [e.g., see Pincus et al. (2003), Räisänen et al. (2004), Alexandrov et al. (2010), and references therein].

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APPENDIX

Mathematical Formulas and Computations

a. Stationary pdf

To find the stationary distribution of (2), the potential conditions are used, as described in Gardiner (2009), section 6.2.2. A potential function, $\Phi(\mathbf{q})$, is sought such that

$$\partial_{q_{i,j}} \Phi(\mathbf{q}) = b(q_{i+1,j} + q_{i,j+1} - 2q_{i,j} + q_{i-1,j} + q_{i,j-1}) - \frac{1}{\tau} (q_{i,j} - q^*) + F.$$
(A1)

Then the stationary distribution is

$$\rho(\mathbf{q}) = Z^{-1} \exp[D_*^{-2} \Phi(\mathbf{q})]. \tag{A2}$$

Equation (A1) is integrated to obtain the potential function

$$\Phi(\mathbf{q}) = F \sum_{i,j=0}^{N-1} q_{i,j} - \frac{1}{2\tau} \sum_{i,j=0}^{N-1} (q_{i,j} - q^*)^2 - b \sum_{\langle (i,j), (i',j') \rangle} (q_{i,j} - q_{i',j'})^2, \qquad (A3)$$

where $\langle \cdot, \cdot \rangle$ denotes nearest neighbors. Furthermore, the normalizing constant Z is

$$Z = (2\pi)^{N/2} \prod_{k,\ell=0}^{N-1} \frac{D_*^2}{2c_{k,\ell}},$$
 (A4)

which is most easily determined from the Fourier space perspective in (16).

b. Power spectral density

The Fourier transform in time of $Q_{k,\ell}(t)$ [(13)], denoted $a_{k,\ell}(\omega)$, is

$$-i\omega a_{k,\ell}(\omega) = F\delta_{k0}\delta_{\ell 0}\delta(\omega) - c_{k,\ell}a_{k,\ell}(\omega) + D_*\hat{\tilde{W}}_{k,\ell}(\omega),$$
(A5)

where $\hat{W}_{k,\ell}(\omega)$ is the discrete Fourier transform in space, and continuous Fourier transform in time, of

white noise $\dot{W}_{i,j}(t)$. The power spectrum is defined as $P_{k,\ell}(\omega)\delta(\omega - \omega') = E[|a_{k,\ell}(\omega)a_{k,\ell}^*(\omega')|]$. Using the identity $E[\hat{\tilde{W}}_{k,\ell}(\omega)\hat{\tilde{W}}_{k,\ell}^*(\omega')] = \delta(\omega - \omega')$, the power spectral density is

$$P_{k,\ell}(\omega) = \frac{D_*^2}{2(\omega^2 + c_{k,\ell}^2)} + \frac{F^2}{c_{0,0}^2} \delta(\omega) \delta_{k0} \delta_{\ell 0}.$$
 (A6)

c. Space and time correlations

The Wiener–Khinchin theorem is used to easily extract the spatiotemporal correlations in the stationary state from the properties of the complex-valued Ornstein–Uhlenbeck process in (13). Briefly, it states that the space correlation, denoted C(i - i', j - j'), is the Fourier transform of the correlation in Fourier space. That is,

$$C(i - i', j - j') = FT\{E(|Q_{k,\ell}|^2)\}(i - i', j - j'), \quad (A7)$$

where FT{·} the discrete Fourier transform. [See Gardiner (2009), section 1.5.2, for more detail and simple one-dimensional examples. See Yaglom (1962) for a more complete description.] Note that this is the steady-state correlation function and is written as a function of i - i' and j - j' using the fact that the process is spatially homogeneous.

d. Marginal and conditional statistics

Let $\mathbf{q} \in \mathbb{R}^{N^2}$ be the vector of $q_{i,j}$ with rowwise ordering. It is a Gaussian random variable with mean Fand one time covariance C(i - i', j - j'). Therefore, $q_{i,j}$ are jointly Gaussian random variables and the marginal distribution of $q_{i,j}$ is Gaussian. The joint distribution of $q_{i,j}$ and the k = 0 mode, $q_{M_{n,m}} = \sum_{(k,\ell) \in M_{n,m}} q_{k,\ell}$, is found in the following way. Using an affine transformation of the variables $\mathbf{q} = \{q_{i,j}\}_{i,j \in M_{n,m}}$, define the matrix,

$$\tilde{\mathbf{A}} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix}.$$
(A8)

