

EXACT AND APPROXIMATE SOLUTIONS FOR ELASTIC INTERACTIONS IN A NEMATIC LIQUID CRYSTAL*

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Abstract. Anisotropic fluids appear in a diverse array of systems, from liquid crystal displays to bacterial swarms, and are characterized by orientational order. Large colloidal particles immersed in such environments disturb the medium’s orientational order, resulting in a stored elastic energy within the bulk. As a consequence, multiple immersed bodies interact at equilibrium through fluid-mediated forces and torques, which depend on the bodies’ positions, orientations, and shapes. We provide the equilibrium configuration of a model nematic liquid crystal with multiple immersed bodies or inclusions in two dimensions, as well as the associated body forces, torques, and surface tractions. A complex variables approach is taken which leans on previous work by Crowdy [*Solving Problems in Multiply Connected Domains*, SIAM, Philadelphia, 2020] for describing solutions with multiply connected domains. Free periods of a complex director field, which correspond to topological defect positioning and net topological charge, are determined numerically to minimize a global stored elastic energy, including a contribution of a weak (finite) anchoring strength on the body surfaces. Finally, a general, analytical description of two-body far-field interactions is provided, along with examples using two cylindrical inclusions of arbitrary position and size, and two triangles of arbitrary position and orientation.

Key words. liquid crystals, conformal mapping, multiply connected

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1. Introduction. Many fluids are host to a suspension of elongated bodies which show a preference toward orientational alignment. In a uniaxial liquid crystal, the local molecular orientation, averaged over a small control volume, is represented as a director field $\mathbf{n}(\mathbf{x}, t)$, with spatial position \mathbf{x} , time t , and $|\mathbf{n}(\mathbf{x}, t)| = 1$. Deformations of the director field away from uniformity result in an elastic stress response [18, 85]. Such fluids have been of great industrial interest for decades due to their optical properties [102], applicability to medical science [101], chemical and biological sensing [11], and the design of soft active materials [10]. Active biological systems have been similarly described [51, 21], from the dynamic ordering of mucus [93], biofilms [94], and tissues [74, 53] to suspensions of swimming bacteria [71, 38, 72] and the interior of cells [9, 23, 61].

Among the most alluring (and analytically challenging) features of liquid crystals is the prevalence of topological singularities, which satisfy global conservation laws [46, 3]. The locations of the defects on the surface or in the fluid depend on the relationship between the bulk elastic energy and the surface anchoring conditions on any domain boundaries. In addition to focusing elastic stress on immersed surfaces, topological defects are important sites in biological settings for the onset of cell death and extrusion [73], layer formation [14], cell accumulation [37], cell sorting [5], and morphogenesis [52, 92, 97]. They have also been considered for directed self-assembly

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[55, 96] and control [62, 28, 47, 49, 25]. Analytical insight into defect positioning and its consequences for locally stored elastic energy is, thus, of broad interest.

Bodies immersed in a liquid crystal (that are much larger than the liquid crystal constituents) disturb the orientational order of the bulk liquid crystal. Confining or immersed boundaries introduce preferential orientations of the director field with a given strength (for instance, a tangential anchoring condition); these generally lie in competition with the preferred uniformity of the orientation field [83]. If there are multiple immersed bodies or boundaries, the elastic energy may be reduced by altering their relative positions and orientations. Dipolar and quadrupolar far-field interactions between colloids (depending on normal or tangential anchoring conditions) have been investigated in three dimensions [66, 70, 27, 2], and similarly between a colloid and a confining boundary [26]. When many colloids are introduced to a liquid crystal they can self-assemble into linear chains [64, 44, 76, 20]. When the bodies are sufficiently well separated, their long-range interactions conjure a related problem, the interaction of topological defects themselves [86, 36].

Near-field interactions, meanwhile, can be strongly nonlinear due to the interaction and positional rearrangement of topological defects [87, 4, 13, 36]. The self-assembly of colloids in liquid crystals has seen wide use in the engineering of smart materials, with applications ranging from biosensors to dynamic porous membranes [7]. Rather than colloid translations and rotations to reduce the system energy, a separate path toward relaxation is available if the immersed particles are deformable [50, 56, 103, 59, 75].

As a consequence of defect repositioning in near-field interactions, spherical colloids with tangential anchoring can settle into a configuration with broken symmetry, and multiple colloids can self-assemble into a chain aligned at an angle of 30° with the alignment axis of the liquid crystal [65, 63, 80, 88, 19] or into kinked chains [77, 29]. Crystal lattice configurations have also been observed [60, 55, 29]. More exotic interactions include particle binding via Saturn-ring defect interactions [33, 90, 89, 79]. In addition to their positioning relative to the director field alignment axis, colloid interactions through the liquid crystal also depend on the particle geometry and relative orientation. Two triangular bodies, for instance, can be arranged such that they are either attractive or repulsive just by rotating them relative to one another [41, 79].

While a variety of numerical methods for exploring liquid crystal configurations have been developed [95], analytical solutions of the equilibrium director field configuration are needed in order to better understand the geometry-dependent, liquid crystal-mediated elastic body interactions. Even though the equilibrium director field is a harmonic function in the single Frank elastic constant approximation [18], these body interactions are not simple to determine due to nonlinear anchoring boundary conditions and topological defects, whose positions are unknown a priori.

The equilibrium director field around a single immersed body already introduces a number of important features, which inform the question of body interactions. In Chandler and Spagnolie [12], we used complex variables techniques to find analytical solutions in the asymptotic regime of large surface anchoring strengths. Among our findings, we showed that topological singularities are preferentially positioned at or near sharp corners of an immersed body, depending on whether the anchoring strength is infinite or finite. When multiple bodies are immersed in the fluid, or if a nearby wall or other boundary is present, the problem tends that much further from tractability. The complex variables approach for interactions was used to similarly characterize the interactions of two topological defects [86].

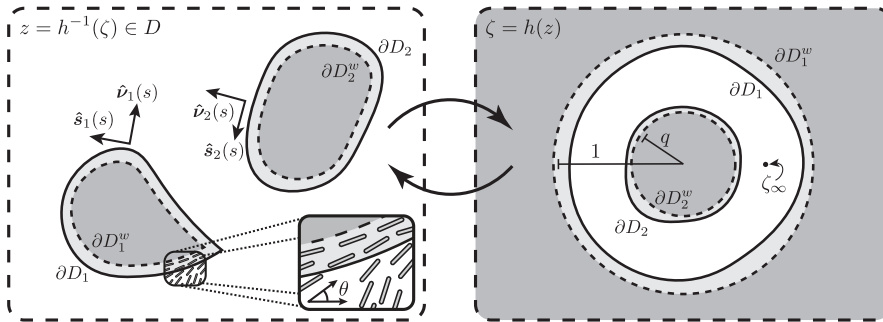


FIG. 1. Left: The physical z -domain with two rigid bodies immersed in a two-dimensional nematic liquid crystal, where $z = x + iy$. The liquid crystal is described by a director field $\mathbf{n} = (\cos \theta, \sin \theta, 0)$ with director angle $\theta(z) \in [0, \pi]$ for $z \in D$. The boundaries of the two bodies are shown as solid curves, ∂D_1 and ∂D_2 , with unit normal and tangent vectors $\hat{\mathbf{v}}_k(s)$ and $\hat{\mathbf{s}}_k(s)$, respectively. The effective (or virtual) boundaries are shown as dotted curves, ∂D_1^w and ∂D_2^w . Right: The conformally mapped ζ -domain. The pole at ζ_∞ corresponds to $z \rightarrow \infty$ in the physical domain.

The problem of determining harmonic functions with generic boundary conditions in multiply connected domains has been explored in great depth by Crowdy [17]. Using complex variables, the problem can be recast as a search for a locally holomorphic function with particular boundary conditions. The physical domain is first conformally mapped to a multiconnected annulus, and then a series of images of a free-space Green's function across all of the (now circular) boundaries leads in the direction of the solution, although additional care must be taken to monitor the periods of the holomorphic function and related auxiliary functions around each boundary.

In this paper, we use the approach put forth by Crowdy [17] to analyze multibody interactions in a nematic liquid crystal. Figure 1 provides a schematic of the general problem. Just as in the case of a single immersed body, the nonlinear boundary conditions and topological defect positions in the strong anchoring limit pose additional challenges, which are overcome upon appeals to the energy. Although a single body has no force or torque acting upon it [12], analogous to d'Alembert's paradox in classical potential flow theory, two bodies can impose forces and torques on each other through the fluid, as has been observed experimentally [41]. These interactions are generically shape and orientation dependent. A number of additional nonlinear phenomena will be examined along the way, including a symmetry breaking instability when two cylinders are drawn nearer to each other, corresponding to a discontinuous jump in the topological defect positions.

This paper is organized as follows. We begin in section 2 with a review of the mathematical model, including a discussion of boundary conditions and surface tractions, and we recall from [12] the effective boundary technique that allows for the solution of a weak (finite) anchoring problem based on the solution of a strong (infinite) anchoring problem with a slightly different boundary. Analytical solutions for two immersed bodies are then provided in section 3. Two worked examples of multiple-body interactions are then presented, which demonstrate the above methodology for determining the two-dimensional director field at equilibrium, including the selection of the topological charges and defect positions on the body surfaces. The first of these two examples is given in section 4, where we investigate two immersed cylinders with tangential anchoring, which includes the case of a single cylinder near an infinite wall as a limiting case. We consider a more involved example in section 5, the interactions between two triangular prisms, where we again provide formulae for the body forces

and torques, and observe how defect positioning and particle interactions are orientation dependent, reproducing experimental findings. When the distance between the bodies is large, asymptotically valid approximations may be derived, as described in section 6. Finally, in section 7, we provide a closing summary and directions of future applications.

2. Mathematical formulation. We begin with a description of the general problem and recall the relevant structure developed for the case of a single immersed body [12]. Consider a two-dimensional nematic liquid crystal outside N simply connected bodies, as illustrated in Figure 1 for $N = 2$. The liquid crystal domain and the boundary of the k th body are denoted by D and ∂D_k , respectively. Assuming the one-constant approximation, the director angle, $\theta(x, y)$, is described by the Dirichlet free energy $\mathcal{F}_{\text{surface}} := K|\nabla\theta|^2/2$, where K is the single Frank elastic constant. In general there are distinct elastic moduli penalizing liquid crystal bend and splay deformations, but they tend to be comparable [8, 104], and the single constant model is often used to simplify mathematical analysis [18].

At the boundaries, the Rapini–Papoular form of the surface anchoring energy is given by $\mathcal{F}_{\text{surface}} := W_k \sin^2(\theta - \phi_k)/2$, where W_k is the anchoring strength and ϕ_k is the preferred orientation defined on ∂D_k [67]. Examples will be provided for the important case where ϕ_k represents the tangent angle on the surface of the k th body, but the formulation below is valid for general ϕ_k .

