An enumeration degree perspective for effective mathematics

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Abstract

- The *Turing degrees* measure the computability-theoretic complexity of elements of 2^ω (or ω^ω).
- We can code other mathematical objects as binary sequences and use the Turing degrees to measure their complexity.
- However, this does not always lead to a coherent measure of complexity; there may not be a "canonical" coding.
- The *enumeration degrees*, a natural extension of the Turing degrees, work in some circumstances where the Turing degrees fail.
- E.g., the enumeration degrees can measure the complexity of continuous functions $f\colon [0,1] \to \mathbb{R}$. In fact, we get a proper subclass of the enumeration degrees: the *continuous degrees*.
- A larger subclass, the *cototal degrees*, arises naturally in symbolic dynamics and computable structure theory.

Example: every real number has a Turing degree

In computable analysis, coding is done via *names*.

Definition

$\lambda\colon \mathbb{Q}^+ \to \mathbb{Q}$ is a *name* of a real $x \in \mathbb{R}$ if for all rationals $\varepsilon > 0$ we have $|\lambda(\varepsilon) - x| < \varepsilon$.

Names can be easily coded as binary sequences, allowing us to transfer computability-theoretic notions to computable analysis. For example:

Definition

A function $f\colon \mathbb{R} \to \mathbb{R}$ is *computable* if there is a Turing functional that takes a name for any real $x \in \mathbb{R}$ to a name for $f(x)$.

- The binary expansion of a real x is computable from every name. (But this is nonuniform because of the dyadic rationals!)
- The binary expansion of x computes a name for x.
- This is the least Turing degree name for x; it is natural to take this as the *Turing degree* of x.

The Turing degrees are not always sufficient: Computable structure theory

Let $\mathcal{A}$ be a countable structure in a finite language L. A *presentation* of $\mathcal{A}$ is an isomorphic copy of $\mathcal{A}$ with universe ω.

Definition

The *degree spectrum* of a countable structure $\mathcal{A}$ is the collection $\mathrm{Spec}(\mathcal{A})$ of Turing degrees of presentations of $\mathcal{A}$.

When $\mathrm{Spec}(\mathcal{A})$ has a least element, we call it the *Turing degree* of $\mathcal{A}$.

Not all countable structures have a Turing degree.

Theorem (Richter 1977)

If a linear ordering $\mathcal{L}$ has Turing degree, then it is computable.

In this case, the enumeration degrees won't help much.

Torsion-free abelian groups of rank 1

A torsion free abelian group of rank 1 $\mathcal{G}$ is a subgroup of $(\mathbb{Q}, +)$.

- Baer introduced *the type* of a group $\mathcal{G}$, $\chi(\mathcal{G})$, in terms of divisibility properties of nonzero elements of $\mathcal{G}$.
- Two torsion free abelian groups of rank 1 are isomorphic if and only if they have the same type.
- There is a standard representation of $\chi(\mathcal{G})$ as a set of natural numbers $S(\mathcal{G})$, *the standard type* of $\mathcal{G}$.
- Every set of natural numbers can be associated with the standard type of some group $\mathcal{G}$.

Theorem (Downey, Jockusch 1997)

The degree spectrum of $\mathcal{G}$ is precisely $\{d_T(Y) \colon S(\mathcal{G}) \text{ is c.e. in } Y\}$.

Following Richter, not every set has a least Turing degree enumeration.

The enumeration degrees

Friedberg and Rogers introduced enumeration reducibility in 1959.

Informally: $A \subseteq \omega$ is *enumeration reducible* to $B \subseteq \omega$ ($A \leqslant_e B$) if there is a uniform way to compute an enumeration of A from an enumeration of B. (Selman proved that the uniformity condition can be dropped.)

Definition

$A \leqslant_e B$ if there is a c.e. set W such that

$$A = \{n \colon (\exists e)\, \langle n, e \rangle \in W \text{ and } D_e \subseteq B\},$$

where D_e is the eth finite set in a canonical enumeration.

The degree structure $\mathcal{D}_e$ induced by $\leqslant_e$ is called the *enumeration degrees*. It is an upper semi-lattice with a least element (the degree of all c.e. sets).

The total enumeration degrees

Proposition. $A \leqslant_T B$ iff $A \oplus \overline{A}$ is B-c.e. iff $A \oplus \overline{A} \leqslant_e B \oplus \overline{B}$.

This suggests a natural embedding of the Turing degrees into the enumeration degrees.

