Math 240/CS 240 Spring 2018 — February 20, 2018

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I. (15 points.) Each question has only one correct answer. Circle the one you think is correct.

1. Which of the following correctly formalizes the negation of the statement "Every student who practiced for the exam got a good grade"?
   (a) $\forall x(S(x) \wedge P(x) \rightarrow \exists y(Gr(x,y) \wedge G(y)))$.
   (b) $\exists x(S(x) \wedge P(x) \wedge \forall y(Gr(x,y) \rightarrow \neg G(y)))$.
   (c) $\exists x(S(x) \wedge \neg P(x) \wedge \exists y(\neg Gr(x,y) \vee \neg G(y)))$.

2. Which of the following formulas is not satisfiable?
   (a) $(\neg p \vee q) \wedge p \wedge \neg q$.
   (b) $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$.
   (c) $(p \wedge \neg q) \vee (q \wedge \neg r) \vee (r \wedge \neg p)$.

3. If A and B are subsets of U then:
   (a) Either A or B must be nonempty.
   (b) Neither $A \in B$, nor $B \in A$.
   (c) Both A and B are in $\mathcal{P}(U)$.

4. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a bijective function.
   (a) $f \circ f$ is injective but not necessarily surjective.
   (b) f^{-1} is not necessarily defined.
   (c) There is no function $g : \mathbb{N} \rightarrow \mathbb{R}$, such that $g \circ f$ is a one to one and onto.

5. Let $f(x) = \sqrt{5}x^5 - 7x^4 + x^2 + 8$.
   (a) f is $\Theta(\sqrt{x^5})$.
   (b) f is $O(x^5 \log x)$.
   (c) f is not $O(2^x)$.

II. (15 points.) True or False? Explain why!

1. "p is necessary for q to be true" means that $p \to q$.
2. The set of all words in the English language is countable.
3. The recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 1$ is satisfied by a sequence starting with $-2, 1, -1, -2$.
4. The formula $\forall y \exists x_1 \exists x_2 (y = x_1^2 x_2)$ interpreted in the reals expresses the fact that every real number can be represented as the product of two different numbers one of which is positive.
5. If A is not countable then neither is its powerset $\mathcal{P}(\mathcal{A})$.

III. (20 points.) For each of the following formulas determine if it is a validity or not. Verify your answer.

1. $\forall x(P(x) \leftrightarrow Q(x) \wedge R(x)) \leftrightarrow (\forall x P(x) \leftrightarrow \forall x Q(x) \wedge \forall x R(x))$.
2. $\neg(\exists x P(x) \to \exists x Q(x)) \to \exists x P(x) \wedge \forall x \neg Q(x)$.

IV. (20 points.) Let $f : \mathbb{R} \to \mathbb{Z}$ be defined as $f(x) = \lfloor x \rfloor * 2 - 1$.

1. Compute the sets $f([2,7])$ (here $[2,7]$ is the closed interval of real numbers with endpoints 2 and 7) and $f^{-1}(\{2,3,4\})$.
2. Is f injective, surjective, or bijective? Does f have an inverse? Verify your answer.
3. Compute $f \circ g$ where $g(x) = \lceil x \rceil$.

V. (20 points.) Prove or disprove each of the following

1. $A \cup (X \setminus (B \cup C)) = (A \cup (X \setminus B)) \cap (A \cup (X \setminus C))$.
2. $\mathcal{P}(A \times (B \cap C)) = \mathcal{P}(A \times B \cap A \times C)$.

VI. (10 points.) Consider the following matrices:

$$A = \begin{bmatrix} 4 & -2 & 5 \\ 2 & 3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1. Compute each of the following terms that are well defined: $A.B$, $B.A$, A^t, $C + D$.
2. Compute each of the following terms that are well defined: $C \wedge D$, $B \vee C$, $C^{[81]}$, $B \odot D$.