Combining the bulk and surface energies yields the net free energy

$$(2.1) \quad \mathcal{E} := \frac{K}{2} \iint_D |\nabla\theta|^2 dA + \sum_{k=1}^N \frac{W_k}{2} \int_{\partial D_k} \sin^2(\theta - \phi_k) ds,$$

where s is an anticlockwise arc length parameterization of the bodies, and dA and ds are the infinitesimal surface area and arc length elements, respectively. The principle of virtual work applied to (2.1) yields the equilibrium equation for the director angle

$$(2.2) \quad \nabla^2\theta = 0 \quad \text{in } D,$$

subject to the weak anchoring boundary conditions,

$$(2.3) \quad -K \frac{\partial\theta}{\partial\hat{\nu}_k} + \frac{W_k}{2} \sin[2(\theta - \phi_k)] = 0 \quad \text{on } \partial D_k,$$

for $k \in \{1, \dots, N\}$, where $\hat{\nu}_k = -\mathbf{x}_s^\perp$ is the fluid-pointing unit normal on the k th body, as depicted in Figure 1. The traction on the k th surface due to the liquid crystal is also determined in this process (see [12]) and is given by

$$(2.4) \quad \mathbf{t}_k = K \left(\frac{1}{2} |\nabla\theta|^2 \hat{\nu}_k - \frac{\partial\theta}{\partial\hat{\nu}_k} \nabla\theta \right) + \frac{W_k}{2} \left(\sin(\theta - \phi_k)^2 \hat{\mathbf{s}}_k + \sin[2(\theta - \phi_k)] \hat{\nu}_k \right)_s,$$

where $\hat{\mathbf{s}}_k = \mathbf{x}_s$ is the unit tangent vector on the k th body, and the subscript s denotes an arc length derivative. Given a director field that satisfies (2.2) and (2.3), the energy and surface traction associated with the liquid crystal can be computed using (2.1) and (2.4), respectively.

The problem is made dimensionless by scaling all lengths upon a characteristic length scale associated with the immersed bodies, a , and defining a dimensionless free energy, $\hat{\mathcal{E}} := \mathcal{E}/K$, and tractions, $\hat{\mathbf{t}}_k := a^2 \mathbf{t}_k/K$. The resulting equations are governed by the dimensionless anchoring strengths $w_k := aW_k/K$. The dimensionless

free energy of the liquid crystal may be written as a boundary integral using the divergence theorem, i.e.,

$$(2.5) \quad \hat{\mathcal{E}} = \frac{1}{2} \sum_{k=1}^N \int_{\partial D_k} -\theta \frac{\partial \theta}{\partial \hat{\nu}_k} + w_k \sin^2(\theta - \phi_k) \, ds.$$

Henceforth, we shall only work in these dimensionless variables.

2.1. Complex variable representation. To access a wide range of complex variable techniques, we introduce the complex coordinate $z := x + iy$ and complex director angle

$$(2.6) \quad \Omega(z) := \tau(x, y) - i\theta(x, y),$$

where $\tau(x, y) = \operatorname{Re} \Omega(z)$ is a harmonic conjugate of $\theta(x, y) = -\operatorname{Im} \Omega(z)$ (i.e., $\tau_x = -\theta_y$ and $\tau_y = \theta_x$) [12]. Since $\theta(x, y)$ is harmonic in D , $\theta_x - i\theta_y$ must be holomorphic in D and $\Omega(z)$ is at least locally holomorphic. In general, Ω may not be single-valued around each immersed body, thus the period around each must be defined. We write

$$(2.7) \quad \oint_{\partial D_k} d\Omega \equiv \frac{1}{i} \int_{\partial D_k} \theta_x - i\theta_y dz = \Upsilon_k - 2\pi i M_k \quad \text{for } k \in \{1, \dots, N\},$$

for some given real constants Υ_k and half-integers M_k , which correspond to the topological charge of the k th body.

In these complex variables, the boundary condition (2.3) is equivalent to the constraints

$$(2.8) \quad \left(|e^{\Omega(z)}|^2 \right)_s + w_k \operatorname{Im} \left[e^{2i\phi_k} e^{2\Omega(z)} \right] = 0 \quad \text{on } \partial D_k,$$

for $k \in \{1, \dots, N\}$; the net free energy (2.5) may be written as

$$(2.9) \quad \hat{\mathcal{E}} = \frac{1}{4} \sum_{k=1}^N \oint_{\partial D_k} \operatorname{Im} \left[\left(\Omega(z) - \overline{\Omega(z)} \right) \Omega'(z) z_s \right] + w_k \operatorname{Re} \left[1 - e^{\Omega(z) - \overline{\Omega(z)}} e^{2i\phi_k} \right] ds,$$

where the bar denotes a complex conjugate, and the surface traction on the k th body, $\hat{\mathbf{t}}_k \equiv (\hat{t}_k^x, \hat{t}_k^y)$ given by (2.4), satisfies

$$(2.10) \quad \hat{t}_k^x - i\hat{t}_k^y = \frac{1}{2i} \Omega'(z)^2 z_s + \frac{w_k}{8} \left[\left(2 + e^{-2i\phi_k} e^{\overline{\Omega(z)} - \Omega(z)} - 3e^{2i\phi_k} e^{\Omega(z) - \overline{\Omega(z)}} \right) \bar{z}_s \right]_s.$$

Integrating the traction around ∂D_k yields the net dimensionless force, $(\hat{F}_k^x, \hat{F}_k^y)$, and torque, \hat{T}_k , acting on the k th body:

$$(2.11a) \quad \hat{F}_k^x - i\hat{F}_k^y = \oint_{\partial D_k} \hat{t}_k^x - i\hat{t}_k^y ds = \frac{1}{2i} \oint_{\partial D_k} \Omega'(z)^2 dz,$$

$$(2.11b) \quad \hat{T}_k = \oint_{\partial D_k} (x - x_k) \hat{t}_k^y - (y - y_k) \hat{t}_k^x ds = \frac{1}{2} \operatorname{Re} \left[\oint_{\partial D_k} (z - z_k) \Omega'(z)^2 dz \right] + \Upsilon_k,$$

where $z_k = x_k + iy_k$ is the center of torque of the k th body.

3. Analytical solutions for two immersed bodies. In this section, we consider the interaction of two immersed bodies ($N = 2$). The liquid crystal is assumed to be oriented with the x -axis in the far-field and subject to finite tangential anchoring on both ∂D_1 and ∂D_2 , with dimensionless anchoring strengths w_1 and w_2 , respectively; that is, $\Omega(z) \rightarrow 0$ as $|z| \rightarrow \infty$ and $\Omega(z)$ satisfies (2.8) with $\phi_k = \arg(z_s) \pmod{\pi}$.

We first show that the two bodies appear as equal and opposite topological charges in the far-field, and that their periods similarly sum to zero. Consider the contour integral $\oint_C d\Omega$ for a closed contour C which encircles both bodies. Since $\Omega'(z)$ is holomorphic in D , the contour can be deformed within D via Cauchy's integral theorem. By taking the contour to infinity and imposing the far-field condition, $\Omega(z) \rightarrow 0$ as $|z| \rightarrow \infty$, we find that the integral vanishes and, thus, Ω is single-valued outside the two bodies. However, Ω may be multivalued along contours that pass between the bodies. It follows that the period of Ω around ∂D_1 must be the additive inverse of the period around ∂D_2 , i.e., $\Upsilon := \Upsilon_1 = -\Upsilon_2$ and $M := M_1 = -M_2$ in (2.7). The two bodies, thus, appear as topological defects of charge $M_1 = M$ and $M_2 = -M$ in the far-field. We shall focus our attention on the case $M = 0$ since this is known to minimize the free energy for an isolated body [12].

At large anchoring strengths, subjecting a director field to finite-strength tangential anchoring on a boundary ∂D_k (i.e., (2.8) with $\phi_k = \arg(z_s) \pmod{\pi}$) is asymptotically equivalent to subjecting it to strong (exact) tangential anchoring on an effective interior boundary ∂D_k^w , i.e.,

$$(3.1) \quad \text{Im} \left[e^{\Omega(z)} z_s \right] = \mathcal{O}(1/w_k^3) \quad \text{on } \partial D_k^w,$$

as $w_k \rightarrow \infty$ [12]. The effective (or virtual) boundary, ∂D_k^w , is found by displacing the physical boundary, ∂D_k , by $-\hat{\nu}_k(s)/w_k - \hat{s}'_k(s)/(2w_k^2) + \mathcal{O}(1/w_k^3)$. This asymptotic equivalence was termed the "effective boundary technique" [12].

Analytical progress can be made by writing the complex director as

$$(3.2) \quad \Omega(z) = \log f'(z) + g(z),$$

where $g(z)$ is any locally holomorphic function that accounts for the periods in (2.7), i.e.,

$$(3.3a,b) \quad \oint_{\partial D_1} dg = \Upsilon \quad \text{and} \quad \oint_{\partial D_2} dg = -\Upsilon,$$

and $f'(z)$ is a single-valued holomorphic function that accounts for the boundary conditions (3.1). Without loss of generality, we may choose $g(z)$ such that its imaginary part vanishes on ∂D_1^w and is constant on ∂D_2^w . The boundary conditions for $f(z)$ then follow from integrating (3.1) with respect to arc length. Together these yield the following problem: find functions $f(z)$ and $g(z)$ such that

$$(3.4a) \quad g(z) \text{ locally holomorphic} \quad \text{in } D^w,$$

$$(3.4b) \quad \text{Im } g(z) = 0 \quad \text{on } \partial D_1^w,$$

$$(3.4c) \quad \text{Im } g(z) = \alpha \quad \text{on } \partial D_2^w,$$

$$(3.4d) \quad g(z) \rightarrow i\beta \quad \text{as } |z| \rightarrow \infty,$$

with periods (3.3) and real constants α and β , which are to be determined, and

$$(3.5a) \quad f(z) \text{ locally holomorphic} \quad \text{in } D^w,$$

$$(3.5b) \quad \text{Im } f(z) = 0 \quad \text{on } \partial D_1^w,$$

$$(3.5c) \quad \text{Im} [e^{i\alpha} f(z)] = C \quad \text{on } \partial D_2^w,$$

$$(3.5d) \quad f(z) \sim e^{-i\beta} z \quad \text{as } |z| \rightarrow \infty,$$

for some unknown constant C . Here, we have fixed the gauge of f such that the constant in (3.5b) vanishes. A unique solution is selected by specifying the period of f around each body, i.e.,

$$(3.6) \quad \oint_{\partial D_1^w} df = \Gamma_1 \quad \text{and} \quad \oint_{\partial D_2^w} e^{i\alpha} df = \Gamma_2,$$

for real periods Γ_1 and Γ_2 .

By an extension of the Riemann mapping theorem for multiconnected domains (due to Koebe, see discussion in [30]), there exists a conformal map, $z = h(\zeta)$, from the doubly connected effective domain $z \in D^w$ to the annulus $q \leq |\zeta| \leq 1$, with ∂D_1^w mapped onto $|\zeta| = 1$, ∂D_2^w onto $|\zeta| = q$, and $z = \infty$ to an interior point $\zeta = \zeta_\infty$ such that $z = h(\zeta) \sim C_\infty/(\zeta - \zeta_\infty)$ as $\zeta \rightarrow \zeta_\infty$. According to the Riemann mapping theorem, there are three real degrees of freedom for any conformal map of a simply connected domain. In the case of a doubly connected domain, however, two of these degrees of freedom are needed to ensure the annulus is concentric. Thus, only a rotational degree of freedom remains, which we use to place $\zeta = \zeta_\infty$ on the positive real axis, so that $0 < q < \zeta_\infty < 1$. The remaining parameters q , ζ_∞ , and C_∞ are dependent on the geometry and positions of the two bodies and must be determined.

