

FINAL EXAM
MATH 770 FALL 2019
DUE DEC 17TH

Instructions. Write out clear rigorous arguments for each problem. If you prefer handwritten solutions, make sure that your writing is legible. Work alone! Scan your solutions and email them to *msoskova@math.wisc.edu* no later than 11:59 PM on December 17th. No extensions will be granted for the final, so plan accordingly.

Problem 1. Let T be a theory in a computable language L . Let Σ_T be the set of all L -sentences which are true in some finite model of T .

- (1) Show that if T is finitely axiomatizable then Σ_T is computably enumerable.
- (2) Show that if T is decidable, then Σ_T is computably enumerable. (Note, the language of T might be infinite and T might not be finitely axiomatizable).
- (3) Give an example of a computably enumerable theory T in the language of pure equality such that the set Σ_T defined as above is not computably enumerable.

Problem 2. Prove that for every infinite regular cardinal κ there is a cardinal λ such that $\lambda = \omega_\lambda$ and λ has cofinality κ .

Problem 3. Suppose that T is a consistent theory in a countable language with fewer than 2^ω nonisomorphic countable models. Show that there is a sentence θ such that the set of consequences of $T \cup \{\theta\}$ is complete and consistent.

Problem 4. Fix a linearly ordered set M . Suppose that in M there are no increasing or decreasing sequences of size ω_1 and that further there are no subsets of M that are isomorphic to the rationals. Show that M is at most countable.

Hint: Consider the equivalence relation on M defined by $a \sim b$ if and only if the interval $[\min(a, b), \max(a, b)]$ is countable. Reason about the equivalence classes of this relation.

Problem 5. A theory T is κ -categorical if all models of T of size κ are isomorphic.

- (1) Prove that every κ -categorical theory T (in a countable language) which has no finite models is complete.
- (2) Let $L = \{E\}$, where E is a binary relation symbol. An L -structure can be thought of as a directed graph. Consider a set of axioms Σ that say:
 - (a) The universe is infinite.
 - (b) There are no (directed) paths of length two: $\neg \exists x \exists y \exists z [E(x, y) \wedge E(y, z)]$.
 - (c) Every node is on some path of length one: $\forall x \exists y [E(x, y) \vee E(y, x)]$.
 - (d) Every node has out-degree 0 or 1: $\neg \exists x \exists y \exists z [E(x, y) \wedge E(x, z) \wedge y \neq z]$.
 - (e) Every node has in-degree 0 or 2: $\forall x [\neg \exists y E(y, x) \vee \exists y \exists z [E(y, x) \wedge E(z, x) \wedge y \neq z \wedge \forall u [E(u, x) \rightarrow u = y \vee u = z]]]$.

Prove that the set of consequences of Σ is a complete theory.

Problem 6. Prove in set theory without the axiom of choice (ZF) that the following two principles are equivalent.

- (1) Every set can be linearly ordered.
- (2) For any set X the partial order $(\mathcal{P}(X), \subseteq)$ (the power set of X ordered by inclusion) contains a maximal linearly ordered (under inclusion) subset.

Problem 7. Prove that every nonstandard model of true arithmetic (the theory of $\mathfrak{N} = (\mathbb{N}, 0, s, +, *, <)$, where s is the successor function and $+$, $*$ and $<$ are the standard addition, multiplication and less than) has a substructure which is not a model of true arithmetic.