

Name:

I	40 Points	
II	10 Points	
III	12 Points	
IV	16 Points	
V	10 Points	
VI	12 Points	
Total	100 Points	

1. Write your name.
2. Your answers must be accompanied by rigorous arguments!

I. (40 points.) Let $\mathfrak{N} = (\mathbb{N}, 0, S, +, *, E, <)$. True or False? Explain why!

1. It is possible to have sentences σ, τ such that $\sigma \models \tau$ and τ has no model.

Answer:

2. If t is a term that is substitutable for x in φ then φ_t^x is a validity.

Answer:

3. If $\mathfrak{A}$ is a structure that can be isomorphically embedded in a structure $\mathfrak{B}$ then $\mathfrak{A}$ is elementary equivalent to $\mathfrak{B}$.

Answer:

4. There is a sentence σ that has models of every finite size and no infinite model.

Answer:

5. If Γ is a consistent set of sentences then Γ can be extended to a complete set of sentences in the same language.

Answer:

6. There is a finite structure in a finite language with a theory that is not decidable.

Answer:

7. If D is a definable set in a structure $\mathfrak{A}$ then D is countable.

Answer:

8. Let Σ be a finite set of sentences in a finite language. We can always find two different extensions $\Sigma \subseteq T_1$ and $\Sigma \subseteq T_2$, such that T_1 and T_2 are complete theories.

Answer:

9. If T is an inconsistent theory then T is finitely axiomatizable.

Answer:

10. If T is a theory with a finite model then T is consistent.

Answer:

11. Let T_G be the set of sentences (in the language of groups $L_G = \{e, *, {}^{-1}\}$) that are satisfied by every group G . T_G is a complete theory.

Answer:

12. Let $\mathfrak{N} = (\mathbb{R}, 0, S, +, *, E, <)$. The structure $\mathfrak{N}$ has only uncountable models.

Answer:

13. $Th(\mathfrak{N})$ proves that every nonzero natural number is in the range of the successor function.

Answer:

14. There is a nonstandard model $\mathfrak{M} = (M, 0, S, +, *, E, <)$ of number theory ($\mathfrak{M} \equiv \mathfrak{N}$), which contains an infinite descending chain (an infinite sequence $a_0 > a_1 > \cdots > a_n > \dots$).

Answer:

15. The function $f(x, y) = 2^{y+1}(2x + 1)$ is representable in A_E .

Answer:

16. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined from $g : \mathbb{N}^2 \rightarrow \mathbb{N}$ as follows: $f(x)$ is the least u such that $g(x, u) = 5$. If g is representable in A_E and f is defined for every natural number x then f is also representable.

Answer:

17. Every theory T in the language of number theory that contains A_E is either incomplete, or inconsistent.

Answer:

18. The set $\#Th(\mathfrak{N})$ is not definable in $\mathfrak{N}$.

Answer:

19. The set of all codes $\#\sigma$ of sentences σ in the language of number theory is decidable.

Answer:

20. The theory of $\mathfrak{N}$ is effectively enumerable.

Answer:

II. (10 points.) Let Γ be a set of formulas.

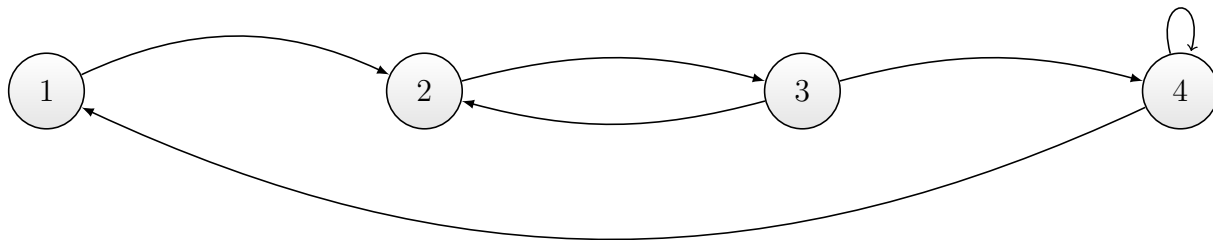
1. Is it true that if $\Gamma \vdash \varphi$ and $\Gamma \not\vdash \psi$ then $\Gamma \not\vdash \neg\varphi \vee \psi$. Provide a proof for your answer.

Answer:

2. Is it true that if $\Gamma \not\vdash \varphi$ and $\Gamma \vdash \psi$ then $\Gamma \vdash \neg\varphi \wedge \psi$. Provide a proof for your answer.

Answer:

III. (12 points.) Let L be the language with one predicate symbol: E with 2 arguments. Structures for this language are called directed graphs. For example, the structure $\mathfrak{A} = (\{1, 2, 3, 4\}, E^{\mathfrak{A}} = \{(1, 2), (2, 3), (3, 2), (3, 4), (4, 4), (4, 1)\})$ is a graph that can be visualized as:



1. Prove that the class of all directed graphs with no loops is EC_{Δ} . A loop of size $n \geq 3$ in a structure $\mathfrak{A}$ is a sequence of n distinct elements of $|\mathfrak{A}|$: $a_1, \dots, a_n$, such that $(a_i, a_{i+1}) \in E^{\mathfrak{A}}$ for every $i < n$ and $(a_n, a_1) \in E^{\mathfrak{A}}$. For example 1, 2, 3, 4 is a loop in the graph above.

Answer:

2. Is the class of all graphs with a loop EC_{Δ} ? Provide a proof for your answer.

Answer:

IV. (16 points.) Let $\mathfrak{A} = (\mathbb{Q}; *, E; L_1, =)$, where:

- $*$ is a function symbol with two arguments and $q_1 *^{\mathfrak{A}} q_2$ is the product of q_1 and q_2 ;
- E is a function symbol with two arguments such and $q_1 E^{\mathfrak{A}} q_2$ is $q_1^{q_2}$ **if** q_2 is an integer ($q_2 \in \mathbb{Z}$) and 0 otherwise. (E.g. $7E(-2) = \frac{1}{49}$ and $7E\frac{1}{2} = 0$).
- L_1 is a predicate symbol with 1 argument such and $q \in L_1^{\mathfrak{A}}$ if $q < 1$.

Show that each of the following is definable in the structure $\mathfrak{A}$.

1. The set of all integers.

Answer:

2. The relation $\{(z_1, z_2) \mid z_1 \in \mathbb{Z} \wedge z_2 \in \mathbb{Z} \wedge z_1 = -z_2\}$.

Answer:

3. The relation $\{(z_1, z_2) \mid z_1 \in \mathbb{Z} \wedge z_2 \in \mathbb{Z} \wedge z_2 = z_1 + 1\}$.

Answer:

4. For every integer z the set $\{z\}$.

Answer:

5. The set of all positive integers.

Answer:

V. (10 points.) Consider the structure $\mathfrak{B} = (\mathbb{Q}; L_1, =)$, where

- L_1 is a predicate symbol with 1 arguments such and $q \in L_1^{\mathfrak{B}}$ if $q < 1$.

Prove that for every rational number q the set $\{q\}$ is not definable.

Answer:

VI. (12 points.) Let $\mathfrak{A} = (\mathbb{N}, P)$, where $P^{\mathfrak{A}} = \{(n, n+1) \mid n \in \mathbb{N}\}$. Prove that there is a structure $\mathfrak{B}$, such that $\mathfrak{B} \equiv \mathfrak{A}$ such that $\mathfrak{B} \not\cong \mathfrak{A}$.

Answer:

Scratch Paper!

Nothing written on this paper will be graded.