In the ζ -plane, the two potentials $G(\zeta) := g(z(\zeta))$ and $F(\zeta) := f(z(\zeta))$ satisfy

$$(3.7a) \quad G(\zeta) \text{ locally holomorphic} \quad \text{in } q < \zeta < 1,$$

$$(3.7b) \quad \text{Im } G(\zeta) = 0 \quad \text{on } |\zeta| = 1,$$

$$(3.7c) \quad \text{Im } G(\zeta) = \alpha \quad \text{on } |\zeta| = q,$$

$$(3.7d) \quad G(\zeta) = i\beta \quad \text{at } \zeta = \zeta_\infty,$$

with periods $\oint_{|\zeta|=1} dG = -\Upsilon$ and $\oint_{|\zeta|=q} dG = -\Upsilon$ and

$$(3.8a) \quad F(\zeta) \text{ locally holomorphic} \quad \text{in } q < \zeta < 1,$$

$$(3.8b) \quad \text{Im } F(\zeta) = 0 \quad \text{on } |\zeta| = 1,$$

$$(3.8c) \quad \text{Im} [e^{i\alpha} F(\zeta)] = C \quad \text{on } |\zeta| = q,$$

$$(3.8d) \quad F(\zeta) \sim C_\infty e^{-i\beta}/(\zeta - \zeta_\infty) \quad \text{as } \zeta \rightarrow \zeta_\infty,$$

with periods $\oint_{|\zeta|=1} dF = -\Gamma_1$ and $\oint_{|\zeta|=q} e^{i\alpha} dF = \Gamma_2$.

The analytical solution to (3.7) and (3.8) can be found by using the method of images to construct functions akin to generalized Green's functions [17]. The full derivation is provided in section SM1 in the supplemental material. We find that

$$(3.9) \quad G(\zeta) = -\frac{\Upsilon}{2\pi i} \log \zeta,$$

which yields the constants $\alpha = \Upsilon \log q/(2\pi)$ and $\beta = \Upsilon \log \zeta_\infty/(2\pi)$, and

$$(3.10) \quad F(\zeta) = \frac{C_\infty e^{-i\beta}}{\zeta_\infty} K(\zeta/\zeta_\infty) - \frac{\overline{C_\infty} e^{i\beta}}{\zeta_\infty} K(\zeta_\infty \zeta) - \frac{\Gamma_1}{2\pi i} \log \frac{P(\zeta/\zeta_\infty)}{P(\zeta_\infty \zeta)} - \frac{\Gamma_2 e^{-i\alpha}}{2\pi i} \log \frac{P(\zeta/\zeta_\infty)}{P(\zeta_\infty \zeta/q^2)},$$

where

$$(3.11a) \quad P(\zeta) := (1 - \zeta) \prod_{k=1}^{\infty} (1 - q^{2k}\zeta)^{e^{2ki\alpha}} (1 - q^{2k}/\zeta)^{e^{-2ki\alpha}},$$

$$(3.11b) \quad \text{and } K(\zeta) := \frac{\zeta P'(\zeta)}{P(\zeta)} = \frac{\zeta}{\zeta - 1} + \sum_{k=1}^{\infty} \left(\frac{e^{-2ki\alpha} q^{2k}}{\zeta - q^{2k}} - \frac{q^{2k} e^{2ki\alpha}}{1/\zeta - q^{2k}} \right).$$

Note that, since $0 < q < \zeta_{\infty} < 1$, this infinite product and summation converge absolutely within the annulus, and only a few terms are needed to obtain accurate approximations. The solution (3.10) can be understood as follows: the first two terms account for the pole at $\zeta = \zeta_{\infty}$ by introducing an infinite cascade of images across the two body boundaries, $|\zeta| = 1$ and $|\zeta| = q$, and the final two terms account for the periods around the two bodies by introducing logarithmic cuts between $\zeta = \zeta_{\infty}$ and $\zeta = 1/\zeta_{\infty}$ (i.e., across $|\zeta| = 1$) and $\zeta = \zeta_{\infty}$ and $\zeta = q^2/\zeta_{\infty}$ (i.e., across $|\zeta| = q$).

With these two potentials, the complex director angle is

$$(3.12) \quad \Omega(z) = \log f'(z) + g(z) = \log [F'(\zeta)/h'(\zeta)] - \Upsilon \log \zeta / (2\pi i),$$

where $\zeta = h^{-1}(z)$ and

$$(3.13) \quad F'(\zeta) = C_{\infty} \frac{e^{-i\beta}}{\zeta_{\infty}^2} K'(\zeta/\zeta_{\infty}) - \overline{C_{\infty}} e^{i\beta} K'(\zeta_{\infty}\zeta) + \frac{\Gamma_2 e^{-i\alpha}}{2\pi i \zeta} + \frac{\Gamma_1 + \Gamma_2 e^{i\alpha}}{2\pi i \zeta} K(\zeta_{\infty}\zeta) - \frac{\Gamma_1 + \Gamma_2 e^{-i\alpha}}{2\pi i \zeta} K(\zeta/\zeta_{\infty}).$$

The above expression has been simplified using the identity $P(\zeta/q^2) = -\zeta P(\zeta) e^{2i\alpha}/q^2$. Finally, the director angle is given by $\theta = -\text{Im}\Omega(z)$, and the free energy, surface tractions, body forces, and body torques are computed by inserting (3.12) into (2.9)–(2.11). Analyzing these solutions further, however, requires the specification of the physical domain (i.e., the conformal map $h(\zeta)$). We therefore proceed to consider two concrete examples: two interacting circular cylinders and two interacting triangular prisms. Following these, we will provide a more general analytical perspective which is available when the bodies are well separated.

4. Example 1: Two cylinders with tangential anchoring. Consider a liquid crystal outside two immersed cylinders, one centered at $z = 0$ with dimensionless unit radius, and the other centered at $z = de^{i\chi}$ with dimensionless radius b . We denote the boundaries of these cylinders as ∂D_1 and ∂D_2 , respectively. The liquid crystal is assumed to be oriented with the x -axis in the far-field and is subject to finite tangential anchoring on each cylinder: $\Omega(z) \rightarrow 0$ as $|z| \rightarrow \infty$ and $\Omega(z)$ satisfies (2.8) with $\phi = \arg(z_s) \pmod{\pi}$ on the surfaces $|z| = 1$ and $|z - de^{i\chi}| = b$. We shall also assume that both cylinders have vanishing topological charge, i.e., $\oint_{\partial D_k} d\theta = 0$. This configuration is plotted in Figure 2.

For large anchoring strengths (i.e., $w_1 \gg 1$ and $w_2 \gg 1$), the effective boundary technique may be implemented and the solutions derived in section 3 may be used. The first step is to find the effective boundaries corresponding to the two cylinders, $|z| = 1$ and $|z - de^{i\chi}| = b$. In [12], we showed that the effective boundary corresponding to a unit cylinder with anchoring strength w is a cylinder of radius $|z| = \rho(w) := (\sqrt{1 + 4/w^2} - 2/w)^{1/2}$. This effective boundary is not only consistent

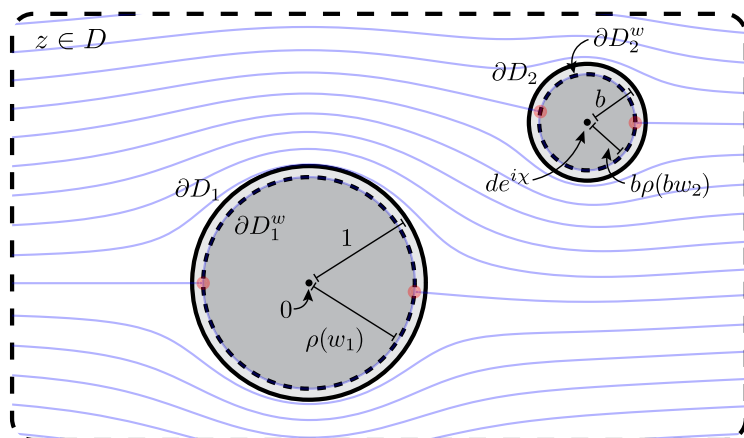


FIG. 2. Example 1. Two-dimensional liquid crystal outside a unit cylinder centered at $z = 0$ and a cylinder of radius b centered at $z = de^{i\chi}$ (∂D_1 and ∂D_2 , black solid curves). The effective domain boundaries are shrunk cylinders of radii $\rho(w_1)$ and $b\rho(bw_2)$, respectively (∂D_1^w and ∂D_2^w , black dotted curves). Integral curves of the director field are shown in blue for $w_1 = w_2 = 10$, $d = 2.75$, $b = 0.5$, $\chi = \pi/6$, and using numerically determined energy-minimizing periods: $\Gamma_1^{\min} \approx -0.3097$, $\Gamma_2^{\min} \approx 0.3786$, and $\Upsilon^{\min} = -0.0225$.

with the asymptotic expression in (3.1) for large w , but it in fact holds for all anchoring strengths (i.e., $w \geq 0$). It follows that, here, a suitable choice for the effective boundaries of the two cylinders is $|z| = \rho(w_1)$ and $|z - de^{i\chi}| = b\rho(bw_2)$, which we denote as ∂D_1^w and ∂D_2^w , respectively.

The next step is to find a conformal map, $z = h(\zeta)$, which maps the effective domain $z \in D^w$ to the annulus $q \leq |\zeta| \leq 1$. Consider the Möbius transformation,

$$(4.1) \quad z = h(\zeta) = \rho(w_1)e^{i\chi} \frac{\zeta_\infty \zeta - 1}{\zeta - \zeta_\infty}.$$

This map is a composition of three conformal maps: $Z := ze^{-i\chi}/\rho(w_1)$ rotates and expands the domain so that the primary cylinder is of unit size and the secondary cylinder lies on the positive real axis; $\eta := 1/Z$ reflects the exterior of the unit cylinder into the interior; and $\eta := (\zeta - \zeta_\infty)/(\zeta_\infty \zeta - 1)$ is an automorphism of the unit disc, which centers the eccentric circles. Without loss of generality, we place ζ_∞ on the positive real axis. The resulting transformation, (4.1), maps $z = \infty$ onto $\zeta = \zeta_\infty$ and ∂D_1^w onto $|\zeta| = 1$, while ζ_∞ and q are chosen such that ∂D_2^w is mapped onto $|\zeta| = q$.

Using the results of section 3, the complex director angle, $\Omega = \tau - i\theta$, is given by the expression (3.12) with $\zeta = h^{-1}(z) = (\zeta_\infty z - \rho(w_1)e^{i\chi})/(z - \zeta_\infty \rho(w_1)e^{i\chi})$ and $C_\infty = -(1 - \zeta_\infty^2)\rho(w_1)e^{i\chi}$. The three periods in (3.12), i.e., Γ_1 , Γ_2 , and Υ , still remain unknown. Determining these requires the computation and minimization of the free energy (2.9). Before we address this, however, it is useful to analyze the singularities of $\Omega(z)$ (i.e., the topological defects) within the effective domain, D^w .

4.1. Topological defects. The director field corresponding to (3.12) does not contain defects in the fluid domain since $\Omega(z)$ is analytic by construction. There are, however, two -1 defects on the boundary of each effective cylinder, that is,

$$(4.2a) \quad \Omega(z) \sim \log \left[z - \rho(w_1)e^{ia_1^\pm} \right] \quad \text{as } z \rightarrow \rho(w_1)e^{ia_1^\pm},$$

$$(4.2b) \quad \text{and } \Omega(z) \sim \log \left[z - de^{i\chi} - b\rho(bw_2)e^{ia_2^\pm} \right] \quad \text{as } z \rightarrow de^{i\chi} + b\rho(bw_2)e^{ia_2^\pm},$$

for some real constants a_1^\pm and a_2^\pm . These defects tend to points on the body surfaces in the limits as $\rho(w_1 \rightarrow \infty) \rightarrow 1$ and $\rho(bw_2 \rightarrow \infty) \rightarrow 1$. Furthermore, we will show that their positions (i.e., the arguments a_1^\pm and a_2^\pm) are of the utmost importance for understanding body interactions within the liquid crystal, and we shall refer to these as “effective-boundary defects.” (Note that there are in fact a countably infinite number of “defects” within the two cylinders, corresponding to the singularities of the analytical continuation of $\Omega(z)$ within $|z| \leq \rho(w_1)$ and $|z - de^{i\chi}| \leq b\rho(bw_2)$. These singularities are the images of the above four effective-boundary defects and $\zeta = \zeta_\infty$ across $|\zeta| = 1$ and $|\zeta| = q$ —a consequence of the method of images.)

