

HONORS ASSIGNMENT II
MATH 571 FALL 2019
DUE NOV 12TH

Problem 1. Prove that the class of dense linear orderings without endpoints (i.e. without least and greatest element) is an elementary class.

Problem 2. Prove that every countable dense linear ordering without endpoints is isomorphic to the structure $(\mathbb{Q}; \leq)$.

Hint: Use the *back and forth method* to construct an isomorphism: If $\mathfrak{A} = (A; L)$ is a countable dense linear ordering without endpoints then fix a listing of its domain: $A = \{a_0, a_1, \dots, a_n, \dots\}$. Fix a listing of the rationals $\mathbb{Q} = \{q_0, q_1, \dots, q_n, \dots\}$. Build an isomorphism h as the union of finite approximations $h_0, h_1, \dots, h_n, \dots$, where h_n is a finite function $h_n : A \rightarrow \mathbb{Q}$ that preserves the ordering (i.e. if $a, b \in \text{dom}(h_n)$ and $L(a, b)$ then $h_n(a) \leq h_n(b)$), that is one-to-one and such that $\{a_0, \dots, a_n\} \subseteq \text{dom}(h_n)$ and $\{q_0, \dots, q_n\} \subseteq \text{range}(h_n)$.

Problem 3. Are all countable dense linear orderings with endpoints isomorphic? Explain your answer!

Problem 4. A linear ordering is discrete if every element x has an *immediate successor*, an element y larger than x (in the sense of the ordering) such that the interval (x, y) (in the sense of the ordering) is empty. Prove that there are countable non-isomorphic discrete linear orderings without endpoints.

Problem 5. Let $\mathfrak{A}$ be the structure $(\mathbb{N}, D^{\mathfrak{A}})$, where $D^{\mathfrak{A}}$ is the set of all pairs of natural numbers (m, n) such that m divides n . Show that the set of prime numbers is definable in $\mathfrak{A}$. Prove that a singleton $\{n\}$ is definable in the structure $\mathfrak{A}$ if and only if $n = 0$ or $n = 1$. (Hint: define an appropriate automorphism to prove that other singletons cannot be definable.)