

I. (30 points.) True or False? Explain why!

1. A first order language always has at least one constant and equality.

Answer: False

There are languages without constants or $=$.

2. Every first order language L comes with a unique structure $\mathcal{A}$.

Answer: False There are a class of structures for a language L .

3. If $\mathcal{A} \models \varphi[s]$ and x is a variable that does not occur free in φ then $\mathcal{A} \models \exists x \varphi[s]$.

Answer: True

Let a be an el of $|\mathcal{A}|$ (which is always a nonempty set). By the coincidence lemma $\mathcal{A} \models \varphi[s]$ iff $\mathcal{A} \models \varphi[s(x/a)]$. Hence $\mathcal{A} \models \exists x \varphi[s]$.

4. If $\Gamma \vdash \varphi \vee \psi$ and $\Gamma \not\vdash \psi$ then $\Gamma \vdash \varphi$.

Answer: False. $\Gamma \vdash \psi$ does not necessarily imply that $\Gamma \vdash \neg \psi$.

5. There is no sentence φ , whose models are structures with exactly 2 or exactly 5 elements.

Answer: False

$\varphi = \exists x_1 \exists x_2 (x_1 \neq x_2 \wedge \forall y (y = x_1 \vee y = x_2)) \vee$
 $\exists x_1 \exists x_2 \dots \exists x_5 (\bigwedge_{1 \leq i < j \leq 5} x_i \neq x_j \wedge \forall y (\bigvee_{1 \leq i \leq 5} y = x_i))$

6. If D_1 and D_2 are definable sets in a structure $\mathcal{A}$ then $D_1 \cap D_2$ is also definable in $\mathcal{A}$.

Answer: True
 If $D_1 = \{a \in |\mathcal{A}| : \mathcal{A} \models \varphi_1(a)\}$ and $D_2 = \{a \in |\mathcal{A}| : \mathcal{A} \models \varphi_2(a)\}$ then
 $D_1 \cap D_2 = \{a \in |\mathcal{A}| : \mathcal{A} \models \varphi_1 \wedge \varphi_2(a)\}$.

7. If $D_1 \cap D_2$ is a definable set in a structure $\mathcal{A}$ then D_1 is also definable in $\mathcal{A}$.

Answer: False Consider $\mathcal{A} = (\mathbb{Z}, <)$.
 $D_1 = \{z + z : z \in \mathbb{Z}\}$ is not definable (take the automorphism $h(z) = z + 1$ then $2 \in D_1$ but $h(2) = 3 \notin D_1$).
 $D_2 = \emptyset$ Then $D_1 \cap D_2 = \emptyset$ is definable: $\{x : x < x\}$.

8. If D is definable in a structure $\mathcal{A}$ and $\mathcal{B}$ is isomorphic to $\mathcal{A}$ via an isomorphism $h : |\mathcal{A}| \rightarrow |\mathcal{B}|$ then $h(D) = \{h(d) \mid d \in D\}$ is definable in $\mathcal{B}$.

Answer: True If h is an isomorphism then
 $\mathcal{A} \models \varphi(a)$ iff $\mathcal{B} \models \varphi(h(a))$
 Hence if D is defined by φ then $h(D)$ is defined by φ in $\mathcal{B}$.

9. If $\mathcal{A}$ is a substructure of $\mathcal{B}$ then $\mathcal{A} \models \forall x P(x)$ if and only if $\mathcal{B} \models \forall x P(x)$.

Answer: False $\mathcal{B} = (\mathbb{N}, P^{\mathcal{B}})$ where $P^{\mathcal{B}} = 2/\mathbb{N}$
 Let $\mathcal{A} \subseteq \mathcal{B}$ be $(2\mathbb{N}, P^{\mathcal{B}} \cap 2\mathbb{N})$
 $\mathcal{A} \models \forall x P(x)$ but $\mathcal{B} \not\models \forall x P(x)$.

10. If $\Gamma \vdash \alpha$ and $\alpha \vdash \forall x \gamma$ then $\Gamma \cup \{\neg \gamma\}$ is inconsistent. True.