For a single immersed body, in [12], we showed that the positions of two effective-boundary defects are set by a single unknown period Γ , which is determined by minimizing the free energy of the liquid crystal. Analogously, here the positions of the four effective-boundary defects are set by minimizing the energy for the three unknown periods, Γ_1 , Γ_2 , and Υ . By inserting the Schwarz functions of the two circles, $\bar{z} = S_1(z) = 1/z$ for ∂D_1 and $\bar{z} = S_2(z) := de^{-i\chi} + b^2/(z - de^{i\chi})$ for ∂D_2 , into the expression (2.9) with $e^{i\phi_k} = z_s = 1/\sqrt{S'_k(z)}$, the net free energy appears as the real part of a sum of two closed contour integrals (corresponding to the two bodies). These integrals are computed numerically in order to determine the energy-minimizing values $\Gamma_1 = \Gamma_1^{\min}$, $\Gamma_2 = \Gamma_2^{\min}$, and $\Upsilon = \Upsilon^{\min}$, which we pursue using the Nelder–Mead simplex search method (`fminsearch` in MATLAB) [40]. Figure 2 shows integral curves of the energy-minimizing director field in blue for the given physical configuration, while the effective-boundary defects are shown as red dots. A loss of symmetry due to the two bodies is apparent.

When the two effective cylinders have equal radii (i.e., $\rho(w_1) = b\rho(bw_2)$, for example when $b = 1$ and $w_1 = w_2$), the domain is symmetric across the line $y \sin \chi = -x \cos \chi$ and one finds that $\zeta_\infty = \sqrt{q}$. After minimizing the energy, the positions of the effective-boundary defects are also found to be symmetrically located with $\Gamma_1^{\min} = -\Gamma_2^{\min}$ and $\Upsilon^{\min} = 0$. The complex director angle, (3.12), then takes the simplified form

$$(4.3) \quad \Omega(z) = \log \left(\sum_{k=-\infty}^{\infty} \left[\frac{q^{2k}}{(\zeta - q^{2k}\zeta_\infty)^2} - \frac{q^{2k}e^{-2i\chi}}{(\zeta_\infty\zeta - q^{2k})^2} \right] + \frac{ie^{-i\chi}G^{\min}}{\zeta} \right) + 2 \log(\zeta - \zeta_\infty),$$

where $\zeta(z) = h^{-1}(z) = (\zeta_\infty z - \rho(w_1)e^{i\chi})/(z - \zeta_\infty\rho(w_1)e^{i\chi})$ and $\Gamma_1^{\min} = -\Gamma_2^{\min} = 2\pi(1 - \zeta_\infty^2)\rho(w_1)G^{\min}$.

4.2. Body forces and symmetry breaking configurations. Changing the body positions results in a change in the total elastic energy stored in the fluid—external forces are, thus, required to keep the bodies fixed in place. The force acting on a body is found by integrating $\Omega'(z)^2/(2i)$ around a closed contour containing it, i.e., (2.11a). Integrating around a closed contour containing both bodies yields the total force acting on the system, but since $\Omega(z) \sim 0$ as $|z| \rightarrow \infty$, this integral must vanish [12]. It follows that the force acting on one of the cylinders is equal and opposite to the force acting on the other cylinder, i.e., $(\hat{F}_2^x, \hat{F}_2^y) = -(\hat{F}_1^x, \hat{F}_1^y)$. We compute this force using adaptive quadrature in MATLAB.

The force acting on ∂D_2 is plotted in the phase portrait in Figure 3(a) for $b = 1$, $w_1 = w_2 = 10$, and the numerically determined energy-minimizing periods $\Upsilon^{\min} = 0$ and $\Gamma_1^{\min} = -\Gamma_2^{\min}$. Furthermore, three examples of the forces on the cylinders at different body configurations, as well as the quasi-static liquid crystal director field, are presented in Figure 3(1)–(3). If the line of centers between the particles is

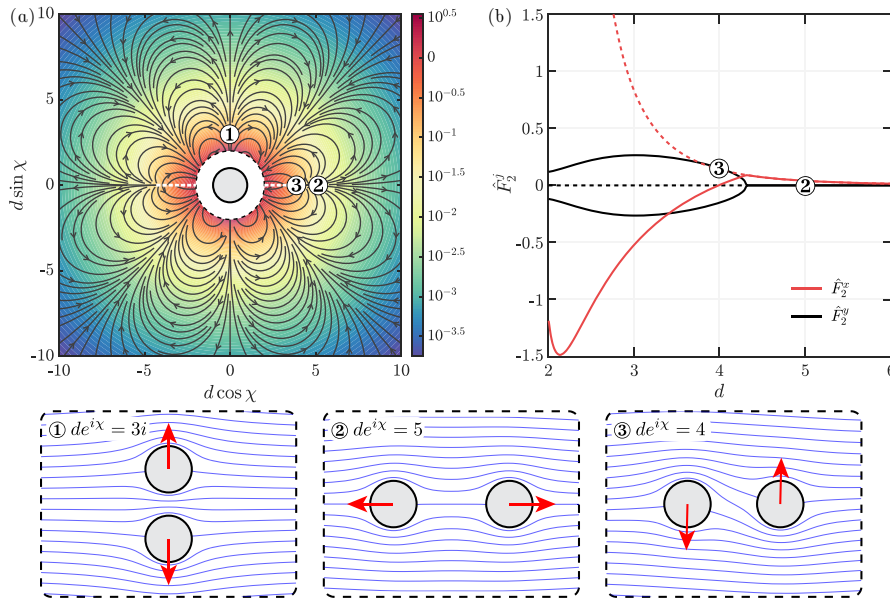


FIG. 3. Example 1. (a) A contour plot showing the dimensionless net force, $(\hat{F}_x^2, \hat{F}_y^2)$, acting on a cylinder placed at $|z - de^{i\chi}| = b$ due to a unit cylinder at $z = 0$ for $w_1 = w_2 = 10$, $b = 1$, and energy-minimizing periods $\Gamma_1^{\min} = -\Gamma_2^{\min}$ and $\Upsilon^{\min} = 0$. Arrows denote the direction of the force and color denotes the magnitude, $|\hat{F}_x^2 - i\hat{F}_y^2|$. The cylinder cannot be placed inside $|z| = 1 + b = 2$ (dashed curve) due to the unit cylinder at $z = 0$ (solid curve). (b) The dimensionless net force acting on inline cylinders ($\chi = 0$) is shown as solid lines for the energy-minimizing periods and dashed lines for vanishing periods ($\Gamma_1 = \Gamma_2 = \Upsilon = 0$). These solutions diverge as d decreases, resulting in a supercritical pitchfork bifurcation. The multiple energy-minimizing states are delineated by the white dashed lines in (a). In ①–③, the integral curves of the director field are shown as blue curves for $de^{i\chi} = 3i$, 5 , and 4 , while the direction of the force acting on the two cylinders is shown by the red arrows.

either parallel or antiparallel to the alignment axis, the bodies experience a repulsion from one another, provided they are sufficiently separated (Figure 3①–②). But this interaction is unstable to symmetry breaking perturbations. For instance, if the angle between the body centers is small, but nonzero, body forces would seek to increase this angle. Additionally, when the cylinders are inline ($\chi = 0$ or π), the energy-minimizing configuration undergoes a supercritical pitchfork bifurcation as the separation distance, d , decreases; this is delineated by the white dashed line in Figure 3(a) and is shown explicitly in Figure 3(b). This bifurcation is a result of the effective-boundary defects transitioning from being up-down symmetric (Figure 3②) to being off-axis (Figure 3③). Setting $\Upsilon = \Gamma_1 = \Gamma_2 = 0$, instead of minimizing the energy, fixes the locations of the defects to always be up-down symmetric, resulting in the cylinders being repulsive at all separation distances—see the dashed line in Figure 3(b). This symmetry breaking, and associated snapping from repulsion to attraction, has been observed for spheres with tangential anchoring as well [80, 88].

If the bodies were free to move (and the relaxation time of the liquid crystal was sufficiently small so that a quasi-static approximation could be made), the cylinders would eventually attract each other along a path diagonal to the preferred orientation of the liquid crystal. Thus, while spheres with strong tangential anchoring have been found to align experimentally at a 30° offset from the alignment axis [65, 63, 80], cylinders are predicted to align at a 45° offset. More generally, the offset angle of the

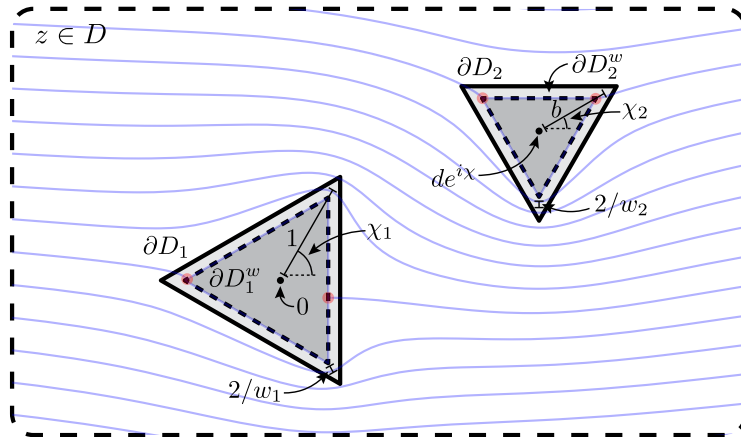


FIG. 4. Example 2. Two-dimensional liquid crystal outside two triangles with corners at the roots of $z^3 = e^{3i\chi_1}$ and $(z - de^{i\chi})^3 = b^3 e^{3i\chi_2}$ (∂D_1 and ∂D_2 , black solid lines). The effective domain boundaries are similar triangles with corners at the roots of $z^3 = (1 - 2/w)^3 e^{3i\chi_1}$ and $(z - de^{i\chi})^3 = (b - 2/w)^3 e^{3i\chi_2}$ (∂D_1^w and ∂D_2^w , black dotted lines). Integral curves of the director field are shown in blue for $w = 10$, $d = 2.5$, $b = 0.75$, $\chi_1 = \pi/3$, $\chi_2 = \pi/6$, $\chi = \pi/6$, and energy-minimizing periods: $\Gamma_1^{\min} = 0.00$, $\Gamma_2^{\min} = 2.41$, and $\Upsilon^{\min} = -0.01$.

stable configuration is dependent on the anchoring strengths and the ratio of cylinder radii, b . For example, when $w_1 = w_2 = 10$ and $b = 1$ (as pictured in Figure 3) the stable configurations are at a much smaller offset angle of approximately 8° . This is substantially smaller than the chaining angle observed in two dimensions for two sharp bodies [19], suggesting that corners, which promote defect repositioning, can have an outsized effect. Fixing the locations of the defects, instead of minimizing the energy for their locations, results in a comparable offset angle.

At large separation distances, the bodies only weakly interact and the force appears to resemble an asymptotic pole of the form

$$(4.4) \quad \hat{F}_1^x - i\hat{F}_1^y = -\hat{F}_2^x + i\hat{F}_2^y \sim -C / (de^{i\chi})^5 \quad \text{as } d \rightarrow \infty$$

for some $C > 0$, which is consistent with previous work on far-field quadrupolar interactions in a nematic liquid crystal [42]. We will return to this topic for arbitrary particle shapes and sizes in section 6, but first we explore a second example.