Then, the new variables $\mathbf{u} = \tilde{\mathbf{A}}\mathbf{q}/M^2$ have a first component of

$$u_1 = \frac{1}{M^2} \sum_{k,\ell \in M_{n,m}} q_{k,\ell} = q_{M_{n,m}}.$$
 (A9)

From the properties of jointly Gaussian random variables, **u** is jointly Gaussian with mean $\mathbf{F} = (F, F, F, \dots, F)$ and covariance matrix

$$\begin{cases} C_{ij}, & i, j \neq 1 \\ \Sigma_{k,\ell} C_{k,\ell}, & i = j = 1 \\ \Sigma_k C_{i,k}, & j = 1, i \neq 1 \\ \Sigma_k C_{k,j}, & i = 1, j \neq 1 \end{cases}$$
(A10)

where $C_{i,j}$ is the original covariance matrix of **q**.

For the joint distribution of Q, q_i , consider i > 1 (for convenience). The variables are jointly Gaussian with covariance matrix,

$$\tilde{\mathbf{C}} = \begin{pmatrix} \Sigma_{k,\ell} C_{k,\ell} & \Sigma_k C_{i,k} \\ \Sigma_k C_{i,k} & C_{i,i} \end{pmatrix}.$$
(A11)

The stationary density for the anomalies $\overline{q}_i = q_i - \tau F$ and $\overline{Q} = Q - \tau F$ is

$$\rho(\overline{Q},\overline{q}_i) = Z_{Q,q_i}^{-1} \exp\left(-\frac{1}{2}\tilde{C}_{1,1}^{-1}\overline{Q}^2 + \tilde{C}_{1,2}^{-1}\overline{Q}\overline{q}_i - \frac{1}{2}\tilde{C}_{2,2}^{-1}\overline{q}_i^2\right),$$
(A12)

where Z_{Q,q_i} is the normalizing constant for the marginal variables q_i and Q.

The statistic $E(\sigma_i | \Sigma_i q_i = R)$ is found by using the above distribution. First note,

$$E\left(\sigma_{i} \mid \sum_{i} q_{i} = Q = R\right) = 1 \times \frac{\Pr(q_{i} > q^{*}, Q = R)}{\Pr(Q = R)} + 0 \times \frac{\Pr(q_{i} < q^{*}, Q = R)}{\Pr(Q = R)}.$$
(A13)

Using the above density, along with the fact that $q_i > q^*$ implies $\overline{q}_i > -\tau F$ yields

$$E(\sigma_i \mid Q = R) = \frac{\int_{-\tau F}^{\infty} Z_{Q,q_i}^{-1} \exp\left[-\frac{1}{2}\tilde{C}_{1,1}^{-1}(R - \tau F)^2 + \tilde{C}_{1,2}^{-1}(R - \tau F)\overline{q}_i - \frac{1}{2}\tilde{C}_{2,2}^{-1}\overline{q}_i^2\right] d\overline{q}_i}{Z_Q^{-1} \exp\left[-\frac{1}{2}\tilde{C}_{1,1}^{-1}(R - \tau F)^2\right]}.$$
 (A14)

After simplification, it is

$$E(\sigma_i \mid Q = R) = \frac{1}{2} \left(1 + \operatorname{erf}\left\{ \frac{(\Sigma_{k,\ell} C_{k,\ell})F + (\Sigma_k C_{i,k})(R - \tau F)}{\sqrt{2(\Sigma_{k,\ell} C_{k,\ell})[\Sigma_{k,\ell} C_{i,i} C_{k,\ell} - (\Sigma_k C_{i,k})^2]}} \right\} \right).$$
(A15)

Using the above methods, $E(q_i\sigma_i | Q = R)$ can also be computed. Thus analytical formulas for $E(|F|\sigma | Q = R)$, $E[(q_i - q^*)\tau^{-1}\sigma_i | Q = R]$, and $E(r_i | Q = R)$ can be recovered.

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