I. (15 points.) Each question has only one correct answer. Circle the one you think is correct.

1. Which of the following correctly formalizes the negation of the statement "Every student who practiced for the exam got a good grade"?
   (a) $\forall x(S(x) \wedge P(x) \rightarrow \exists y(Gr(x,y) \wedge G(y)))$.
   (b) $\exists x(S(x) \wedge P(x) \wedge \forall y(Gr(x,y) \rightarrow \neg G(y)))$.
   (c) $\exists x(S(x) \wedge \neg P(x) \wedge \exists y(\neg Gr(x,y) \vee \neg G(y)))$.

2. Which of the following formulas is not satisfiable?
   (a) $(\neg p \vee q) \wedge p \wedge \neg q$.
   (b) $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$.
   (c) $(p \wedge \neg q) \vee (q \wedge \neg r) \vee (r \wedge \neg p)$.

3. If A and B are subsets of U then:
   (a) Either A or B must be nonempty.
   (b) Neither $A \in B$, nor $B \in A$.
   (c) Both A and B are in $\mathcal{P}(U)$.

4. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a bijective function.
   (a) $f \circ f$ is injective but not necessarily surjective.
   (b) f^{-1} is not necessarily defined.
   (c) There is no function $g : \mathbb{N} \rightarrow \mathbb{R}$, such that $g \circ f$ is a one to one and onto.

5. Let $f(x) = \sqrt{5}x^5 - 7x^4 + x^2 + 8$.
   (a) f is $\Theta(\sqrt{x^5})$.
   (b) f is $O(x^5 \log x)$.
   (c) f is not $O(2^x)$.

II. (15 points.) True or False? Explain why!

1. "p is necessary for q to be true" means that $p \to q$.

   Answer: False! If p is necessary for q to be true then whenever q is true, p must also be true. So $q \to p$ is the relationship expressed by the statement. p might not be sufficient! For example "I will be buy a new computer if my old computer breaks and I have enough money". Both "My old computer broke" and "I have enough money" are necessary conditions for "I will buy a new computer", but neither is sufficient.

2. The set of all words in the English language is countable.

   Answer: True! There are finitely many words in the the English language, and finite sets are countable.

3. The recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 1$ is satisfied by a sequence starting with $-2, 1, -1, -2$.

   Answer: True! If $a_0 = -2$, $a_1 = 1$, then $a_3 = 2.a_1 + a_0 - 1 = 2 - 2 - 1 = -1$, $a_4 = 2.a_2 + a_1 - 1 = -2 + 1 - 1 = -2$.

4. The formula $\forall y \exists x_1 \exists x_2 (y = x_1^2 x_2)$ interpreted in the reals expresses the fact that every real number can be represented as the product of two different numbers one of which is positive.

   Answer: False! Nothing in the formulas implies that $x_1^2 \neq x_2$, also x_2^1 can be 0.

5. If A is not countable then neither is its powerset $\mathcal{P}(A)$.

   Answer: True! There is an injective function $f : A \to \mathcal{P}(A)$ given by $f(x) = \{x\}$. Hence $|\mathcal{P}(A)| > |A| > |\mathbb{N}|$, and so $\mathcal{P}(A)$ is not countable.

III. (20 points.) For each of the following formulas determine if it is a validity or not. Verify your answer.

1. $\forall x(P(x) \leftrightarrow Q(x) \wedge R(x)) \leftrightarrow (\forall x P(x) \leftrightarrow \forall x Q(x) \wedge \forall x R(x))$.

   Answer: This is not a validity. To see this consider the natural numbers and let $P(x)$ be true if $x = 3k$ for some k, $Q(x)$ if $x = 3k + 1$ for some k and $R(x)$ if $x = 3k + 2$ for some k. Then $\forall x(P(x) \leftrightarrow Q(x) \wedge R(x))$ is false, as can be seen when one takes $x = 3$. On the other hand each of $\forall x P(x)$, $\forall x Q(x)$, and $\forall x R(x)$ is false (e.g. it is not true that all natural number are divisible by 3) . This means that $(\forall x P(x) \leftrightarrow \forall x Q(x) \wedge \forall x R(x))$ is true, because it is if the form $F \leftrightarrow F$. The whole statement is now $F \leftrightarrow T$ which has value false and so the statements is not a validity.

2. $\neg(\exists x P(x) \to \exists x Q(x)) \to \exists x P(x) \wedge \forall x \neg Q(x)$. *Answer:* This is a validity. We will use a sequence of equivalences based on well established rules.

   By the rule for implication: $p \to q \equiv \neg p \vee q$ we see that the statement is equivalent to

   $$\neg\neg(\exists x P(x) \to \exists x Q(x)) \vee (\exists x P(x) \wedge \forall x \neg Q(x))$$

   Next we use the rule for double negation to get:

   $$(\exists x P(x) \to \exists x Q(x)) \vee (\exists x P(x) \wedge \forall x \neg Q(x))$$

   We use the implication rule again to get :

   $$\neg(\exists x P(x)) \vee \exists x Q(x) \vee (\exists x P(x) \wedge \forall x \neg Q(x))$$

   We use the rule for negating quantifies to get:

   $$\neg(\exists x P(x)) \vee \neg\forall x \neg Q(x) \vee (\exists x P(x) \wedge \forall x \neg Q(x))$$

   We use DeMorgans law to get:

   $$\neg((\exists x P(x)) \wedge \forall x \neg Q(x)) \vee (\exists x P(x) \wedge \forall x \neg Q(x))$$

   Finally we use the rule that says that $\neg p \vee p \equiv T$ to conclude that this is a vlidity.

