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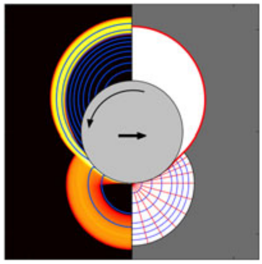
Dropping slender-body theory into the mud



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The equations describing classical viscous fluid flow are notoriously challenging to solve, even approximately, when the flow is host to one or many immersed bodies. When an immersed body is slender, the smallness of its aspect ratio can sometimes be used as a basis for a 'slender-body theory' describing its interaction with the surrounding environment. If the fluid is complex, however, such theories are generally invalid and efforts to understand the dynamics of immersed bodies are almost entirely numerical in nature. In a valiant effort, Hewitt & Balmforth (*J. Fluid Mech.*, vol. 856, 2018, pp. 870–897) have unearthed a theory to describe the motion of slender bodies in a viscoplastic fluid, 'fluids' such as mud or toothpaste which can be coaxed to flow, but only with a sufficiently large amount of forcing. Mathematical theories for some tremendously complicated physical systems may now be within reach.

Key words: complex fluids, propulsion, slender-body theory

1. Introduction

Among the great achievements of applied mathematics in the twentieth century was the development of asymptotic methods, means by which solutions to ordinary and partial differential equations can be rigorously approximated by devious exploitation of small parameters. One technique, the method of matched asymptotics, allows for a rigorous stitching together of two solutions to a partial differential equation: an 'inner solution' for regions where variables (perhaps a flow field) change over a very small length scale, and an 'outer solution' where changes occur over a much longer scale.

Some of the most useful theoretical advances to come from this period in fluid mechanics were 'slender-body theories' to describe the mechanics of a very thin fibre in a viscous flow. First used by Gray & Hancock (1955) to study the swimming of sea-urchin spermatozoa, significant improvements to the theory were derived in the 1970s and have since been used with great success to understand transport by flagellar and ciliary activity (Lauga & Powers 2009), elastic fibres in flows (Lindner & Shelley 2015), and other complex systems.

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In the earliest theories, the fluid flow is solved at each cross-section of the body assuming that it may be treated locally as a long, straight cylinder. The result is a drag law which relates the local filament speed to the local viscous force per unit length, and which distinguishes motions parallel and perpendicular to the long axis. Important improvements to the local theory were carried out by Keller & Rubinow (1976) and Johnson (1980) by the inclusion of non-local self-interactions, the effects of local curvature, and end effects. Interest in the classical theory continues to this day (Koens & Lauga 2018; Mori, Ohm & Spirn 2018).

While these theories have been exceedingly useful to understand such problems as microorganism locomotion, many worms and microbes spend their lives squirming through mud, mucus, and sediment, and the consequences of fluid complexity are often hard to guess. In viscoelastic fluids, for instance, swimming speeds can be reduced by viscoelasticity at small helical amplitudes (Fu, Wolgemuth & Powers 2009), but enhanced at large helical amplitudes (Liu, Powers & Breuer 2011; Spagnolie, Liu & Powers 2013). For reviews of swimming in complex fluids, see Elfring & Lauga (2015) and Sznitman & Arratia (2015). But what happens if the fluid is not remotely Newtonian? Is it even sensible to hope for an analytical theory describing the dynamics of a microorganism, or even a simple cylinder, moving through sludge?

2. Overview

Hewitt & Balmforth (2018) have pushed resistive force theory into new territory, replacing a simple Newtonian fluid with a Bingham model fluid describing viscoplastic flow with no inertia. A viscoplastic (or yield-stress) material is one that only deforms beyond a sufficiently large amount of forcing, a large class which can include toothpaste, mud, mucus, and flowing lava. For detailed reviews, see Balmforth, Frigaard & Ovarlez (2014) and Coussot (2014). The force per unit area needed to budge the material is the yield stress. Dragging a body through such an environment results in a particularly complicated flow.

In the Bingham model, the stress tensor $\boldsymbol{\tau}$ is given by

$$\boldsymbol{\tau} = \mu \left(1 + \frac{\mathrm{Bi}}{\dot{\gamma}}\right) \dot{\boldsymbol{\gamma}} \quad \text{for } \tau > \mathrm{Bi}, \tag{2.1}$$

and $\dot{\boldsymbol{\gamma}} = \mathbf{0}$ if $\tau \leqslant \mathrm{Bi}$. Here, $\dot{\boldsymbol{\gamma}}$ is the symmetric rate-of-strain tensor, $\dot{\gamma} = \sqrt{(1/2)\dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}}}$, $\tau = \sqrt{(1/2)\boldsymbol{\tau} : \boldsymbol{\tau}}$, and $\mathrm{Bi} = \tau_Y R/(\mu U)$ is the Bingham number, with R the filament radius, U the translational velocity, μ the fluid viscosity, and τ_Y the critical yield stress below which the material remains undeformed.

