

I. (20 points.) Each question has only one correct answer. The correct answer is highlighted.

1. Let W consist of all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $b = 8$. Which of the following is correct?
 - (a) W is a subspace of M_{22} with dimension 3.
 - (b) W is closed under addition, but not closed under scalar multiplication.
 - (c) W does not contain the zero vector and so is not a subspace of M_{22} .
2. Let U be the set of polynomials of the form $at^2 + c$, where a and c are real numbers. Which of the following is correct?
 - (a) U is a subspace of P_3 with dimension 2.
 - (b) U is a subspace of P_2 with dimension 3.
 - (c) U is not a subspace of P_3 or P_2 .
3. Let Y be the span of $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$. Which of the following is correct?
 - (a) Y has exactly two elements.
 - (b) Y is a subspace of $\mathbb{R}^2$ with dimension 1.
 - (c) $Y = \mathbb{R}^2$.
4. Let v_1, v_2 be vectors in a vector space V and a, b be scalar constants. Suppose that $av_1 + bv_2 = 0$. Which of the following is correct?
 - (a) If $a \neq 0$ then v_1 and v_2 are linearly dependent.
 - (b) If $a = b = 0$ then v_1 and v_2 are linearly independent.
 - (c) If $a = -b$ then $v_1 = -v_2$.

5. The system: $\begin{cases} 3x_1 - 2x_2 = 0 \\ x_1 + 5x_2 = 0 \end{cases}$ has a unique solution. Which statement is correct?
- (a) The matrix of the system has rank 0.
 - (b) The vectors $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right\}$ are linearly independent.
 - (c) The solution is nonzero.
6. Consider the vector spaces P_9 (polynomials of degree less than or equal to 9) and M_{32} (all 3×2 matrices). Which of the following is correct?
- (a) P_9 and M_{32} are isomorphic.
 - (b) P_9 is isomorphic to a subspace of M_{32} .
 - (c) M_{32} is isomorphic to a subspace of P_9 .
7. Let $S = \{3t^2 - 1, 2t + 2, 5t - 1, t^2, t, 1\}$. Which of the following is correct?
- (a) S is a basis for P_2 .
 - (b) S spans P_2 .
 - (c) S is linearly independent.
8. Suppose that A is a 3×3 matrix with rank 2. Which of the following is correct?
- (a) The system $Ax = 0$ has a unique solution.
 - (b) The solution space of the system $Ax = 0$ has dimension 2.
 - (c) The solution space of the system $Ax = 0$ has dimension 1.
9. If A is an invertible 2×2 matrix. What can we say about the matrix transformation $L(x) = Ax$?
- (a) L is onto.
 - (b) L is not linear.
 - (c) The kernel of L is empty.

10. Let L be a linear transformation $L : \mathbf{R}_2 \rightarrow \mathbf{R}_2$. Suppose that L is represented by the matrix A with respect to the standard basis S and $B = P^{-1}AP$, where $P = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$. Which is correct?

- (a) L is represented by B with respect to $T = \{[1, 2], [-1, 0]\}$.
- (b) L is represented by B with respect to $T = \{[1, -1], [2, 0]\}$.
- (c) L is represented by B with respect to the standard basis.

II. (15 points.) Let $A = \begin{bmatrix} 3 & -2 & -1 \\ 2 & 1 & -1 \\ -1 & -2 & 1 \end{bmatrix}$.

1. Compute the determinant of A ?

$$\begin{aligned} \det(A) &= 3 \cdot 1 \cdot 1 + (-2) \cdot (-1) \cdot (-1) + (-1) \cdot 2 \cdot (-2) - (-1) \cdot 1 \cdot (-1) - 3 \cdot (-1) \cdot (-2) - (-2) \cdot 2 \cdot 1 = \\ &= 3 - 2 + 4 - 1 - 6 + 4 = 2 \end{aligned}$$

2. Check your answer: compute the determinant of A in a different way using the cofactor expansion of A along the first or third column.

$$\begin{aligned} A_{11} &= 1 \cdot 1 - (-2) \cdot (-1) = -1 \quad A_{21} = -((-2) \cdot 1 - (-2) \cdot (-1)) = 4 \quad A_{31} = (-2) \cdot (-1) - 1 \cdot (-1) = 3 \\ \det(A) &= 3 \cdot A_{11} + 2 \cdot A_{21} + (-1) \cdot A_{31} = -3 + 8 - 3 = 2 \end{aligned}$$

3. Solve the linear system $Ax = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$ using either Cramer's Rule or the formula for the inverse of A that uses determinants.

$$x_1 = \det \left(\begin{bmatrix} 0 & -2 & -1 \\ 0 & 1 & -1 \\ -3 & -2 & 1 \end{bmatrix} \right) / \det(A) = ((-3)A_{31})/2 = \frac{-9}{2}.$$

$$x_2 = \det \begin{pmatrix} 3 & 0 & -1 \\ 2 & 0 & -1 \\ -1 & -3 & 1 \end{pmatrix} / \det(A) = ((-3)A_{32})/2 = \frac{-3}{2}.$$

$$x_3 = \det \begin{pmatrix} 3 & -2 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & -3 \end{pmatrix} / \det(A) = ((-3)A_{33})/2 = \frac{-21}{2}.$$

III. (15 points.) Let $L : P_2 \rightarrow P_2$ be the linear transformation defined by

$$L(at^2 + bt + c) = (3b - a)t^2 + (2c - b)t.$$

1. Find the matrix A that represents L with respect to the ordered basis $\{t^2, t, 1\}$.

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

2. What are the eigenvalues of L ?

$$\det(I\lambda - A) = \det \begin{pmatrix} \lambda + 1 & -3 & 0 \\ 0 & \lambda + 1 & -2 \\ 0 & 0 & \lambda \end{pmatrix} = (\lambda + 1)^2 \lambda$$

$$\lambda_{1,2} = -1, \lambda_3 = 0$$

3. For each eigenvalue, compute the associated eigenvectors of L . (Remember! They must be polynomials in P_2 !)

For $\lambda_{1,2} = -1$, we look at the homogeneous system with matrix $\begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. Every

solution of this system is of the form $\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$. When $r = 1$ this corresponds to the eigenvector

with coordinates $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, which is t^2 .