5. Example 2: Two triangles with tangential anchoring. In this section, we consider two equilateral triangles, one with corners at the roots of $z^3 = e^{3i\chi_1}$ and the other with corners at the roots of $(z - de^{i\chi})^3 = b^3 e^{3i\chi_2}$. We denote these triangles as ∂D_1 and ∂D_2 , respectively. As before, the liquid crystal is assumed to be oriented with the x -axis in the far-field and subject to finite tangential anchoring on each triangle. Both triangles are also assumed to have vanishing topological charge, i.e., $\oint_{\partial D_k} d\theta = 0$. This configuration is plotted in Figure 4.

Here we again make use of the effective boundaries, internal to the physical boundaries, upon which the anchoring is strong (i.e., perfect tangential anchoring). Since angles are preserved under the effective boundary technique, the triangle ∂D_1 is mapped to a similar triangle with corners at the roots of $z^3 = (1 - 2/w_1)^3 e^{3i\chi_1}$ [12], which we denote as ∂D_1^w . Likewise, the triangle ∂D_2 is mapped to a similar triangle with corners at the roots of $(z - de^{i\chi})^3 = (b - 2/w_2)^3 e^{3i\chi_2}$, which we denote as ∂D_2^w . These effective triangles are shown in Figure 4 as dotted lines.

Next, we seek a conformal map, $z = h(\zeta)$, that maps the effective domain $z \in D^w$ onto the annulus $q \leq |\zeta| \leq 1$. This is achieved by using an extension to the Schwarz–Christoffel mapping for multiply connected polygonal domains [16]. The mapping takes the form

$$(5.1) \quad z = h(\zeta) = A + B \int^\zeta \frac{\prod_{k=1}^3 [P_0(s/a_1^k)P_0(s/a_2^k)]^{2/3}}{[sP_0(s/\zeta_\infty)P_0(s\bar{\zeta}_\infty)]^2} ds,$$

where A and B are complex constants and

$$(5.2) \quad P_0(\zeta) := (1 - \zeta) \prod_{k=1}^{\infty} (1 - q^{2k}\zeta)(1 - q^{2k}/\zeta),$$

i.e., (3.11a) with $\alpha = 0$. Here, $\zeta = \zeta_\infty$ is the image of $z = \infty$ and $\zeta = a_1^k$ and $\zeta = a_2^k$ are the images of the corners of ∂D_1^w and ∂D_2^w on $|\zeta| = 1$ and $|\zeta| = q$, respectively. Without loss of generality, we shall place $\zeta = \zeta_\infty$ on the positive real axis by setting the rotational degree of freedom of the annulus. The remaining 12 accessory parameters (A , B , q , $\arg a_j^k$, and $|\zeta_\infty|$) are determined by ensuring the six vertices are mapped correctly.

Determining the accessory parameters of a Schwarz–Christoffel mapping is itself a challenging problem [22], and we will turn to numerical techniques. The MATLAB toolbox `plgcirmap` [58] computes the conformal mapping from a given multiply connected polygonal domain onto a circular domain. We use this package to compute the mapping from the effective domain, $z \in D^w$, onto an auxiliary domain outside two circular cylinders, $|\eta| = r_1$ and $|\eta - De^{iX}| = r_2$, while preserving orientation at infinity, i.e., $z \sim \eta + \mathcal{O}(1/\eta)$ as $|\eta| \rightarrow \infty$. Here, r_1 , r_2 , D , and X are numerically determined real numbers. We then apply a Möbius transformation that maps this auxiliary domain onto the annulus $q \leq |\zeta| \leq 1$, that is,

$$(5.3) \quad \eta(\zeta) = r_1 e^{iX} \frac{\zeta_\infty \zeta - 1}{\zeta - \zeta_\infty},$$

where ζ_∞ and q are chosen such that $|\eta - De^{iX}| = r_1$ is mapped onto $|\zeta| = q$ —this final mapping is analogous to (4.1). The composition of these two conformal maps is equivalent to computing the Schwarz–Christoffel mapping (5.1), by the uniqueness of conformal mappings [1, 30].

5.1. Topological defects. Equipped with the above conformal map, the complex director angle, $\Omega(z) = \tau(x, y) - i\theta(x, y)$, can be expressed as (3.12) with $C_\infty = -r_1 e^{iX} (1 - \zeta_\infty^2)$. By construction, $\Omega(z)$ is analytic outside ∂D_1^w and ∂D_2^w , thus there are no topological defects in the fluid domain. However, defects appear at the corners of the effective triangles (a consequence of the Schwarz–Christoffel mapping), as well as two additional -1 defects on each effective boundary. These “effective-boundary defects” are akin to those found for the two immersed cylinders, (4.2), and their positions are controlled by the three periods, Γ_1 , Γ_2 , and Υ . As before, these periods are determined by numerical minimization of the free energy, (2.9). This is achieved by parameterizing each side of the triangle and evaluating the integral and conformal map numerically for given b , d , χ , χ_1 , χ_2 , w_1 , w_2 , Γ_1 , Γ_2 , and Υ . The resulting energy is then minimized to determine $\Gamma_1 = \Gamma_1^{\min}$, $\Gamma_2 = \Gamma_2^{\min}$, and $\Upsilon = \Upsilon^{\min}$.

An example of a director field found in the manner above is shown as blue curves in Figure 4, while the effective-boundary defects are shown as red dots. Note that

three defects are located at the corners of the effective boundaries, while the fourth sits along one straight edge. Corners are natural locations for topological defects to reside in order to reduce the total elastic energy in the liquid crystal, just as the Kutta condition selects the circulation (by the placement of a surface stagnation point at a sharp edge) in potential flow theory [12]. In general, the final effective-boundary defect is not found at a corner due to the constraint of horizontal alignment in the far-field. Thus, one triangle is allowed to have both defects at corners (with three possible pairs) and the other triangle is only allowed one defect at a corner (again, with three possible choices). Each of the $3 \times 3 \times 2 = 18$ possible combinations of corners for the effective-boundary defects to be located correspond to local energy-minimizing configurations. Determining precisely which three corners yield the global energy minimum requires direct comparison. The liquid crystal configuration around a single regular polygon has also been studied using a reduced Landau–de Gennes framework [35, 34].

In the case shown in Figure 4, the three corners that provide a global energy minimum as $w_1, w_2 \rightarrow \infty$ are the three closest to the horizontal axes passing through the triangle centers (i.e., the preferred alignment of the liquid crystal). This heuristic has at least appeared valid at large separation distances, d , and holds asymptotically as $d \rightarrow \infty$ (see section 6.6 and section SM2 in the supplemental material). Thus, the defect locations are predominantly dependent on the orientations of each triangle, i.e., χ_1 and χ_2 . Denoting the six corners of the two effective triangles as $\hat{c}_k = (1 - 2/w_1)e^{i\chi_1}e^{2i\pi k/3}$ and $\hat{C}_k = de^{i\chi_2} + (b - 2/w_2)e^{i\chi_2}e^{2i\pi k/3}$ for $k \in \{-1, 0, 1\}$, we find that the three select corners are delineated by the parameter-space plot in Figure 5.

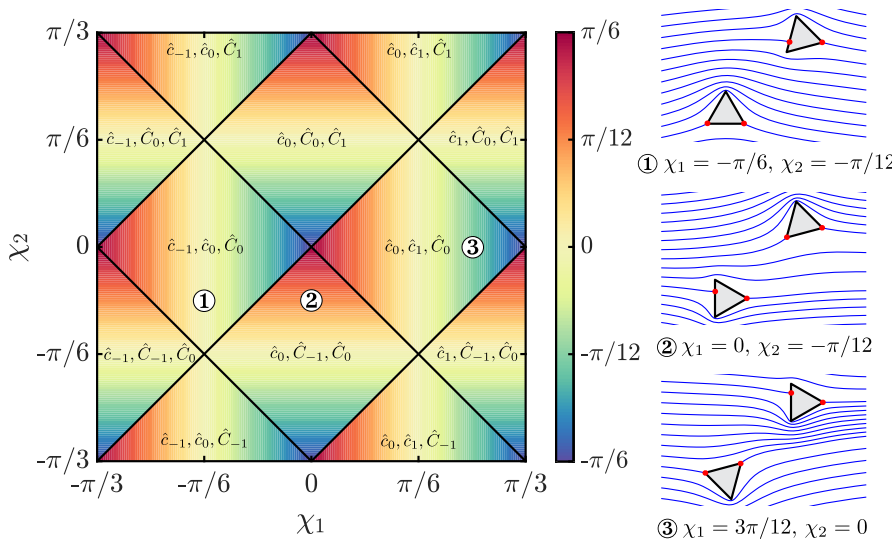


FIG. 5. Example 2. Plot of the triangles’ orientation (χ_1, χ_2) -space showing the three triangle corners at which an effective-boundary defect is located in the strong anchoring limit, $w_1, w_2 \rightarrow \infty$. Here, the black solid lines partition the parameter space, while $\hat{c}_k = (1 - 2/w_1)e^{i\chi_1}e^{2i\pi k/3}$ and $\hat{C}_k = de^{i\chi_2} + (b - 2/w_2)e^{i\chi_2}e^{2i\pi k/3}$ with $k \in \{-1, 0, 1\}$ denote the six corners of the effective triangles. Color is proportional to the asymptotic torques $\hat{T}_1 \sim -\hat{T}_2$ as $d \rightarrow \infty$, as defined in (6.27). These torques drive the triangles to individually rotate until they are either pointed upward ($\gamma_k = -\pi/6$) or downward ($\gamma_k = \pi/6$). In ①–③, the integral curves of the director field outside triangles oriented at the labeled angles are shown as blue curves, while the effective-boundary defects are shown as red dots, for $w = 100$, $d = 5$, $b = 1$, $\chi = \pi/4$, and energy-minimizing periods: $\Gamma_1 = \Gamma_1^{\min}$, $\Gamma_2 = \Gamma_2^{\min}$, and $\Upsilon = \Upsilon^{\min}$.

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For example, if $\chi_1 = \pi/3$ and $\chi_2 = \pi/6$, then the three effective-boundary defects are located at \hat{c}_1 , \hat{C}_0 , and \hat{C}_1 , as shown in Figure 4 by the red dots. (Note that 6 out of the 18 possible corner combinations are never global energy minima.) For finite anchoring strengths, the sharp transitions between the 12 regions in Figure 5 are smoothed out, with the defects lying close to, but not exactly at, the corresponding corners [12].

5.2. Body forces and torques. As in our first example, the dimensionless net force and torque exerted on each of the triangles can be computed by evaluating the contour integrals in (2.11). Since $\Omega'(z)^2$ is analytic outside the effective triangles, the integration contours can be freely deformed within the liquid crystal bulk. As before, it follows that the force acting on each triangle is equal and opposite, i.e., $(\hat{F}_2^x, \hat{F}_2^y) = -(\hat{F}_1^x, \hat{F}_1^y)$. We compute the forces and torques integrals using adaptive quadrature in MATLAB.

