

Name:

I	30 Points	
II	10 Points	
III	20 Points	
IV	20 Points	
V	10 Points	
VI	10 Points	
Total	100 Points	

1. Write your name.
2. Your answers must be accompanied by rigorous arguments!

I. (30 points.) True or False? Explain why!

1. If α and β are wffs and Σ is a set of wffs such that $\Sigma \models \alpha \wedge \beta$ then $\Sigma \models \alpha \vee \beta$.

Answer:

2. If α and β are wffs and Σ is a set of wffs such that $\Sigma \not\models \alpha$ then $\Sigma \not\models \alpha \vee \beta$.

Answer:

3. If α is a tautology then for every wff β we have that $\alpha \models \beta$.

Answer:

4. $(\neg A) \rightarrow A$ is a tautology.

Answer:

5. If A and B are decidable sets of words on the alphabet $\{a, b\}$ then the symmetric difference of A and B : the set $C = (A \cup B) \setminus (A \cap B)$, is also decidable.

Answer:

6. If G is a Boolean function on n arguments then there is a unique wff α that realizes it.

Answer:

7. If $\Sigma \models \tau$ and $\Sigma_0 \subseteq \Sigma$ then $\Sigma_0 \models \tau$.

Answer:

8. If C is the smallest set of integers that contains the set $\{3\}$ and is closed under $f(x) = x + 4$ and $g(x) = x - 2$ then C is freely generated from $\{3\}$ by f and g .

Answer:

9. If C is the smallest set of integers that contains the set $\{3\}$ and is closed under $f(x) = x + 4$ and $g(x) = x - 2$ then C contains the set of odd natural numbers.

Answer:

10. If Σ is a set of wffs then Σ is decidable.

Answer:

II. (10 points.) Let $\Sigma = \{A \vee B, A \vee \neg B, (\neg A \rightarrow A) \rightarrow \neg(C \vee \neg B)\}$

1. Prove that $\Sigma \models \neg C \wedge B$.

Answer:

2. Prove that $\Sigma \not\models C \leftrightarrow B$

Answer:

III. (20 points.) Consider the set U of all words in the alphabet $\{X, Y, Z, (,), \top, \perp\}$. Let $B = \{X, Y, Z\}$ and $\mathcal{F} = \{D_{\top}(\alpha) = \top(\alpha), D_{\perp}(\alpha) = \perp(\alpha)\}$. Let $C \subseteq U$ be the smallest set that contains B and is closed under the operations in $\mathcal{F}$.

1. Prove that C is freely generated from B using the functions in $\mathcal{F}$.

Answer:

2. Is C is decidable? Give an informal argument for your answer.

Answer:

3. Using recursion define a function $l : C \rightarrow \mathbb{N}$ that on argument α gives the number of symbols $s \in \{ (,), \top, \perp \}$ that appear in α .

Answer:

IV. (20 points.) Prove each of the following statements.

1. The set $\{\neg, \nrightarrow\}$ is complete. Here $\nrightarrow$ is a new connective such that $\bar{v}(\alpha \nrightarrow \beta) = T$ if and only if $\bar{v}(\alpha) = T$ and $\bar{v}(\beta) = F$ for all truth assignments v .

Answer:

2. The set $\{\vee, \wedge\}$ is not complete.

Answer:

V. (10 points.) Find a wff α in disjunctive normal form that realizes the function $M(X, Y, Z) = X \oplus Y \oplus Z$, where $\oplus$ is addition modulo 2, i.e. $T \oplus T = F \oplus F = F$ and $T \oplus F = F \oplus T = T$. Explain your reasoning!

Answer:

VI. (10 points.) Let $\Sigma = \{\alpha_0, \alpha_1, \dots, \alpha_n \dots\} = \{\alpha_i \mid i \in \mathbb{N}\}$ be a set of wffs. Let $\Sigma_n = \{\alpha_0, \alpha_1, \dots, \alpha_n\} = \{\alpha_i \mid i \leq n\}$ for every n . Prove or disprove the following statement: If for every odd number n the set Σ_n is satisfiable then Σ is satisfiable.

Answer:

Scratch Paper!

Nothing written on this paper will be graded.