For $\lambda_{1,2} = 0$, we look at the homogeneous system with matrix $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$. Every solution

of this system is of the form $\begin{bmatrix} 6r \\ 2r \\ r \end{bmatrix}$. When $r = 1$ this corresponds to the eigenvector with

coordinates $\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$, which is $6t^2 + 2t + 1$.

4. Is L diagonalizable?

No, because it does not have a basis consisting of eigenvectors.

IV. (20 points.) Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

1. Find the eigenvalues of the matrix A .

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{bmatrix} = (\lambda - 3)^2 - 1 = (\lambda - 3 - 1)(\lambda - 3 + 1) = (\lambda - 4)(\lambda - 2).$$

$$\lambda_1 = 4, \lambda_2 = 2$$

2. Find the eigenvectors associated to each eigenvalue. For $\lambda_1 = 4$, we look at the homogeneous system with matrix $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Every solution of this system is of the form $\begin{bmatrix} r \\ r \end{bmatrix}$.

When $r = 1$ this corresponds to the eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda_2 = 2$, we look at the homogeneous system with matrix $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$. Every solution of this system is of the form $\begin{bmatrix} r \\ -r \end{bmatrix}$. When $r = 1$ this corresponds to the eigenvector $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

3. Find matrices P and D such that D is diagonal and $A = P^{-1}DP$.

$P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, the matrix with columns the eigenvectors v_1 and v_2 of A .

$D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$, the diagonal matrix with diagonal elements the eigenvalues of A λ_1 and λ_2 .

Note that the order of the eigenvectors must correspond to the order of the eigenvalues.

$P = (P^{-1})^{-1}$. To find it we use the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. So $P = -\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

4. Find matrices Q and E such that E is diagonal and $A^4 = Q^{-1}EQ$.

$$A^4 = (P^{-1}DP)^4 = P^{-1}DPP^{-1}DPP^{-1}DPP^{-1}DP = P^{-1}D^4P$$

$$\text{So } Q = P \text{ and } E = D^4 = \begin{bmatrix} 4^4 & 0 \\ 0 & 2^4 \end{bmatrix} = \begin{bmatrix} 256 & 0 \\ 0 & 16 \end{bmatrix}.$$

V. (15 points.) An inner product for P_1 is defined. The matrix of this inner product with respect to the basis $T = \{t-1, t+1\}$ is $C = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

1. Find the inner product of $5t-1$ and $3t+1$.

First we need to find the coordinates $\begin{bmatrix} a \\ b \end{bmatrix}$ of $5t-1$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ of $3t+1$ with respect to the basis T :

$$5t-1 = a(t-1) + b(t+1) \Rightarrow \begin{cases} a+b=5 \\ -a+b=-1 \end{cases} \Rightarrow \begin{cases} a=3 \\ b=2 \end{cases}.$$

$$3t+1 = c(t-1) + d(t+1) \Rightarrow \begin{cases} c+d=3 \\ -c+d=1 \end{cases} \Rightarrow \begin{cases} c=1 \\ d=2 \end{cases}$$

Now we have that the inner product $(5t-1, 3t+1) = \begin{bmatrix} a \\ b \end{bmatrix}^T C \begin{bmatrix} c \\ d \end{bmatrix} = [3 \ 2] \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [4 \ -1] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2$.

2. What are the lengths of $5t - 1$ and $3t + 1$ when computed using the inner product defined above.

The length of $5t - 1$ is the square root of $(5t - 1, 5t - 1) = \begin{bmatrix} a \\ b \end{bmatrix}^T C \begin{bmatrix} a \\ b \end{bmatrix} = [3 \ 2] \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = [4 \ -1] \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 10$. So the length of $5t - 1$ is $\|5t - 1\| = \sqrt{10}$.

The length of $3t + 1$ is the square root of $(3t + 1, 3t + 1) = \begin{bmatrix} c \\ d \end{bmatrix}^T C \begin{bmatrix} c \\ d \end{bmatrix} = [1 \ 2] \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [0 \ 1] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2$. So the length of $3t + 1$ is $\|3t + 1\| = \sqrt{2}$.

3. What is the angle between the polynomials $5t - 1$ and $3t + 1$ when computed using the inner product defined above. . The cosine of the angle between $5t - 1$ and $3t + 1$ is given by

$$\frac{(5t - 1, 3t + 1)}{\|5t - 1\| \|3t + 1\|} = \frac{2}{\sqrt{10}\sqrt{2}} = \frac{1}{\sqrt{5}}.$$

So the angle is $\cos^{-1}(\frac{1}{\sqrt{5}})$.

VI. (15 points.) Find an orthonormal basis for the solution space of the homogeneous system

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x = 0.$$

First we need to find a basis for the solution space of the homogeneous system. Every solution has the form $\begin{bmatrix} r - s \\ s \\ r \end{bmatrix}$. This means that a basis is given by $\{u_1, u_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$. We use the Gram Schmidt process. We set $v_1 = u_1$. Then

$$v_2 = u_2 - \frac{(v_1, u_2)}{(v_1, v_1)} v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{1 \cdot (-1) + 0 \cdot 1 + 1 \cdot 0}{1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Finally, we get an orthonormal basis $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|} \right\} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$.