

Name: _____

Solutions

I	40 Points	
II	10 Points	
III	12 Points	
IV	16 Points	
V	10 Points	
VI	12 Points	
Total	100 Points	

1. Write your name.
2. Your answers must be accompanied by rigorous arguments!

I. (40 points.) Let $\mathfrak{N} = (\mathbb{N}, 0, S, +, *, E, <)$. True or False? Explain why!

1. It is possible to have sentences σ, τ such that $\sigma \models \tau$ and τ has no model.

Answer: True. If σ has no model then $\sigma \models \tau$ for all τ .

2. If t is a term that is substitutable for x in φ then φ_t^x is a validity.

Answer: False. Consider $\varphi: x \neq x$. $t = c$
 t is substitutable for x in φ , because φ is atomic
 $\varphi_t^x = c \neq c$ is not a validity.

3. If $\mathfrak{A}$ is a structure that can be isomorphically embedded in a structure $\mathfrak{B}$ then $\mathfrak{A}$ is elementary equivalent to $\mathfrak{B}$.

Answer: False. $(\mathbb{N}, <)$ can be isomorphically embedded in $(\mathbb{Z}, <)$ by the identity.
 But the two structures are not el. equivalent:
 $(\mathbb{N}, <) \models \exists x \forall y (y \neq x \rightarrow x < y)$ $(\mathbb{Z}, <) \not\models \exists x \forall y (y \neq x \rightarrow x < y)$

4. There is a sentence σ that has models of every finite size and no infinite model.

Answer: False. Let $\varphi_n: \exists x_1 \dots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$
 $\{\sigma \vee \exists \varphi_n: n \in \mathbb{N}\}$ is finitely satisfiable
 By compactness, it is satisfiable by, say σ
 Then σ is infinite and $\sigma \models \sigma$

5. If Γ is a consistent set of sentences then Γ can be extended to a complete set of sentences in the same language.

Answer: True. Let $\Gamma_0 = \Gamma$ and
 $\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent} \\ \Gamma_n \cup \{\neg \varphi_n\} & \text{o.w.} \end{cases}$, where $\varphi_0, \varphi_1, \dots$
 is a listing of all φ -as in the language
 Consider $\Delta = \bigcup_{n \in \mathbb{N}} \Gamma_n$

6. There is a finite structure in a finite language with a theory that is not decidable.

Answer: False. If $\mathcal{A}$ is a finite structure in a finite language then $\text{Th } \mathcal{A}$ is axiomatized by a single sentence φ . Since the set of consequences of φ ($\text{Th } \mathcal{A}$) is both complete and

7. If D is a definable set in a structure $\mathcal{A}$ then D is countable. enumerable, it is decidable.

Answer: False. Consider $R = (\mathbb{R}, 0, <)$
 $D = \{x : x \neq 0\}$ is definable by $\varphi(x): x \neq 0$
It is not countable.

8. Let Σ be a finite set of sentences in a finite language. We can always find two different extensions $\Sigma \subseteq T_1$ and $\Sigma \subseteq T_2$, such that T_1 and T_2 are complete theories.

Answer: False. If Σ is inconsistent then the only theory T s.t. $\Sigma \subseteq T$ is the set of all sentences in the language.

9. If T is an inconsistent theory then T is finitely axiomatizable.

Answer: True. If φ is a validity in the language of T then T is the set of all consequences of $\top \varphi$.

10. If T is a theory with a finite model then T is consistent.

Answer: True. By the completeness theorem every satisfiable set of f-las is consistent.

11. Let T_G be the set of sentences (in the language of groups $L_G = \{e, *, ^{-1}\}$) that are satisfied by every group G . T_G is a complete theory.

Answer: False. There are abelian (commutative) and nonabelian groups. Hence T_G does not contain φ or $\neg\varphi$, where $\varphi: \forall x \forall y (x * y = y * x)$

12. Let $\mathfrak{R} = (\mathbb{R}, 0, S, +, *, E, <)$. The structure $\mathfrak{R}$ has only uncountable models.

Answer: False. By the Lowenheim-Skolem theorem if $\text{Th}(\mathfrak{R})$ has an infinite model (such as $\mathfrak{R}$) then it has a countable model.

13. $\text{Th}(\mathbb{N})$ proves that every nonzero natural number is in the range of the successor function.

Answer: True. $\text{Th}(\mathbb{N}) \ni \forall x (x \neq 0 \rightarrow \exists y (S(y) = x))$

14. There is a nonstandard model $\mathfrak{M} = (M, 0, S, +, *, E, <)$ of number theory ($\mathfrak{M} \equiv \mathbb{N}$), which contains an infinite descending chain (an infinite sequence $a_0 > a_1 > \dots > a_n > \dots$).

