

# Complex Fluids in Biological Systems

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Saverio E. Spagnolie  
Editor

# Complex Fluids in Biological Systems

Experiment, Theory, and Computation



Springer

*Editor*

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*To life, in all its wondrous  
and stupefying complexity*



# Preface

The complexity of biological systems, even on the smallest length scales, is staggering. Biological systems are replete with active functionality, heterogeneity, memory, and interconnectedness on a vast spectrum of length and time scales. With our ever-advancing abilities to observe nature *in vivo* at the microscale, and with continuing developments of mathematical and numerical machinery for understanding multiscale physical systems, the fields of complex fluids and biological systems are ripe for fruitful cross-pollination. There have already been many successful scientific advances along these lines, as will be made clear in the chapters that follow. The aim of this book is to introduce the reader to many of the exciting directions that this research is taking and to provide a valuable reference on fundamental phenomena, models, and analysis of complex fluids in a variety of biological systems.

The book is organized into four parts. In Part I, Newtonian and complex fluids are introduced, along with the terminology and models that will appear frequently throughout the book. The first chapter provides the mathematical framework of continuum mechanics and presents common constitutive laws used to describe fluids with such properties as shear-dependent viscosity and viscoelasticity. Classical rheological flows frequently used in experiments are introduced. The second chapter lays the foundations for the topics to be covered in the book and explores critical functional roles played by complex fluids in a familiar biological system, the human body. Using mucus as an illustrative example, a multidisciplinary approach to studying and modeling soft, complex biological matter is emphasized.

In Part II, the measurement of biological material properties, or *rheology*, takes center stage. The first chapter is devoted to microrheology, wherein the behavior of small immersed particles is used to infer material properties of the surrounding environment. Both passive microrheology and active microrheology are discussed, beginning with the famed Stokes-Einstein relation and marching through a history of the field towards a “nonequilibrium equation of state.” The following two chapters return to specific biological structures, namely the cell membrane and cell cytoskeleton. Microrheology is revisited as a means of studying the viscoelastic properties of molecularly thin shells, and the intricate biopolymer network internal

to individual cells is introduced. A final chapter in this section explores a variety of challenges faced by experimentalists in the study of complex biological fluids and shows how a misinterpretation of data can suggest complex fluid properties when there are none and vice versa.

Part III focuses on the locomotion of microorganisms through complex biological fluids, as described from experimental, analytical, and numerical perspectives. The first chapter reviews the recent experimental studies of biolocomotion in viscous and viscoelastic fluids and then turns to intriguing experimental results on the propulsion of a model organism, the roundworm *C. elegans*. This sets the stage for the following chapter, which covers a detailed mathematical theory of locomotion in complex fluids, and connections between microrheology and biolocomotion are described. The interaction of swimming organisms in complex fluids is also discussed, which leads naturally into the final chapter of the part. In the last chapter, the focus turns to a model of large collections of such swimming organisms, or an *active suspension*, which can exhibit large-scale correlated motions, pattern formation, and complex fluid properties including normal stress differences. The model is extended to the study of other systems, including the interaction of microtubules and translocating motor proteins as found in individual cells.

Finally, Part IV covers methods for computing fluid flows with intricate immersed boundaries. Common numerical approaches are made considerably more challenging when the fluid is highly elastic. The first chapter describes many of these challenges, including the catastrophic high-Weissenberg number problem, and offers solutions. The immersed boundary method is introduced, and the locomotion of *C. elegans* in viscoelastic fluids is revisited as a test problem from a numerical perspective. The final chapter of the book presents a cell-level numerical study of blood flow, where the shapes and dynamics of individual cells and their interactions are captured in a boundary integral formulation of the problem. The numerical method is used to understand physical effects well known to physiologists such as the Fåhræus effect, Fåhræus-Lindqvist effect, and the margination of leukocytes and platelets.

The chapters contained herein will provide the reader with an overview as well as a detailed inspection of the challenges and opportunities that await us in the coming decades of research in complex biological flows, and the observations, methods, and tools available for their study. Active areas of exploration are presented by many of the world's foremost experts in their respective fields. Consequently, each chapter both provides a substantial review of the literature and delivers the very cutting edge of our current knowledge. The book was developed with advanced undergraduate and early graduate students in the engineering, biological, and mathematical sciences in mind, but it will appeal to anyone interested in the intricate and beautiful nature of complex fluids in the context of living systems.

Numerous acknowledgements are in order. It has been a great pleasure to work with the many authors of this book, who continue to forge new paths in their respective fields and to inspire with their creativity and remarkably hard work. It is immensely gratifying to toil as a member of an extended scientific family that knows no geographical borders. I am particularly indebted to Harvey Segur, Michael

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Madison, WI, USA  
July, 2014

Saverio E. Spagnolie



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# List of Symbols

## Fluid Properties

$\rho$	Density
$\mu$	Viscosity of a Newtonian fluid

## Fluid Dynamics

$\mathbf{u}$	Fluid velocity
$p$	Pressure
$\sigma$	Total stress tensor ( $= -p\mathbf{I} + \boldsymbol{\tau}$ )
$\boldsymbol{\tau}$	Deviatoric stress tensor
$D/Dt$	Substantial/material time derivative ( $= \partial/\partial_t + \mathbf{u} \cdot \nabla$ )
$\overset{\nabla}{\boldsymbol{\tau}}$	Upper convected time derivative of $\boldsymbol{\tau}$ ( $\equiv (D/Dt)\boldsymbol{\tau} - [(\nabla \mathbf{u})^T \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \nabla \mathbf{u}]$ )

## Kinematic Tensors

$\nabla \mathbf{u}$	Velocity gradient tensor [ $\nabla \mathbf{u}]_{ij} = \partial u_j / \partial x_i$ ]
$\dot{\gamma}$	Rate-of-strain tensor ( $= \nabla \mathbf{u} + (\nabla \mathbf{u})^T$ )
$\omega$	Vorticity tensor ( $= \nabla \mathbf{u} - (\nabla \mathbf{u})^T$ )

## Quantities in Constitutive Equations

$\eta_0$	Zero-shear-rate viscosity
$\eta_\infty$	Infinite-shear-rate viscosity
$\lambda$	Relaxation time
$G(t - t')$	Relaxation modulus
$M(t - t')$	Memory function

## Material Functions

$\eta( \dot{\gamma} )$	Non-Newtonian viscosity ( $= \eta \left( \sqrt{\text{tr}(\dot{\gamma}^2)} / 2 \right)$ )
$N_1$	First normal stress difference
$N_2$	Second normal stress difference
$\Psi_1$	First normal stress difference coefficient

$\Psi_2$	Second normal stress difference coefficient
$\eta^*(\omega)$	Complex viscosity ( $\eta^* = \eta' - i\eta''$ )
$G^*(\omega)$	Complex modulus ( $G^* = G' + iG''$ )

### Dimensionless Numbers

De	Deborah number
Re	Reynolds number
Wi	Weissenberg number

### General

$k_B$	Boltzmann's constant
T	Absolute temperature
$N_A$	Avogadro's number

### Mathematical Symbols

$\Re(z)$	Real part of complex number $z$
$\Im(z)$	Imaginary part of complex number $z$