The torques acting on each body desire to rotate the triangles until one of their sides is aligned with the preferred axis of the liquid crystal (i.e., the horizontal, $\chi_k = \pm\pi/6$). Furthermore, the equal and opposite forces acting on each body want the triangles to rotate around a midpoint until they are horizontally ($\chi = 0$ or π) or vertically ($\chi = \pm\pi/2$) aligned. These observations suggest that if the bodies were free to move and the relaxation time of the liquid crystal was sufficiently small, then the two triangles would ultimately be aligned vertically or horizontally and pointed upward or downward. Once in this configuration, the interaction force is dependent on the triangles' orientations, as shown in Figure 6. In particular, we observe that in line triangles ($\chi = 0$ or π) *repel* each other when pointed in the same direction (e.g., $\chi_0 = \chi_1 = -\pi/6$, Figure 6(a)), but *attract* when oriented in opposing directions (e.g., $\chi_0 = -\chi_1 = -\pi/6$, Figure 6(b)). Inversely, parallel triangles ($\chi = \pm\pi/2$) *attract* each other when both pointed in the same direction (e.g., $\chi_0 = \chi_1 = -\pi/6$, Figure 6(c)), but *repel* when oriented in opposing directions (e.g., $\chi_0 = -\chi_1 = \pi/6$, Figure 6(d)). These numerical results converge to the general far-field asymptotics derived in section 6 as $d \rightarrow \infty$, in particular (6.28), as shown in Figure 6(e). This orientation-dependent interaction between triangles has previously been observed experimentally by Lapointe, Mason, and Smalyukh [41].

More will be said about the interactions between two triangular bodies below, but it will first be of use to examine the interactions of two bodies that are well separated more generally.

6. Far-field interactions between two general bodies. An isolated body immersed in a liquid crystal only locally disturbs the director field with the director angle decaying according to $\theta \sim \mathcal{O}(a/|z|)$ as $|z| \rightarrow \infty$ (a dipole) in general, where a is a length scale associated with the body. If there is no period around the body, instead the director field decays more rapidly, as a quadrupole, $\theta \sim \mathcal{O}(a^2/|z|^2)$ [12]. It follows that two immersed bodies separated by a large distance will only weakly interact. In this section, we shall analyze this weak interaction for two arbitrary bodies immersed in a director field that is oriented with the x -axis in the far-field. The challenge comes from the periods in (2.7). Such periods are not possible for an isolated body since it results in a logarithmic growth in the director angle [12]. However, this can be avoided in the case of two immersed bodies by introducing opposing periods, as in (3.3), including the case in which the second "body" is an outer boundary or infinite wall. Importantly, it is thus possible that the solution for an isolated body is not recovered in the large separation distance limit.

We begin in section 6.1 by formally introducing the separation distance, d , by considering the conformal map, $z = h(\zeta)$, which was introduced in section 3. Then, in

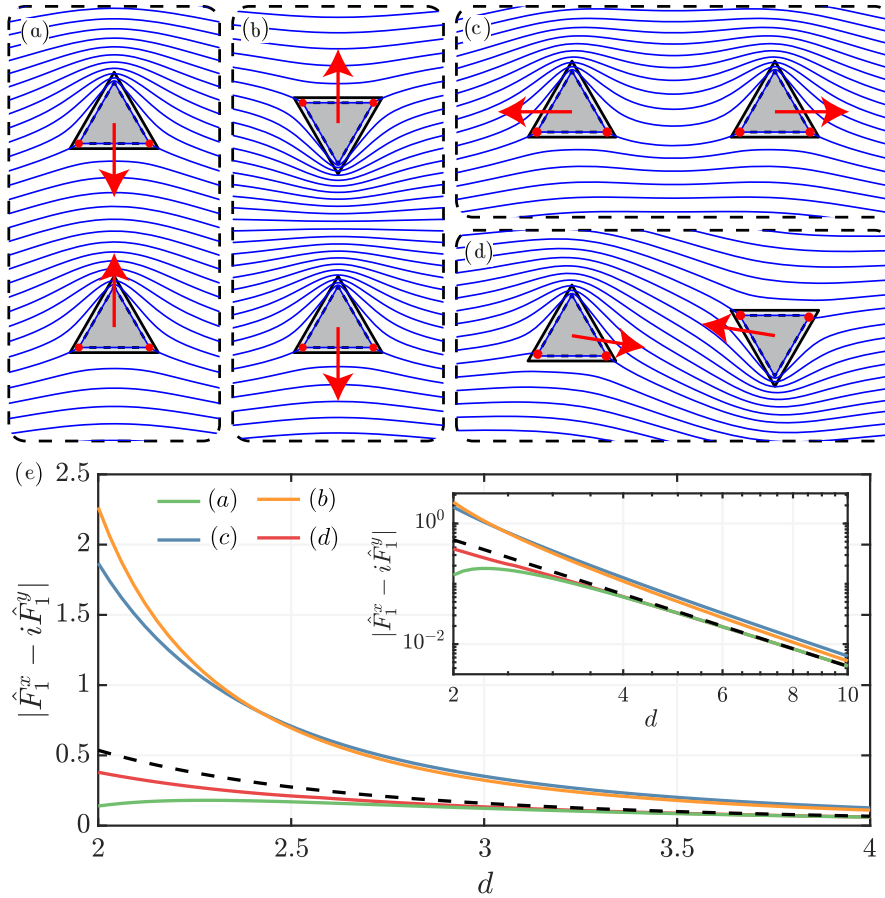


FIG. 6. Example 2. Integral curves of the director field outside two identical triangles held in parallel (a), (b) with $\chi = \pi/2$ and series (c), (d) with $\chi = 0$, for $d = 2$, $b = 1$, $w = 10$, and energy-minimizing periods: $\Gamma_1 = \Gamma_1^{\min}$, $\Gamma_2 = \Gamma_2^{\min}$, and $\Upsilon = \Upsilon^{\min}$. In (a), (c) the triangles are pointing in the same direction ($\chi_1 = \chi_2 = -\pi/6$), while in (b), (d) the triangles are pointing in opposing directions ($\chi_1 = -\chi_2 = \pi/6$). The red arrows point in the direction of the force acting on each triangle. The magnitude of the force, $|\hat{F}_1^x - i\hat{F}_1^y| = |\hat{F}_2^x - i\hat{F}_2^y|$, versus the separation distance, d , in configurations (a)–(d) is plotted as colored curves in (e). As d increases, all four curves converge to the asymptotic solution (6.28), which is shown as a black dashed line. The logarithmically scaled plot is shown in the inset.

section 6.2, we derive an expansion for the director field as $d \rightarrow \infty$ using the general expression (3.12). Finally, in sections 6.3 and 6.4, we compute the resulting force and torque acting on each body. These asymptotic results are applied to the two examples: two immersed cylinders in section 6.5 and two immersed triangles in section 6.6.

6.1. Asymptotic conformal map. In section 3, we introduced the conformal map $z = h(\zeta)$, which maps the doubly connected effective domain, $z \in \partial D_w$, to the annulus, $q \leq |\zeta| \leq 1$, with ∂D_1^w mapped to $|\zeta| = 1$, ∂D_2^w mapped to $|\zeta| = q$, and $z = \infty$ mapped to $\zeta = \zeta_\infty \in (q, 1)$. By construction, this map is analytic in the annulus except for a first-order pole at $\zeta = \zeta_\infty$. It can, thus, be expressed as the Laurent expansion

$$(6.1) \quad z = h(\zeta) = \frac{C_\infty}{\zeta - \zeta_\infty} + \sum_{j=0}^{\infty} h_j \zeta^j + \sum_{j=1}^{\infty} \frac{H_j}{\zeta^j}$$

for some complex coefficients h_j and H_j . (For example, for the two cylinders considered in section 4, $C_\infty = -\rho(w_1)e^{i\chi}(1 - \zeta_\infty^2)$, $h_0 = \rho(w_1)e^{i\chi}\zeta_\infty$, and $h_j = H_j = 0$ for $j \geq 1$.)

If we instead consider the two bodies as isolated, then, according to the Riemann mapping theorem, there exist two conformal maps, $z = a(\eta)$ and $z = A(\eta)$, from the exterior of ∂D_1 and the exterior of ∂D_2 to the interior of a unit circle $|\eta| < 1$, where η is a new complex variable. These maps may always be defined such that $z = \infty$ is mapped to $\eta = 0$ and can be written as the series expansions

$$(6.2) \quad z = a(\eta) = \frac{a_{-1}}{\eta} + \sum_{j=0}^{\infty} a_j \eta^j \quad \text{and} \quad z = A(\eta) = \frac{A_{-1}}{\eta} + \sum_{j=0}^{\infty} A_j \eta^j$$

for some complex coefficients a_j and A_j . In addition, since the disc has rotational symmetry, $a(\eta)$ and $A(\eta)$ both contain a rotational degree of freedom, we shall fix this by assuming $a_{-1} > 0$ and $A_{-1} > 0$. (For example, for the two cylinders consider in section 4, $a_{-1} = \rho(w_1)$, $A_{-1} = b\rho(bw_2)$, $a_0 = 0$, $A_0 = de^{i\chi}$, and $a_j = A_j = 0$ for $j \geq 1$.) Here, it is useful to introduce the two parameters

$$(6.3a,b) \quad d := |A_0 - a_0| \quad \text{and} \quad e^{i\chi} := \frac{A_0 - a_0}{|A_0 - a_0|},$$

which measure the bodies' separation distance and relative argument, respectively.

As the distance between the two bodies is increased (i.e., $d \rightarrow \infty$), we expect $q/\zeta_\infty \rightarrow 0$ and $\zeta_\infty \rightarrow 0$. Furthermore, the conformal maps from the doubly connected domain to the annulus, $h(\zeta)$, and from the doubly connected domain to the inverted annulus, $h(q/\zeta)$, should recover the maps $a(\eta)$ and $A(\eta)$ up to rotational degree of freedom. That is, $h(\zeta) \sim a(e^{ib}\zeta)$ and $h(q/\zeta) \sim A(e^{ic}\zeta)$ as $d \rightarrow \infty$ for some $b, c \in (-\pi, \pi]$ to be determined. Inserting the series expansions (6.1) and (6.2) yields the leading-order balance

$$(6.4a) \quad \sum_{j=0}^{\infty} h_j \zeta^j + \sum_{j=1}^{\infty} \frac{H_j + C_\infty \zeta_\infty^{j-1}}{\zeta^j} \sim \frac{a_{-1}}{e^{ib}\zeta} + \sum_{j=0}^{\infty} a_j (e^{ib}\zeta)^j,$$

$$(6.4b) \quad \sum_{j=0}^{\infty} \frac{(h_j - C_\infty/\zeta_\infty^{j+1})q^j}{\zeta^j} + \sum_{j=1}^{\infty} \frac{H_j \zeta^j}{q^j} \sim \frac{A_{-1}}{e^{ic}\zeta} + \sum_{j=0}^{\infty} A_j (e^{ic}\zeta)^j,$$

as $d \rightarrow \infty$. Above, we have expanded the pole at $\zeta = \zeta_\infty$ using the fact that $\zeta \ll \zeta_\infty$ and $\zeta \ll q/\zeta_\infty$.

Equating the series coefficients yields the asymptotic expressions for C_∞ , ζ_∞ , q , h_j , and H_j as $d \rightarrow \infty$. The unknown rotation angles, b and c , are then set by enforcing the requirements that $q > 0$ and $\zeta_\infty > 0$, which yields $b = \pi - \chi$ and $c = -\chi$. Together, we find that

$$(6.5a-c) \quad C_\infty \sim -a_{-1}e^{i\chi}, \quad \zeta_\infty \sim \frac{a_{-1}}{d}, \quad q \sim \frac{a_{-1}A_{-1}}{d^2},$$

$$(6.5d,e) \quad h_j \sim a_j (-e^{-i\chi})^j, \quad H_j \sim A_j (qe^{-i\chi})^j,$$

as $d \rightarrow \infty$. It follows that the conformal map has three dominant behaviors,

$$(6.6) \quad z = h(\zeta) \sim \begin{cases} a(\eta) & \text{for } \eta = -e^{-i\chi}\zeta, \\ A(\eta) & \text{for } \eta = qe^{-i\chi}/\zeta, \\ (A_0 - a_0)/(1 - \eta) + a_0 & \text{for } \eta = \zeta/\zeta_\infty, \end{cases}$$

as $d \rightarrow \infty$, corresponding to the three regions: (i) local to the primary body, $|\zeta| \sim 1$; (ii) local to the secondary body, $|\zeta| \sim q = \mathcal{O}(1/d^2)$; and (iii) away from both bodies, $|\zeta| \sim \zeta_\infty = \mathcal{O}(1/d)$.