Answer: True. Add new constants $c_0, c_1, \dots, c_n, \dots$ to the language. For every n $\text{Th}(\mathbb{N}) \cup \{c_0 > c_1, c_1 > c_2, \dots, c_{n-1} > c_n\}$ is satisfiable by $\mathbb{N}$ with $c_i^n = n - i$. By compactness $\text{Th}(\mathbb{N}) \cup \{c_0 > c_1 > \dots > c_n, \dots\}$ is satisfiable by $\mathfrak{B}$. Forgetting the constants we get $\mathfrak{B}' \equiv \mathbb{N}$ with a descending chain.

15. The function $f(x, y) = 2^{y+1}(2x + 1)$ is representable in A_E .

Answer: True. Every computable function is representable in A_E

16. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined from $g : \mathbb{N}^2 \rightarrow \mathbb{N}$ as follows: $f(x)$ is the least u such that $g(x, u) = 5$. If g is representable in A_E and f is defined for every natural number x then f is also representable.

Answer: True. If $\varphi(x, y, z)$ represents g then f is represented by $\psi(x, y)$:

$$\varphi(x, y, s(s(s(s(s(0)))))) \wedge \forall u (u < y \rightarrow \neg \varphi(x, u, s(s(s(s(s(0)))))))$$

17. Every theory T in the language of number theory that contains A_E is either incomplete, or inconsistent.

Answer: False. This would be true if T were also decidable (recursive).

$Th \mathcal{N} \supset A_E$, but $Th \mathcal{N}$ is both complete and consistent.

18. The set $\#Th(\mathcal{N})$ is not definable in $\mathcal{N}$.

Answer: True. This is Tarski's Undefinability Theorem.

19. The set of all codes $\# \sigma$ of sentences σ in the language of number theory is decidable.

Answer: True. Recall that representable = recursive = decidable. a is the code of a sentence if a is a code of a f.l.a (decidable) and for all $b < a$ if b is a code of a variable (decidable) then $(a, b) \in Fr =$ the set of pairs of a f.l.a and a var that appears free in φ , which is also decidable.

20. The theory of $\mathcal{N}$ is effectively enumerable.

Answer: By Tarski's theorem $Th(\mathcal{N})$ is not definable, hence not decidable. Since $Th(\mathcal{N})$ is complete, then if it were eff. enumerable, it would be decidable. So: False

II. (10 points.) Let Γ be a set of formulas.

1. Is it true that if $\Gamma \vdash \varphi$ and $\Gamma \nvdash \psi$ then $\Gamma \nvdash \neg\varphi \vee \psi$. Provide a proof for your answer.

Answer:

Suppose that $\Gamma \vdash \neg\varphi \vee \psi$ and

$\Gamma \vdash \varphi$. The set $\{\varphi, \neg\varphi \vee \psi\}$

tautologically implies ψ .

hence by rule $\vdash$, we would
have that $\Gamma \vdash \psi$.

We know that $\Gamma \nvdash \psi$, hence
our assumption that $\Gamma \vdash \neg\varphi \vee \psi$
is false. It follows that
 $\Gamma \nvdash \neg\varphi \vee \psi$.

2. Is it true that if $\Gamma \not\vdash \varphi$ and $\Gamma \vdash \psi$ then $\Gamma \vdash \neg\varphi \wedge \psi$. Provide a proof for your answer.

Answer: False.

Consider Γ to be the set of sentences true in every partial order

$$\Gamma = \{ \forall x (x \neq x), \forall x \forall y (x < y \rightarrow \neg y < x), \forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z) \}.$$

Let φ be $\forall x \forall y (x < y \vee y < x \vee x = y)$

$$\psi = \forall x (x \neq x) \text{ or: } \forall x \neg (x < x)$$

$\Gamma \vdash \psi$ because $\psi \in \Gamma$.

$\Gamma \not\vdash \varphi$: Let $|O| = \{ \emptyset, 303, 315, 3018 \}$
 $<^O = \subseteq$. Then $O \models \Gamma$ and

$O \not\models \varphi$. It follows that $\Gamma \not\vdash \varphi$

and by the completeness theorem $\Gamma \not\vdash \varphi$.

