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I	30 Points	
II	10 Points	
III	20 Points	
IV	20 Points	
V	10 Points	
VI	10 Points	
Total	100 Points	

1. Write your name.
2. Your answers must be accompanied by rigorous arguments!

I. (30 points.) True or False? Explain why!

1. If α and β are wffs and Σ is a set of wffs such that $\Sigma \models \alpha \wedge \beta$ then $\Sigma \models \alpha \vee \beta$.

Answer: True. If v is a truth assignment st. v satisfies Σ then from $\Sigma \models \alpha \wedge \beta$ we have that $v(\alpha \wedge \beta) = T$ hence $v(\alpha) = T$ and $v(\beta) = T$, but then $v(\alpha \vee \beta) = T$.

2. If α and β are wffs and Σ is a set of wffs such that $\Sigma \not\models \alpha$ then $\Sigma \not\models \alpha \vee \beta$.

Answer: False

Consider the case when β is a validity. Then $\alpha \vee \beta$ is a validity and $\Sigma \models \alpha \vee \beta$ for every Σ .

3. If α is a tautology then for every wff β we have that $\alpha \models \beta$.

Answer: False. Let $\beta = A$ and v be s.t. $v(A) = F$. Then v satisfies α , but v does not satisfy β . Hence $\alpha \not\models \beta$.

4. $(\neg A) \rightarrow A$ is a tautology.

Answer: False

$(\neg A) \rightarrow A \models \neg(\neg A) \vee A \models A$. So if $v(A) = F$ then $v(\neg A \rightarrow A) = F$.

5. If A and B are decidable sets of words on the alphabet $\{a, b\}$ then the symmetric difference of A and B : the set $C = (A \cup B) \setminus (A \cap B)$, is also decidable.

Answer: True. Given a word w we can run the program for A , run the program for B and output 'yes' if the two programs give different answers.

6. If G is a Boolean function on n arguments then there is a unique wff α that realizes it.

Answer: False. Any two logically equivalent wffs realize the same Boolean function on n arguments.

7. If $\Sigma \models \tau$ and $\Sigma_0 \subseteq \Sigma$ then $\Sigma_0 \models \tau$.

Answer: False. Let $\Sigma = \{A \wedge \neg A, B\}$ $\tau = C$
 Σ is not satisfiable so $\Sigma \models \tau$ for every τ
 $\Sigma_0 = \{B\} \subseteq \Sigma$ but $\Sigma_0 \not\models \tau$ as witnessed by
 $\forall s.t. \ v(B) = \text{True}$ but $v(C) = \text{False}$

8. If C is the smallest set of integers that contains the set $\{3\}$ and is closed under $f(x) = x+4$ and $g(x) = x-2$ then C is freely generated from $\{3\}$ by f and g .

Answer: False

$3 = g(g(f(3)))$ hence $\text{range } g \cap C \cap \{3\} \neq \emptyset$

9. If C is the smallest set of integers that contains the set $\{3\}$ and is closed under $f(x) = x+4$ and $g(x) = x-2$ then C contains the set of odd natural numbers.

Answer: True. $1 = g(3)$ Let $S = \{n : n \text{ is odd and } n \text{ is in } C\}$. Then $1, 3 \in S$. If $2n+1 \in S$ then $2n+3 \in S$ because $2n+3 = g(f(2n+1))$. Hence all odd numbers are in S .

10. If Σ is a set of wffs then Σ is decidable.

Answer: False.

There are uncountably many sets of wffs and only countably many decidable sets.

II. (10 points.) Let $\Sigma = \{A \vee B, A \vee \neg B, (\neg A \rightarrow A) \rightarrow \neg(C \vee \neg B)\}$

1. Prove that $\Sigma \models \neg C \wedge B$.

Answer: We write the truth table for all f-las in Σ to see which truth assignments satisfy Σ .

ABC	$A \vee B$	$A \vee \neg B$	$(\neg A \rightarrow A) \rightarrow \neg(C \vee \neg B)$
FFF	F	T	T
FET	F	T	T
FTF	T	F	T
FTT	T	F	T
TFF	T	T	F
TET	T	T	F
<u>TTF</u>	<u>T</u>	<u>T</u>	<u>T</u>
TTT	T	T	F

The only truth assignment that makes Σ true is v where
 $v(A) = T$
 $v(B) = T$
 $v(C) = F$

For this v we have $\neg(C \wedge B) = T$
 hence $\Sigma \models \neg C \wedge B$.

2. Prove that $\Sigma \not\models C \leftrightarrow B$

Answer: Consider v from 1: $v(A) = T, v(B) = T, v(C) = F$. Then v satisfies Σ , but $\neg(C \leftrightarrow B) = F$ hence
 $\Sigma \not\models C \leftrightarrow B$.

III. (20 points.) Consider the set U of all words in the alphabet $\{X, Y, Z, (,), \top, \perp\}$. Let $B = \{X, Y, Z\}$ and $\mathcal{F} = \{D_{\top}(\alpha) = \top(\alpha), D_{\perp}(\alpha) = \perp(\alpha)\}$. Let $C \subseteq U$ be the smallest set that contains B and is closed under the operations in $\mathcal{F}$.