6.2. Asymptotic director field. In section 3, the horizontal liquid crystal outside two immersed bodies was found to have a complex director angle (3.12). Using the fact that $q = \mathcal{O}(1/d^2)$ and $\zeta_\infty = \mathcal{O}(1/d)$ as $d \rightarrow \infty$, i.e., (6.5), the infinite sum in (3.11b) can be written as

$$(6.7) \quad K(\zeta) = -\frac{\zeta}{1-\zeta} - \sum_{j=1}^{\infty} \frac{q^{2j} e^{2i\alpha} \zeta^j}{1-q^{2j} e^{2i\alpha}} + \sum_{j=1}^{\infty} \frac{q^{2j} e^{-2i\alpha} / \zeta^j}{1-q^{2j} e^{-2i\alpha}},$$

provided $1/d^4 \ll |\zeta| \ll d^4$. Inserting this into (3.13) yields the potential expansion

$$(6.8) \quad F'(\zeta) = \frac{\overline{C_\infty} e^{i\beta}}{(\zeta_\infty \zeta - 1)^2} - \frac{C_\infty e^{-i\beta}}{(\zeta - \zeta_\infty)^2} + \frac{1}{2\pi i} \frac{\Gamma_1 + \Gamma_2 e^{i\alpha}}{\zeta - 1/\zeta_\infty} - \frac{1}{2\pi i} \frac{\Gamma_1 + \Gamma_2 e^{-i\alpha}}{\zeta - \zeta_\infty} + \frac{\Gamma_2 e^{-i\alpha}}{2\pi i \zeta} - \sum_{j=1}^{\infty} \frac{q^{2j}}{\zeta_\infty^{j+1}} \left(\overline{F_j} \zeta^{j-1} - \frac{F_j}{\zeta^{j+1}} \right),$$

provided $1/d^3 \ll |\zeta| \ll d^3$, with the $\mathcal{O}(1)$ -constants

$$(6.9) \quad F_j := \frac{1}{e^{2i\alpha} - q^{2j}} \left[j \overline{C_\infty} e^{i\beta} + \frac{\Gamma_1 + \Gamma_2 e^{i\alpha}}{2\pi i} \zeta_\infty \right] - \frac{\zeta_\infty^{2j}}{e^{2i\alpha} - q^{2j}} \left[j C_\infty e^{-i\beta} + \frac{\Gamma_1 + \Gamma_2 e^{-i\alpha}}{2\pi i} \zeta_\infty \right].$$

At this point, we can distinguish the three asymptotic regions described in (6.6). In each of these regions, the expression (6.8) can be simplified further and the corresponding director field, (3.12), can be computed as $d \rightarrow \infty$. Below we provide the resulting complex director field at leading-order:

(i) Local to the primary body, $\eta = -e^{-i\chi} \zeta = \mathcal{O}(1)$, we find that

$$(6.10) \quad \Omega(z) = \log \left[\frac{(1 - e^{i(\beta+\gamma_1)} \eta)(1 + e^{i(\beta-\gamma_1)} \eta)}{-\eta^2 a'(\eta)/a_{-1}} \right] - i\beta - \frac{\Upsilon}{2\pi i} \log \eta + \mathcal{O}(1/d),$$

with $\eta = a^{-1}(z) + \mathcal{O}(1/d)$ and $\beta = -\Upsilon \log(d/a_{-1})/(2\pi) + \mathcal{O}(1/d)$ as $d \rightarrow \infty$, for $\Gamma_1 = 4\pi a_{-1} \sin \gamma_1$.

(ii) Local to the secondary body, $\eta = qe^{-i\chi}/\zeta = \mathcal{O}(1)$, we find that

$$(6.11) \quad \Omega(z) = \log \left[\frac{(1 - e^{i(\delta+\gamma_2)} \eta)(1 + e^{i(\delta-\gamma_2)} \eta)}{-\eta^2 A'(\eta)/A_{-1}} \right] - i\delta + \frac{\Upsilon}{2\pi i} \log \eta + \mathcal{O}(1/d),$$

with $\eta = A^{-1}(z) + \mathcal{O}(1/d)$ and $\delta = \beta - \alpha = \Upsilon \log(d/A_{-1})/(2\pi) + \mathcal{O}(1/d)$ as $d \rightarrow \infty$, for $\Gamma_2 = 4\pi A_{-1} \sin \gamma_2$.

(iii) Away from both bodies, $\eta = \zeta/\zeta_\infty = \mathcal{O}(1)$, we find that

$$(6.12) \quad \Omega(z) = -\frac{\Upsilon}{2\pi i} \log \eta + \frac{\Gamma_1 e^{i\beta}}{2\pi i} \frac{1-\eta}{de^{i\chi}} + \frac{\Gamma_2 e^{i\delta}}{2\pi i} \frac{1/\eta-1}{de^{i\chi}} + \mathcal{O}(1/d^2),$$

with $\eta \sim (z - A_0)/(z - a_0) + \mathcal{O}(1/d^2)$ as $d \rightarrow \infty$.

Overall, as the separation distance increases ($d \rightarrow \infty$), the director fields local to the two bodies, (6.10) and (6.11), do not recover the director field outside an isolated body [12]. Instead, they are coupled by an equal and opposite period Υ , which induces a logarithmic director angle away from the two bodies, that is,

$$(6.13) \quad \theta(z) = -\text{Im} \Omega(z) \sim \log \left| \frac{z - a_0}{z - A_0} \right| + \mathcal{O}(1/d),$$

in the far-field, i.e., (6.12). The period Υ , as well as the two other periods Γ_1 and Γ_2 , also control the locations of the four effective-boundary defects (as introduced in section 4.1); these are located on the effective bodies at $z \sim a(\pm e^{-i(\beta \pm \gamma_1)})$ and $z \sim A(\pm e^{-i(\delta \pm \gamma_2)})$. Determining these three periods requires the minimization of the free energy of the liquid crystal. This energy is given by the boundary integral expression in (2.9), but takes the form

$$(6.14) \quad \hat{\mathcal{E}} = \hat{\mathcal{E}}_1(\Gamma_1, \Upsilon) + \hat{\mathcal{E}}_2(\Gamma_2, \Upsilon) + \mathcal{O}(1/d),$$

as $d \rightarrow \infty$, where $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_2$ are the boundary integral in (2.9) around the two bodies ∂D_1 and ∂D_2 with the director angles (6.10) and (6.11), respectively. These integrals can be evaluated and, hence, minimized for Υ , Γ_1 , and Γ_2 to $\mathcal{O}(1/d)$.

6.3. Asymptotic force. The force acting on a body submerged in a liquid crystal is given by (2.11a). Since $\Omega'(z)$ is analytic within the liquid crystal, it follows from Cauchy’s integral theorem that the force acting on the primary body is equal and opposite to the force acting on the secondary body, which can be expressed as

$$(6.15) \quad \hat{F}_1^x - i\hat{F}_1^y = -\hat{F}_2^x + i\hat{F}_2^y = -\frac{1}{2i} \oint_{|\zeta|=R} \left(\frac{d\Omega}{d\zeta} \right)^2 \frac{d\zeta}{z'(\zeta)}$$

for any $R \in (q, 1)$. If we take $R = \zeta_\infty$, then we can insert the asymptotic expansion (6.12) with $\eta = \zeta/\zeta_\infty$; this yields

$$(6.16) \quad \hat{F}_1^x - i\hat{F}_1^y \sim \frac{1}{8\pi^2 i d e^{i\chi}} \oint_{|\eta|=1} \left(\frac{\Upsilon}{\eta} + \frac{\Gamma_1 e^{i\beta}}{d e^{i\chi}} + \frac{\Gamma_2 e^{i\delta}}{d e^{i\chi}} \frac{1}{\eta^2} \right)^2 (1 - \eta)^2 d\eta,$$

as $d \rightarrow \infty$. The integrand in (6.16) is analytic in $|\eta| \leq 1$ except for a pole at $\eta = 0$. Cauchy’s residue theorem, thus, yields

$$(6.17) \quad \hat{F}_1^x - i\hat{F}_1^y = -\hat{F}_2^x + i\hat{F}_2^y \sim -\frac{\Upsilon^2}{2\pi d e^{i\chi}} + \frac{\Upsilon(\Gamma_1 e^{i\beta} + \Gamma_2 e^{i\delta})}{2\pi (d e^{i\chi})^2} - \frac{\Gamma_1 \Gamma_2 e^{i(\beta + \delta)}}{\pi (d e^{i\chi})^3},$$

as $d \rightarrow \infty$, for $\beta = -\Upsilon/(2\pi) \log(d/a_{-1})$ and $\delta = \Upsilon/(2\pi) \log(d/A_{-1})$.

6.4. Asymptotic torque. The torque acting on a body submerged in the liquid crystal is given by (2.11b). Since $\Omega'(z)$ is analytic within the liquid crystal, it follows from Cauchy’s integral theorem that the torque acting on the primary body is

$$(6.18) \quad \hat{T}_1 = \Upsilon - \frac{1}{2} \text{Re} \left[\oint_{|\zeta|=R} (z(\zeta) - a_0) \left(\frac{d\Omega}{d\zeta} \right)^2 \frac{d\zeta}{z'(\zeta)} \right]$$

for any $R \in (q, 1)$. If we again take $R = \zeta_\infty$, then we can insert the asymptotic expansion (6.12); this yields

$$(6.19) \quad \hat{T}_1 \sim \Upsilon + \frac{1}{8\pi^2} \text{Re} \left[\oint_{|\eta|=1} \left(\frac{\Upsilon}{\eta} + \frac{\Gamma_1 e^{i\beta}}{d e^{i\chi}} + \frac{\Gamma_2 e^{i\delta}}{d e^{i\chi}} \frac{1}{\eta^2} \right)^2 (1 - \eta) d\eta \right].$$

The integrand is again analytic in $|\eta| \leq 1$ except for a pole at $\eta = 0$. Cauchy's residue theorem, thus, yields

$$(6.20) \quad \hat{T}_1 \sim \Upsilon + \frac{\Upsilon\Gamma_1}{2\pi d} \sin(\chi - \beta) - \frac{\Gamma_1\Gamma_2}{2\pi d^2} \sin(2\chi - \beta - \delta),$$

as $d \rightarrow \infty$. The torque on the secondary body is similarly determined to be

$$(6.21) \quad \hat{T}_2 \sim -\Upsilon - \frac{\Upsilon\Gamma_2}{2\pi d} \sin(\chi - \delta) + \frac{\Gamma_1\Gamma_2}{2\pi d^2} \sin(2\chi - \beta - \delta),$$

as $d \rightarrow \infty$.