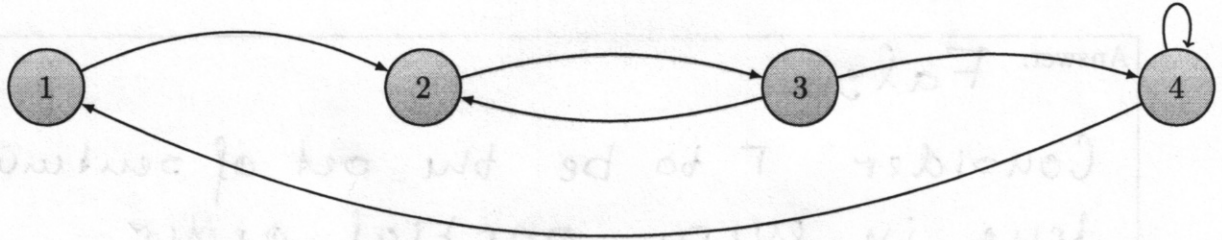
$\Gamma \not\vdash \neg\varphi \wedge \psi$. Let $|B| = \{ 0, 1, 2 \}$

and $<^B = <$ (the standard ordering)

$B \models \varphi \wedge \psi$ since $B \models \neg\varphi \wedge \psi$. $B \models \Gamma$

since $\Gamma \not\vdash \neg\varphi \wedge \psi$. By completeness $\Gamma \not\vdash \neg\varphi \wedge \psi$.

III. (12 points.) Let L be the language with one predicate symbol: E with 2 arguments. Structures for this language are called directed graphs. For example, the structure $\mathfrak{A} = (\{1, 2, 3, 4\}, E^{\mathfrak{A}} = \{(1, 2), (2, 3), (3, 2), (3, 4), (4, 4), (4, 1)\})$ is a graph that can be visualized as:



1. Prove that the class of all directed graphs with no loops is EC_{Δ} . A loop of size $n \geq 3$ in a structure $\mathfrak{A}$ is a sequence of n distinct elements of $|\mathfrak{A}|$: $a_1, \dots, a_n$, such that $(a_i, a_{i+1}) \in E^{\mathfrak{A}}$ for every $i < n$ and $(a_n, a_1) \in E^{\mathfrak{A}}$. For example 1, 2, 3, 4 is a loop in the graph above.

Answer:

$$\text{Let } \varphi_n : \forall x_1 \forall x_2 \dots \forall x_n$$

$$((\exists x_1 x_2 \wedge \exists x_2 x_3 \wedge \dots \wedge \exists x_{n-1} x_n \rightarrow$$

$$\neg \exists x_n, x_1)) \text{ where } n \geq 3.$$

φ_n say "There is no loop of size n "

$\text{Mod} \{ \varphi_n : n \in \mathbb{N}_{n \geq 3} \}$ are exactly the directed graphs with no loops.

2. Is the class of all graphs with a loop EC_{Δ} ? Provide a proof for your answer.

Answer:

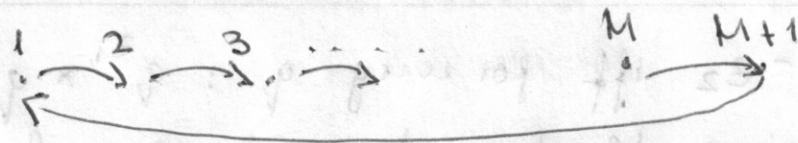
Answer: No. Towards a contradiction let $\mathcal{I}$ be a set of f -ias such that $\text{mod } \mathcal{I} =$ the class of graphs with a loop.

Let φ_n be as in 1.

$\Sigma \{ \exists u : u \in W \}$ is finitely
~~axiomatizable~~ satisfiable. Indeed

if $\Delta \subseteq \mathbb{Z}$ v. l. en: $\frac{n}{n \geq 3}$ et

M be the largest index such that ψ_M is in Δ . The graph $g:$



is a model of Δ . By compactness,

$\exists u \{ \text{len} : \underbrace{u \leq w}_{u \geq 3} \}$ is satisfiable.

Let $\mathcal{A}$ be a model. Then $\mathcal{A} \models \exists$ so $\mathcal{A}$ has a loop. But $\mathcal{A} \models \forall n$ for all $n \geq 3$, hence $\mathcal{A}$ has no loop: contradict

IV. (16 points.) Let $\mathcal{A} = (\mathbb{Q}; *, E; L_1, =)$, where:

- $*$ is a function symbol with two arguments and $q_1 *^{\mathcal{A}} q_2$ is the product of q_1 and q_2 ;
- E is a function symbol with two arguments such and $q_1 E^{\mathcal{A}} q_2$ is $q_1^{q_2}$ if q_2 is an integer ($q_2 \in \mathbb{Z}$) and 0 otherwise. (E.g. $7E(-2) = \frac{1}{49}$ and $7E\frac{1}{2} = 0$).
- L_1 is a predicate symbol with 1 argument such and $q \in L_1^{\mathcal{A}}$ if $q < 1$.

Show that each of the following is definable in the structure $\mathcal{A}$.

