

Name: _____

I	30 Points	
II	20 Points	
III	10 Points	
IV	20 Points	
V	10 Points	
VI	10 Points	
Total	100 Points	

1. Write your name.
2. Your answers must be accompanied by rigorous arguments!

I. (30 points.) True or False? Explain why!

1. A first order language always has at least one constant and equality.

Answer:

2. Every first order language L comes with a unique structure $\mathfrak{A}$.

Answer:

3. If $\mathfrak{A} \models \varphi[s]$ and x is a variable that does not occur free in φ then $\mathfrak{A} \models \exists x\varphi[s]$.

Answer:

4. If $\Gamma \vdash \varphi \vee \psi$ and $\Gamma \not\vdash \psi$ then $\Gamma \vdash \varphi$.

Answer:

5. There is no sentence φ , whose models are structures with exactly 2 or exactly 5 elements.

Answer:

6. If D_1 and D_2 are definable sets in a structure $\mathfrak{A}$ then $D_1 \cap D_2$ is also definable in $\mathfrak{A}$.

Answer:

7. If $D_1 \cap D_2$ is a definable set in a structure $\mathfrak{A}$ then D_1 is also definable in $\mathfrak{A}$.

Answer:

8. If D is definable in a structure $\mathfrak{A}$ and $\mathfrak{B}$ is isomorphic to $\mathfrak{A}$ via an isomorphism $h : |\mathfrak{A}| \rightarrow |\mathfrak{B}|$ then $h(D) = \{h(d) \mid d \in D\}$ is definable in $\mathfrak{B}$.

Answer:

9. If $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ then $\mathfrak{A} \models \forall x P(x)$ if and only if $\mathfrak{B} \models \forall x P(x)$.

Answer:

10. If $\Gamma \vdash \alpha$ and $\alpha \vdash \forall x \gamma$ then $\Gamma \cup \{\neg \gamma\}$ is inconsistent.

Answer:

II. (20 points.) Prove that

1. $\exists x \neg P(x) \rightarrow \forall x Q(x) \models \forall x (\neg Q(x) \rightarrow P(x)).$

Answer:

2. $\forall x(\neg Q(x) \rightarrow P(x)) \not\models \exists x\neg P(x) \rightarrow \forall xQ(x)$.

Answer:

III. (10 points.) Let L be the language without equality and with two predicate symbols: B with 1 argument and Sh with 2 arguments.

1. Prove that the class of all structures whose models can be interpreted as “villages in which every barber shaves those and only those that do not shave themselves” is elementary.

Answer:

2. How many of these structures have barbers? Provide a proof for your answer.

Answer:

IV. (20 points.) Let $\mathfrak{A} = (W; *, \preceq, P_0, P_1, =)$, where:

- W is the set of words in the alphabet $\{0, 1\}$;
- $*$ is a function symbol with two arguments and $w_1 *^{\mathfrak{A}} w_2$ is the concatenation of w_1 and w_2 ;
- $\preceq$ is a binary predicate symbol and $w_1 \preceq^{\mathfrak{A}} w_2$ if and only if w_1 is a prefix of w_2 ;
- P_0 is a unary predicate symbol and $w \in P_0^{\mathfrak{A}}$ if and only if w is a nonempty word that begins with the letter 0;
- P_1 is a unary predicate symbol and $w \in P_1^{\mathfrak{A}}$ if and only if w is a nonempty word that begins with the letter 1;

Prove that each of the following is definable in the structure $\mathfrak{A}$.

1. The set $\{\varepsilon\}$, where ε is the empty word.

Answer:

2. The sets $\{0\}$ and $\{1\}$.

Answer:

3. The set $\{001101\}$.

Answer:

4. The relation $\{(w_1, w_2) \mid w_1 \text{ is lexicographically smaller than } w_2\}$. (Recall that the lexicographical ordering is the ordering we use to order words in the dictionary: w_1 precedes w_2 in the lexicographical ordering if w_1 is a prefix of w_2 or the two words differ in some least letter and that letter is 0 in w_1 and 1 in w_2 .)

Answer:

V. (10 points.) Prove that in the structure $\mathfrak{B} = (W_4, \preceq, =)$, where:

- W_4 is the set of words in the alphabet $\{0, 1\}$ of length ≤ 4 ;
- $\preceq$ is a binary predicate symbol and $w_1 \preceq w_2$ if and only if w_1 is a prefix of w_2 ;

the set $\{1001\}$ is not definable.

Answer:

VI. (10 points.) State and prove the Generalization Theorem.

Answer:

Scratch Paper!

Nothing written on this paper will be graded.