6.5. Far-field interactions between two cylinders. As a first example of far-field interactions, consider the two cylinders introduced in section 4. Here, the asymptotic director field local to $|z| = 1$ and $|z - de^{i\chi}| = b$ is given by (6.10) and (6.11) with the conformal maps

$$(6.22a,b) \quad z = a(\eta) = \rho(w_1)/\eta \quad \text{and} \quad z = A(\eta) = de^{i\chi} + b\rho(bw_2)/\eta,$$

respectively. Due to rotational symmetry of the cylinder, the energy of the liquid crystal local to each body—i.e., $\mathcal{E}_1(\Gamma_1, \Upsilon)$ and $\mathcal{E}_2(\Gamma_2, \Upsilon)$ in (6.14)—is minimized when the four effective-boundary defects, $z \sim \pm\rho(w_1)e^{i(\beta \pm \gamma_1)}$ and $z - de^{i\chi} \sim \pm b\rho(bw_2)e^{i(\delta \pm \gamma_2)}$, are aligned with the preferred axis of the liquid crystal [12]. It follows that the free energy is minimized when $\beta = \delta = \gamma_1 = \gamma_2 = 0$, that is the periods vanish as $d \rightarrow \infty$; thus, the leading-order director fields, (6.10) and (6.11), recover the solution for an isolated cylinder [12]. Furthermore, by considering an expansion of the free energy, we find that, at least, $\Upsilon = o(1/d)$, $\Gamma_1 = o(1/d)$, and $\Gamma_2 = o(1/d)$ as $d \rightarrow \infty$. Higher-order terms are, thus, needed in (6.17) to obtain a leading-order expression for the force acting on the cylinders.

After accounting for the higher-order terms in (6.8), we find that

$$(6.23) \quad \hat{F}_1^x - i\hat{F}_1^y = -\hat{F}_2^x + i\hat{F}_2^y \sim -\frac{\Upsilon^2}{2\pi de^{i\chi}} - \frac{2i\Upsilon(a_{-1}^2 - A_{-1}^2)}{(de^{i\chi})^3} - \frac{48\pi a_{-1}^2 A_{-1}^2}{(de^{i\chi})^5},$$

as $d \rightarrow \infty$, where $a_{-1} = \rho(w_1) \sim 1 - 1/w_1$ and $A_{-1} = b\rho(bw_2) \sim b - 1/w_2$ are the radii of the two effective cylinders. When the radii of the effective cylinders equate, i.e., $a_{-1} = A_{-1}$, the added symmetry implies that $\Upsilon = 0$ and $\Gamma_1 = -\Gamma_2$ for all separation distances, $d > 0$. In this case, the asymptotic force is given by the quadrupolar interaction (4.4) with $C = 48\pi a_{-1}^2 A_{-1}^2$, which is consistent with the results presented in Figure 3.

6.6. Far-field interactions between two triangles. As a second example, consider the two triangular prisms explored in section 5. Here, the asymptotic director fields local to the two triangles, ∂D_1 and ∂D_2 , are given by (6.10) and (6.11) with the conformal maps

$$(6.24a) \quad z = a(\eta) = \left(1 - \frac{2}{w_1}\right) \frac{h(e^{3ix_1}\eta^3)}{h(1)\eta},$$

$$(6.24b) \quad \text{and} \quad z = A(\eta) = de^{i\chi} + \left(b - \frac{2}{w_2}\right) \frac{h(e^{3ix_2}\eta^3)}{h(1)\eta},$$

respectively, for the hypergeometric function $h(\zeta) = {}_2h_1(-2/3, -1/3; 2/3; \zeta)$. The conformal maps $a(\eta)$ and $A(\eta)$ are Schwarz–Christoffel mappings, which map the exterior of the effective polygon onto the unit disc [1, 12]. The corners of the effective polygons, $z = \hat{c}_k = (1 - 2/w_1)e^{i\chi_1}e^{2i\pi k/3}$ and $z = \hat{C}_k = de^{i\chi} + (b - 2/w_2)e^{i\chi_2}e^{2i\pi k/3}$ for $k \in \{-1, 0, 1\}$, are mapped to points on the unit circle, $\eta = \hat{b}_k = e^{-i\chi_1}e^{-2i\pi k/3}$ and $\eta = \hat{B}_k = e^{-i\chi_2}e^{-2i\pi k/3}$, respectively.

For large anchoring strengths ($w_1, w_2 \rightarrow \infty$), the energy of the liquid crystal attains a local minimum when three of the four effective-boundary defects are located at corners of the effective triangles. Furthermore, the energy is globally minimized when the three corners are those closest to the horizontal axes passing through the triangle centers, as delineated in Figure 5. (This observation is shown to hold asymptotically in section SM2 in the supplemental material.) Since the locations of the four effective-boundary defects are known asymptotically, i.e., $z \sim a(\pm e^{-i(\beta \pm \gamma_1)})$ and $z \sim A(\pm e^{-i(\delta \pm \gamma_2)})$, one can apply a simple geometric argument to derive expressions for the three periods as $d \rightarrow \infty$. Below, we present the results of this argument assuming that $|\chi_1|, |\chi_2| \leq \pi/3$, without loss of generality.

If $||\chi_1| - \pi/6| < ||\chi_2| - \pi/6|$, then the two effective-boundary defects on ∂D_1 lie at corners, while the location of the third free defect depends on the orientation of ∂D_2 . It follows that

$$(6.25a,b) \quad \gamma_1 = \text{sgn}(\chi_1)\pi/6, \quad \beta = \chi_1 - \text{sgn}(\chi_1)\pi/6,$$

$$(6.25c) \quad \gamma_2 = \begin{cases} \chi_2 - \delta & \text{if } 0 < |\chi_2| < \pi/6, \\ \delta - \chi_2 + \text{sgn}(\chi_2)\pi/3 & \text{if } \pi/6 < |\chi_2|. \end{cases}$$

Alternatively, if $||\chi_1| - \pi/6| > ||\chi_2| - \pi/6|$, then the two effective-boundary defects on ∂D_2 lie at corners and we find that

$$(6.26a,b) \quad \gamma_2 = \text{sgn}(\chi_2)\pi/6, \quad \delta = \chi_2 - \text{sgn}(\chi_2)\pi/6,$$

$$(6.26c) \quad \gamma_1 = \begin{cases} \chi_1 - \beta & \text{if } 0 < |\chi_1| < \pi/6, \\ \beta - \chi_1 + \text{sgn}(\chi_1)\pi/3 & \text{if } \pi/6 < |\chi_1|. \end{cases}$$

With these variables, the three periods can be computed using $\Gamma_1 \sim 4\pi \sin \gamma_1/a_{-1}$, $\Gamma_2 \sim 4\pi b \sin \gamma_2/A_{-1}$, and $\Upsilon \sim -2\pi\beta/\log|d/a_{-1}| \sim 2\pi\delta/\log|d/A_{-1}|$, for $a_{-1} = (1 - 2/w_1)/h(1)$ and $A_{-1} = (b - 2/w_2)/h(1)$. The asymptotic forces and torques are then given by (6.17), (6.20), and (6.21).

As an example, consider the case of two identical triangles ($b = 1$) with large anchoring strengths ($w = w_1 = w_2 \gg 1$). Here, the torques acting on each triangle satisfy

$$(6.27) \quad \hat{T}_1 \sim -\hat{T}_2 \sim \frac{2\pi}{\log|d/a_{-1}|} \begin{cases} \text{sgn}(\chi_1)\pi/6 - \chi_1 & \text{if } ||\chi_1| - \pi/6| < ||\chi_2| - \pi/6|, \\ \chi_2 - \text{sgn}(\chi_2)\pi/6 & \text{if } ||\chi_1| - \pi/6| > ||\chi_2| - \pi/6|, \end{cases}$$

as $d \rightarrow \infty$, where $a_{-1} = (1 - 2/w)/h(1) \approx 0.73(1 - 2/w)$. (Note that this expression for $\log|d/a_{-1}|\hat{T}_1/(2\pi)$ is shown as the contours in the (χ_1, χ_2) -space in Figure 5.) These torques drive the triangles to individually rotate until they point either upward ($\chi_k = -\pi/6$) or downward ($\chi_k = \pi/6$), with $\Upsilon = 0$. Critically, the torques in (6.27) only decay proportional to $1/\log d$ as $d \rightarrow \infty$, thus one would expect the triangles to experience a rotation even when very well separated. Once oriented with $\chi_k = \pm\pi/6$, the triangles interact according to the force

$$(6.28) \quad \hat{F}_1^x - i\hat{F}_1^y = -\hat{F}_2^x + i\hat{F}_2^y = -\text{sgn}(\chi_1\chi_2) \frac{4\pi a_{-1}^2}{(de^{i\chi})^3} + \mathcal{O}(1/d^4),$$

as $d \rightarrow \infty$, where $a_{-1} = (1-2/w)/h(1) \approx 0.73(1-2/w)$. In particular, this force rotates the triangles around each other until they are in parallel ($\chi = \pm\pi/2$) or series ($\chi = 0$ or π), depending on if they are pointed in the same direction ($\chi_1 = \chi_2 = \pm\pi/6$) or in opposite directions ($\chi_1 = -\chi_2 = \pm\pi/6$), respectively. In either case, the triangles are then attracted to each other, as observed in Figure 6(a), (d). It should be noted that the magnitude of the force in (6.28) is $|\hat{F}_1^x - i\hat{F}_1^y| \sim 4\pi a_{-1}^2/d^3$, which is independent of the triangle orientation. This result is compared to the full numerical solutions in Figure 6(e), suggesting that one obtains a good approximation provided the bodies are separated by (approximately) two body widths. Shape-dependent interactions have indeed been observed experimentally [41, 6].

7. Conclusions. Even though the director angle in a nematic liquid crystal is a harmonic function in the single Frank constant approximation, finding solutions is not a simple task. Nonlinear, Robin boundary conditions provide one challenge, but a far greater difficulty lies in the selection of topological defect locations, either on body surfaces in the strong anchoring limit, or on effective boundaries outside of the fluid domain for weak (finite) anchoring strengths. While this was somewhat straightforward for a single immersed particle [12], multiple bodies demand a more technical analysis. Fortunately, conformal mapping techniques for multiply connected domains could be used effectively as part of this program [17].

Looking ahead, the equilibrium configuration provides a first step in the direction of modeling the anisotropic viscous drag on moving bodies [69, 84, 45, 31] and the dynamics of bodies immersed in active suspensions [43, 68, 98], for applications like microrheology [32, 15] and self-assembly [99] (see also [54, 78]). Fluid anisotropy also impacts individual bacterial trajectories [57, 105, 91, 39, 24, 82], as well as the interactions among nearby bacteria [81]. The locations of topological defects are of particular interest in an effort to template molecular self-assembly [96, 48], and their tendency to reside near sharp boundary features is intriguing [6]. The solutions presented herein may offer a degree of insight on these current scientific pursuits.

Another question of interest, but one which requires different tools to explore, pertains to the relevance of distinct bend and splay moduli. Fortunately, these moduli are comparable in common liquid crystals like PAA, 5CB, and DSCG at room temperature [18, 8, 104], and we suspect that the changes from the present results will be limited. Twist moduli can be substantially smaller, however, and out-of-plane relaxation of stress is another generic possibility (see, for example, [100]) that should be addressed.

Other open questions are of a more analytical variety. The energy change with body rotations has also been recently considered and found to be no worse than Lipschitz continuous in the orientation of bodies in three dimensions [2]. With defects jumping from the corners on one body to another under rotation, as we have observed with two triangular bodies in section 5, we conjecture that no further smoothness in the energy will be possible to show generally.

Although our examples were restricted to the examination of two bodies, there is nothing in the analysis presented here that does not immediately extend to a greater number of bodies. We are eager to see these techniques used to describe many-body elastic interactions, though it may be that the simpler far-field interactions will prove more useful as a starting point for suspension configurations.

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