1. The set of all integers.

Answer: First we define 1 by $\varphi_1(x) \forall y (y * x = y)$.
 Then q is an integer iff $1 E^{\mathcal{A}} q = 1$ by our definition of $E^{\mathcal{A}}$ so

$$\varphi_{\mathbb{Z}}(x) : \forall y (\varphi_1(y) \rightarrow y E x = y)$$

2. The relation $\{(z_1, z_2) \mid z_1 \in \mathbb{Z} \wedge z_2 \in \mathbb{Z} \wedge z_1 = -z_2\}$.

Answer:
 $z_1 = -z_2$ iff for every q : $q^{z_1} * q^{z_2} = 1$
 So using φ_1 from 1 and $\varphi_{\mathbb{Z}}$ from 1:

$$\varphi_{-}(x, y) : \varphi_{\mathbb{Z}}(x) \wedge \varphi_{\mathbb{Z}}(y) \wedge$$

$$\forall u (\varphi_1(u) \rightarrow \forall v (v E x) * (v E y) = u)$$

3. The relation $\{(z_1, z_2) \mid z_1 \in \mathbb{Z} \wedge z_2 \in \mathbb{Z} \wedge z_2 = z_1 + 1\}$.

Answer:

$$z_2 = z_1 + 1 \text{ iff for every } q \quad q * q^{z_1} = q^{z_2}$$

$$\varphi_3(x, y) : \varphi_z(x) \wedge \varphi_z(y) \wedge \\ \forall v (v * (v \in x) = v \in y)$$

4. For every integer z the set $\{z\}$.

Answer:

$$\varphi_0(x) : \varphi_z(x) \wedge \forall u (\varphi_1(u) \wedge \varphi_3(x, u))$$

($0+1=1$). Suppose we have φ_n defining $\{ny\}$ and φ_{-n} defining $\{b-ny\}$

$$\varphi_{n+1}(x) : \exists u (\varphi_n(u) \wedge \varphi_3(u, x))$$

$$\varphi_{-n-1}(x) : \exists u (\varphi_{-n}(u) \wedge \varphi_{-1}(x, u))$$

By induction we have that $\{ny\}, \{b-ny\}$ are definable for every n .

5. The set of all positive integers.

Answer:

$$\varphi_p(x) : \varphi_z(x) \wedge \neg L_1(x)$$

V. (10 points.) Consider the structure $\mathfrak{B} = (\mathbb{Q}; L_1, =)$, where

- L_1 is a predicate symbol with 1 arguments such and $q \in L_1^{\mathfrak{B}}$ if $q < 1$.

Prove that for every rational number q the set $\{q\}$ is not definable.

Answer:

Fix $q \in \mathbb{Q}$.

Case 1. If $q \geq 1$ then let

$f: \mathbb{Q} \rightarrow \mathbb{Q}$ be the function defined by $f(x) = \begin{cases} x & \text{if } x \neq q \text{ and } x \neq 2 \\ 2 & \text{if } x = q \\ q & \text{if } x = 2. \end{cases}$

f is injective and surjective.

f is a homomorphism:

if $L_1(x)$ then $f(x) = x$, so $L_1(f(x))$

if $\neg L_1(x)$ then $f(x) = x$ or 2 or q
so $\neg L_1(x)$. Hence f is an

automorphism

that moves q .

It follows, that

$\{q\}$ is not definable.

Case 2: If $q < 1$ then replace 2 by $1/2$ and use the same argument as in case 1.

VI. (12 points.) Let $\mathcal{A} = (\mathbb{N}, P)$, where $P^{\mathcal{A}} = \{(n, n+1) \mid n \in \mathbb{N}\}$. Prove that there is a structure $\mathcal{B}$, such that $\mathcal{B} \equiv \mathcal{A}$ such that $\mathcal{B} \not\cong \mathcal{A}$.

Answer: We will add two constants c_0 and c_1 to the language. φ_n will be the f.l.a that says $c_0 + n < c_1$ and c_0 is the least el.

$\varphi_n: \exists x_0 \exists x_1 \dots \exists x_{n-1}$

$P(c_0, x_0) \wedge P(x_0, x_1) \wedge \dots$

$P(x_{n-2}, x_{n-1}) \wedge \neg P(x_{n-1}, c_1) \wedge$

$\forall x \neg P(x, c_0).$

$\Sigma = T_n \cup \{ \varphi_n : n \in \mathbb{N} \}$ is finitely satisfiable, because if $\Sigma_0 \subseteq \Sigma$ and n is the max index s.t. $\varphi_n \in \Sigma_0$ then α with $c_0^\alpha = 0$ and $c_1^\alpha = n+1$ satisfies Σ_0 . Let $\mathcal{B}$ be a model of Σ (by the compactness thm). Then $\mathcal{B}' \equiv \alpha$ because $\mathcal{B}' \models T_n(\alpha)$, where $\mathcal{B}'$ is $\mathcal{B}$ with forgotten constants.

$\mathcal{B}' \not\cong \alpha$ because no homomorphism $f: |\alpha| \rightarrow |\mathcal{B}'|$ can have $f(c_1) = c_1$.