IV. (20 points.) Let $f : \mathbb{R} \to \mathbb{Z}$ be defined as $f(x) = \lfloor x \rfloor * 2 - 1$.

1. Compute the sets $f([2,7])$ (here $[2,7]$ is the closed interval of real numbers with endpoints 2 and 7) and $f^{-1}(\{2,3,4\})$.

   Answer: $f([2,7]) = \{3,5,7,9,11,13\}$. This is because we can break up the interval $[2,7] = [2,3) \cup [3,4) \cup [4,5) \cup [5,6) \cup [6,7) \cup \{7\}$. For every point $x \in [n, n+1)$ we have that $f(x) = n * 2 - 1$. So when $n = 2,3,4,5,6,7$ we get the corresponding elements $3,5,7,9,11,13$.

   $f^{-1}(\{2,3,4\})$ is the interval $[2,3)$. No element in the range of f is divisible by 2, so $2,4$ do not contribute anything, 3 as we saw above contributes exactly the interval $[2,3)$, because for all $x \in [2,3)$ we have that $f(x) = 3$.

2. Is f injective, surjective, or bijective? Does f have an inverse? Verify your answer.

   Answer: The function f is not injective because for instance it maps 1 and 1.5 (and any other point in $[1, 2)$) to the same number, namely 1. The function f is not surjective because, as we already saw, no element in its range is divisible by 2 and so 4, for instance, is not in the range of f.

   f does not have an inverse, because only bijective, i.e. injective and surjective, functions have inverses and f is neither.

3. Compute $f \circ g$ where $g(x) = \lceil x \rceil$.

   Answer: $f \circ g(x) = f(g(x)) = f(\lceil x \rceil) = \lfloor \lceil x \rceil \rfloor * 2 - 1 = \lceil x \rceil * 2 - 1$.

V. (20 points.) Prove or disprove each of the following

1. $A \cup (X \setminus (B \cup C)) = (A \cup (X \setminus B)) \cap (A \cup (X \setminus C))$.

   Answer: True: For an arbitrary x we have that

$$x \in A \cup (X \setminus (B \cup C)) \text{ if and only if } x \in A \vee x \in X \setminus (B \cup C)$$

$$\text{if and only if } x \in A \vee x \in X \wedge x \notin (B \cup C)$$

$$\text{if and only if } x \in A \vee x \in X \wedge \neg(x \in B \vee x \in C)$$

$$\text{if and only if } x \in A \vee x \in X \wedge x \notin B \wedge x \notin C$$

$$\text{if and only if } x \in A \vee (x \in X \wedge x \notin B) \wedge (x \in X \wedge x \notin C)$$

$$\text{if and only if } (x \in A \vee (x \in X \wedge x \notin B)) \wedge (x \in A \vee x \in X \wedge x \notin C)$$

$$\text{if and only if } (x \in A \vee x \in X \setminus B) \wedge (x \in A \vee x \in X \setminus C)$$

$$\text{if and only if } (x \in A \cup X \setminus B) \wedge (x \in A \cup X \setminus C)$$

$$\text{if and only if } x \in (A \cup X \setminus B) \cap (A \cup X \setminus C).$$

   It follows that the two sets are equal.

2. $\mathcal{P}(A \times (B \cap C)) = \mathcal{P}(A \times B \cap A \times C)$.

   Answer: True: We will show first that $A \times (B \cap C) = A \times B \cap A \times C$. It then follows that the powerset of these two equal sets is the same set. Let $x = (u, v)$ be an arbitrary ordered pair.

$$(u, v) \in A \times (B \cap C)) \text{ if and only if } u \in A \wedge v \in B \cap C$$

$$\text{if and only if } u \in A \wedge (v \in B \wedge v \in C)$$
$$\text{if and only if } u \in A \wedge v \in B \wedge u \in A \wedge v \in C$$
$$\text{if and only if } (u, v) \in A \times B \wedge (u, v) \in A \times C$$
$$\text{if and only if } (u, v) \in A \times B \cap A \times C.$$

It follows that the two sets are equal.

VI. (10 points.) Consider the following matrices:

$$A = \begin{bmatrix} 4 & -2 & 5 \\ 2 & 3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1. Compute each of the following terms that are well defined: $A.B$, $B.A$, A^t, $C + D$.

   Answer: $A.B = \begin{bmatrix} 7 & 5 \\ 5 & 0 \end{bmatrix}$, $B.A = \begin{bmatrix} 4 & -2 & 5 \\ 4 & -2 & 5 \\ 6 & 1 & 5 \end{bmatrix}$ $A^t = \begin{bmatrix} 4 & 2 \\ -2 & 3 \\ 5 & 0 \end{bmatrix}$, $C + D = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

2. Compute each of the following terms that are well defined: $C \wedge D$, $B \vee C$, $C^{[81]}$, $B \odot D$.

   Answer: $C \wedge D = D$, $B \vee C$ is not defined, $B \odot D$ is not defined.

   $C^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $C^{[3]} = C^{[2]} \odot C = C^{[2]}$. It follows that for all $k \geq 2$ we have that $C^{[k]} = C^{[2]}$. So, in particular $C^{[81]} = C^{[2]}$.