1. Prove that C is freely generated from B using the functions in $\mathcal{F}$.

Answer:

range $D_{\top}$ has words that start with the letter $\top \notin B$ and

range $D_{\perp}$ has words that start with the letter $\perp \notin B$.

$$\text{Hence } B \cap \text{range } D_{\top} = B \cap \text{range } D_{\perp} = \text{range } D_{\top} \cap \text{range } D_{\perp} = \emptyset.$$

$D_{\top}$ is injective because if

$$D_{\top}(\alpha_1) = D_{\top}(\alpha_2) \quad \text{then}$$

$\top(\alpha_1) = \top(\alpha_2)$ deleting the first 2 letters which are the same, as well as the last letter then $\alpha_1 = \alpha_2$.

That $D_{\perp}$ is injective is proved similarly.

2. Is C is decidable? Give an informal argument for your answer.

Answer: Yes.

Given a word α we check:

1. Is it X or Y or Z and if yes output 'yes'. Otherwise, if α starts with $\perp$ go to 2 and if α starts with T go to 3. In all other cases output No.

2. If α starts with $\perp$ (and ends in) then let β be s.t. $\alpha = \perp(\beta)$. Run the program for β starting at 1. Otherwise output No.

3. If $\alpha = T(\beta)$ for some β then run the program for β starting at 1. Otherwise output No.

3. Using recursion define a function $l : C \rightarrow \mathbb{N}$ that on argument α gives the number of symbols $s \in \{(\,,\,), T, \perp\}$ that appear in α .

Answer:

$$l(X) = l(Y) = l(Z) = 0$$

$$l(\perp(\alpha)) = 3 + l(\alpha)$$

$$l(T(\alpha)) = 3 + l(\alpha)$$

IV. (20 points.) Prove each of the following statements.

1. The set $\{\neg, \nrightarrow\}$ is complete. Here $\nrightarrow$ is a new connective such that $\bar{v}(\alpha \nrightarrow \beta) = T$ if and only if $\bar{v}(\alpha) = T$ and $\bar{v}(\beta) = F$ for all truth assignments v .

Answer: $\alpha \wedge \beta \models \alpha \nrightarrow \neg \beta$

We know that $\{\neg, \wedge\}$ is complete
hence $\{\neg, \nrightarrow\}$ is also complete.

2. The set $\{\vee, \wedge\}$ is not complete.

Answer: Claim:

We cannot realize any Boolean function $B^2(x, y)$ s.t. $B(T, T) = F$. There are such functions since the set is not complete.

To prove our claim let

$$S = \{ \alpha : \wedge B_{\alpha}^2(x, T, T) = T \}$$

α is built up from $\vee, \wedge \rightarrow$

Every prop. symbol A we have

$$B_A^2(T, T) = T \rightarrow A \in S.$$

$$\text{If } \alpha, \beta \in S \text{ then } B_{\alpha \wedge \beta}^2(T, T) =$$

$$= B_{A \wedge B}^2(B_{\alpha}^2(T, T), B_{\beta}^2(T, T)) = B_{A \wedge B}^2(T, T) = T$$

$$\rightarrow \alpha \wedge \beta \in S. \quad \text{Similarly } \alpha \vee \beta \in S.$$

If α has a symbol different from $\vee, \wedge$ then $\alpha \in S$ vacuously. $S = \text{WFFS}$.

V. (10 points.) Find a wff α in disjunctive normal form that realizes the function $M(X, Y, Z) = X \oplus Y \oplus Z$, where $\oplus$ is addition modulo 2, i.e. $T \oplus T = F \oplus F = F$ and $T \oplus F = F \oplus T = T$. Explain your reasoning!

Answer: We use the method described in Post's theorem.
 From the truth table for M we build conjunctions $\varphi_{\bar{x}}$ s.t.
 $\bar{v}(\varphi_{\bar{x}}) = T$ iff $v(A_1) = x_1 \vee v(A_2) = x_2$
 and $v(A_3) = x_3$.
 We then form the disjunction of all $\varphi_{\bar{x}}$ s.t. $M(\bar{x}) = T$.

X	Y	Z	$M(X, Y, Z)$	
F	F	F	F	
F	F	T	T	$\neg A_1 \wedge \neg A_2 \wedge A_3 \vee$
F	T	F	T	$\neg A_1 \wedge A_2 \wedge \neg A_3 \vee$
F	T	T	F	
T	F	F	T	$A_1 \wedge \neg A_2 \wedge \neg A_3 \vee$
T	F	T	F	
T	T	F	F	
T	T	T	T	$A_1 \wedge A_2 \wedge A_3$

VI. (10 points.) Let $\Sigma = \{\alpha_0, \alpha_1, \dots, \alpha_n \dots\} = \{\alpha_i \mid i \in \mathbb{N}\}$ be a set of wffs. Let $\Sigma_n = \{\alpha_0, \alpha_1, \dots, \alpha_n\} = \{\alpha_i \mid i \leq n\}$ for every n . Prove or disprove the following statement: If for every odd number n the set Σ_n is satisfiable then Σ is satisfiable.

Answer: The statement is true.

First note that for every n
 $\Sigma_{2n} \subseteq \Sigma_{2n+1}$. If Σ_{2n+1} is
satisfiable then Σ_{2n} is satisfiable.
Hence for all n Σ_n is satisfiable.

Let Σ_0 be some finite subset of Σ

Let n be the largest index of a
formula in Σ_0 . Then

$\Sigma_0 \subseteq \Sigma_n$, since Σ_0 is
satisfiable. This means that

Σ is finitely satisfiable. By
the compactness theorem, Σ
is satisfiable as well.