Answer: $\Gamma, \alpha \vdash \forall x \gamma$ hence $\Gamma \vdash \alpha \rightarrow \forall x \gamma$ by Deduction
 $\Gamma \vdash \alpha$ hence by MP $\Gamma \vdash \forall x \gamma$
 By Axiom 2 + MP $\Gamma \vdash \gamma$ hence $\Gamma \cup \{\neg \gamma\}$
 is inconsistent.

II. (20 points.) Prove that

1. $\exists x \neg P(x) \rightarrow \forall x Q(x) \models \forall x (\neg Q(x) \rightarrow P(x))$.

Answer: Let $\mathcal{A}, \mathcal{B}$ be such that

$$\mathcal{A} \models \exists x \neg P(x) \rightarrow \forall x Q(x) \quad [\mathcal{B}].$$

Fix $a \in |\mathcal{A}|$ if $\mathcal{A} \models \neg Q(a) \exists(x|a)$

then $\mathcal{A} \models \forall x Q(x) [\mathcal{B}]$ hence

$$\mathcal{A} \models \neg \exists x \neg P(x) [\mathcal{B}] \text{ i.e. } \mathcal{A} \models \forall x P(x) [\mathcal{B}]$$

In particular $\mathcal{A} \models P(x) \quad [\mathcal{B}(x|a)]$

Since a was arbitrary

$$\mathcal{A} \models \forall x (\neg Q(x) \rightarrow P(x)).$$

2. $\forall x(\neg Q(x) \rightarrow P(x)) \not\models \exists x \neg P(x) \rightarrow \forall x Q(x)$.

Answer: Let $\mathcal{M} = (\mathbb{N}, P^{\mathcal{M}}, Q^{\mathcal{M}})$
 where $P^{\mathcal{M}} = 2\mathbb{N}$ the set of
 all even numbers and
 $Q^{\mathcal{M}} = 2\mathbb{N} + 1$ is the set
 of all odd numbers.

$$\mathcal{M} \models \forall x (\neg Q(x) \rightarrow P(x))$$

because a number n that is not odd
 is even.

$$\mathcal{M} \models \exists x \neg P(x) \text{ because } \mathcal{M} \models \neg P(x) \text{ [5]}$$

$$\text{But } \mathcal{M} \not\models \forall x Q(x)$$

$$\text{because } \mathcal{M} \models Q(x) \text{ [2]}$$

III. (10 points.) Let L be the language without equality and with two predicate symbols: B with 1 argument and Sh with 2 arguments.

1. Prove that the class of all structures whose models can be interpreted as "villages in which every barber shaves those and only those that do not shave themselves" is elementary.

Answer: This class is $\text{Mod } \varphi$ (all models of the sentence φ) where
 φ is $\forall x (B(x) \Rightarrow \forall y (Sh(x,y) \leftrightarrow \neg Sh(y,y)))$

2. How many of these structures have barbers? Provide a proof for your answer.

Answer: There are 0 barbers because
 $\{ \varphi, \exists x B(x) \}$ is unsatisfiable:
 Suppose $\mathcal{M} \models \exists x B(x)$ and $\mathcal{M} \models \varphi$.
 Let b be s.t. $\mathcal{M} \models B(x) [b]$.
 $\mathcal{M} \models \forall x (B(x) \Rightarrow \forall y (Sh(x,y) \leftrightarrow \neg Sh(y,y)))$
 Hence in particular
 $\mathcal{M} \models B(x) \Rightarrow \forall y (Sh(x,y) \leftrightarrow \neg Sh(y,y)) [b]$
 Hence $\mathcal{M} \models \forall y Sh(b,y) \leftrightarrow \neg Sh(y,y) [b]$
 But now replacing y with b as well
 we get $\mathcal{M} \models Sh(b,b) \leftrightarrow \neg Sh(b,b)$ $\uparrow$.

IV. (20 points.) Let $\mathfrak{A} = (W; *, \preceq, P_0, P_1, =)$, where:

- W is the set of words in the alphabet $\{0, 1\}$;
- $*$ is a function symbol with two arguments and $w_1 * w_2$ is the concatenation of w_1 and w_2 ;
- $\preceq$ is a binary predicate symbol and $w_1 \preceq w_2$ if and only if w_1 is a prefix of w_2 ;
- P_0 is a unary predicate symbol and $w \in P_0$ if and only if w is a nonempty word that begins with the letter 0;
- P_1 is a unary predicate symbol and $w \in P_1$ if and only if w is a nonempty word that begins with the letter 1;

Prove that each of the following is definable in the structure $\mathfrak{A}$.