As in the original classical theories, the approach of the paper is to study the flow near each two-dimensional cross-section under the assumption that the body is a long, straight cylinder, though this is much more challenging in the Bingham fluid since flows due to translations and rotations are nonlinear and coupled. Even the flow around a translating, non-rotating cylinder is quite involved: plugged regions (rigid, undeformed zones) appear in boundary layers at the front and back surfaces of the cylinder and even in the bulk flow where the fluid stress drops below the yield stress (Tokpavi, Magnin & Jay 2008).

In a significant departure from Newtonian fluid flows, there is an intriguing coupling between the effects of rotation and translation: cylinder rotation results in a circular shearing flow which can fluidize the local environment. A fast-rotating cylinder pulled axially through a viscoplastic fluid can thereby enjoy reduced drag; a drilling motion should allow for easier penetration or extraction of a cylinder from mud even if the drill-bit is a perfect cylinder. But lateral motion is also made easier by local rotation. At large yield-stress, Hewitt & Balmforth (2018) show that the size of the viscoplastic boundary layer around the body rotating with speed Ω scales as $O(\mathrm{Bi}^{-1/2}\Omega^{1/2})$ for large rotation rates, which leads to a translational force $F \sim O(\Omega^{-3/2}\mathrm{Bi}^{3/2})$ if the cylinder moves with unit speed, showing the reduction in the force for a fast-rotating cylinder and providing a useful and intuitive physical picture.

The authors also consider some applications of the theory, focusing on the sedimentation of straight rods and V-shaped particles, and then helices, under gravitational forcing. One finding is that for large yield stress the cylinder slides nearly along its long axis for most inclination angles, in stark contrast to cylinders falling in a Newtonian fluid which translate along both axes. When investigating the helical body, the authors find trajectories which are helical themselves.

The problem of swimming through viscoplastic material is also revisited, as the authors apply their framework to locomotion by the passage of helical waves. The authors previously investigated the swimming of the classic Taylor swimming sheet over and inside a viscoplastic medium (Pegler & Balmforth 2013; Hewitt & Balmforth 2017). In the helical case, the swimming speed is found to be monotonically increasing in the yield stress for any pitch angle, tending as $\mathrm{Bi} \to \infty$ towards motion like a corkscrew boring through a solid, similar to that found for helical swimming in a heterogeneous environment (Leshansky 2009). The authors also find an optimal pitch angle for swimming, which is slightly larger than that for swimming in a Newtonian fluid. As in other complex fluids, these consequences on swimming are not simple to guess, and the authors have provided a more tractable means by which to compute and in some limits analyse them.

3. Future

Our understanding of motion through viscoplastic fluids is now poised to see a progression of theoretical advances. Looking back through the Newtonian rear-view mirror, slender-body theories were first used to study microorganism locomotion with prescribed flagellar or ciliary kinematics, then deformable bodies, active internal forcing, and the dynamics of active suspensions. Perhaps similar steps can now be taken to understand more realistic modes of locomotion in yield-stress fluids.

The full range of validity of the theory is not yet clear, and difficulties are sure to arise in some settings; for instance, the shape of the tip of a needle can have a dramatic effect on body motion in second-order and viscoelastic fluids (Leal 1975; Li, Thomases & Guy 2018), and such localized complications might also arise in a Bingham model fluid. The full effects of body curvature and non-local self-interactions may also be required for further advances of the viscoplastic theory. The authors include an appendix on leading-order curvature corrections, and claim (plausibly) that the screening that occurs in a Bingham fluid will sufficiently prohibit the effects of self-interaction. Full numerical simulations will nevertheless be useful for determining the situations in which the theory is essentially complete. Fortunately, the authors have already begun to compare to experimental results and the results appear promising.

It might be surprising that local drag laws continue to emerge for motion in generically nonlinear environments, but recent successes include a resistive force theory for granular media (Hosoi & Goldman 2015), along with some suggestive results in viscoelastic fluids (Martinez *et al.* 2014; Dabade, Marath & Subramanian 2015). These and the recent paper by Hewitt & Balmforth (2018) are expected to spur the development of similar theories in other complex fluids.

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