1. The set $\{\varepsilon\}$, where ε is the empty word.

Answer:

$$\forall x: \neg P_0(x) \wedge \neg P_1(x)$$

If x is not a "nonempty word that starts with 0" or a "nonempty word that starts with 1" then x must be empty!

2. The sets $\{0\}$ and $\{1\}$.

Answer:

$$\{0\}: P_0(x) \wedge \forall y (P_0(y) \rightarrow x \preceq y)$$

" x starts with 0 and x is a prefix of every word that starts with 0"
The only such word is 0.

$$\{1\}: P_1(x) \wedge \forall y (P_1(y) \rightarrow x \preceq y)$$

Similarly

3. The set $\{001101\}$.

Answer: 001101 is the concatenation of 0, 0, 1, 1, 0, 1
 Using the f-las $\psi_0(x)$ and $\psi_1(y)$ (ie. $\psi_1^x(y)$)
 from 2:

$$\exists x \psi_0(x) \wedge \exists y \psi_1(y) \wedge z = x * x * y * y * x * y$$

$$\uparrow$$

 free variable

4. The relation $\{(w_1, w_2) \mid w_1 \text{ is lexicographically smaller than } w_2\}$. (Recall that the lexicographical ordering is the ordering we use to order words in the dictionary: w_1 precedes w_2 in the lexicographical ordering if w_1 is a prefix of w_2 or the two words differ in some least letter and that letter is 0 in w_1 and 1 in w_2 .)

Answer:
 w_1 is lexicographically smaller than w_2
 iff there ~~is~~ w_1 is a prefix of w_2 or
 there is some common prefix x of w_1 and
 w_2 such $x * 0$ is a prefix of w_1 and
 $x * 1$ is a prefix of w_2 .

$$\varphi_{\text{lex}}(w_1, w_2) \quad w_1 \leq w_2 \quad \vee$$

$$\exists x (x * 0 \leq w_1 \wedge x * 1 \leq w_2)$$

V. (10 points.) Prove that in the structure $\mathfrak{B} = (W_4, \preceq, =)$, where:

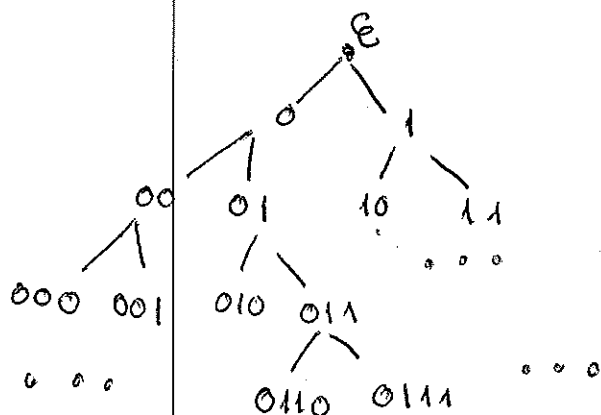
- W_4 is the set of words in the alphabet $\{0, 1\}$ of length ≤ 4 ;
- $\preceq$ is a binary predicate symbol and $w_1 \preceq w_2$ if and only if w_1 is a prefix of w_2 ;

the set $\{1001\}$ is not definable.

Answer:

$W_4 = \{ \epsilon, 0, 1, 00, 01, 10, 11, \dots, 1111 \}$.

We can picture $\preceq$ in $\mathfrak{B}$ using a tree:
 $w \preceq v$ iff w appears above v in the tree



If we exchange the letter 0 by 1 and vice versa, we get an automorphism:

$$h(w) = w'$$

where w' has n 'th letter i iff w has n 'th letter $1-i$.

h is clearly a bijection and if $w_1 \preceq w_2$ then $h(w_1) \preceq h(w_2)$.

$h(1001) = 0110$ hence an automorphism takes a word in the set $\{1001\}$ outside that set. It follows, that $\{1001\}$ is not definable.

VI. (10 points.) State and prove the Generalization Theorem.

Answer:

See Page 117 in the book.

Scratch Paper!

Nothing written on this paper will be graded.