Proposition. The embedding $\iota\colon \mathcal{D}_T \to \mathcal{D}_e$, defined by

$$\iota(d_T(A)) = d_e(A \oplus \overline{A}),$$

preserves the order and the least upper bound (and even the jump).

Definition

A set A is *total* if $A \geqslant_e \overline{A}$ (or equivalently if $A \equiv_e A \oplus \overline{A}$). An enumeration degree is *total* if it contains a total set.

The image of the Turing degrees under the embedding ι is exactly the set of total enumeration degrees.

It is easy to see that there are nontotal enumeration degrees. In fact, every sufficiently generic set has non-total degree.

Back to Torsion-free abelian groups of rank 1

Theorem (Selman 1971)

$A \leqslant_e B$ if and only if $\{d_T(Y) \colon B \text{ is c.e. in } Y\} \subseteq \{d_T(Y) \colon A \text{ is c.e. in } Y\}$.

So every enumeration degree is determined by its enumeration cone, the set of total (Turing) degrees above it.

Theorem (Downey, Jockusch)

The degree spectrum of $\mathcal{G}$ is precisely $\{d_T(Y) \colon S(\mathcal{G}) \text{ is c.e. in } Y\}$.

So every group $\mathcal{G}$ can be characterized by the enumeration degree $d_e(S(\mathcal{G}))$.

And every enumeration degree can be thought of as the degree of some torsion free abelian group of rank 1 $\mathcal{G}$.

Part I: The Cototal Degrees

Symbolic dynamics

The *shift operator* on 2^ω is the map taking an infinite binary sequence $\alpha \in 2^\omega$ to the unique $\beta \in 2^\omega$ such that $\alpha = a\beta$ for some $a \in \{0, 1\}$, i.e., the operator that erases the first bit of the sequence.

Definition

- A *subshift* is closed, shift-invariant subspace X of 2^ω.
- The *degree spectrum* of a subshift X is the set $\mathrm{Spec}(X)$ of Turing degrees of elements of the subshift.
- X is a *minimal subshift* if no nonempty $Y \subset X$ is a subshift.

If $\mathrm{Spec}(X)$ has a least element, then it could be considered as *Turing degree* of the subshift X.

Theorem (Hochman, Vanier 2017)

There is a minimal subshift X with no member of least Turing degree.

The spectrum of a minimal subshift

Given a minimal subshift X, we would like to characterize the set of Turing degrees of members of X.

Definition

The *language* of subshift $X \subseteq 2^\omega$ is the set

$$L_X = \{\sigma \in 2^{<\omega} : (\exists \alpha \in X)\ \sigma \text{ is a subword of } \alpha\}.$$

1. If X is minimal and $\sigma \in L_X$, then for every $\alpha \in X$, σ is a subword of α. So every element of X can enumerate the set L_X.
2. If we can enumerate L_X, then we can compute a member of X.

Theorem (Jeandel)

A Turing degree $\mathbf{a}$ computes a member of the minimal subshift X if and only if $\mathbf{a}$ can enumerate L_X.

So the computability theoretic complexity of a minimal subshift X corresponds exactly to the enumeration degree of L_X.

The cototal enumeration degrees

Jeandel noticed something special about L_X for a minimal subshift X.

- An enumeration of $\overline{L_X}$ allows us to eliminate branches that do not belong to X in a stage by stage manner.
- If w is word that appears along every branch that remains at stage s, then $w \in L_X$.
- The compactness of 2^ω ensures that we won't miss any word from the language using this process of enumeration.

So $L_X \leqslant_e \overline{L_X}$.

Definition

A set A is *cototal* if $A \leqslant_e \overline{A}$. An enumeration degree is *cototal* if it contains a cototal set.

Fact. Every total enumeration degree is cototal: $\overline{A \oplus \overline{A}} \equiv_e A \oplus \overline{A}$.

Characterizations of the cototal enumeration degrees

Definition (Carl von Jaenisch 1862)

Let $G = (V, E)$ be a graph. A set $M \subseteq V$ is *independent*, if no two members of M are edge related. M is *maximal* set, if every $v \in \overline{M}$ is edge related to a vertex in M.

Note. $\overline{M} \leqslant_e M$ because $v \in \overline{M}$ if and only if there is a $w \in M$ such that w and v are edge related.

Theorem (Andrews, Ganchev, Kuyper, Lempp, Miller, A. Soskova, S.)

An enumeration degree is cototal if and only if it contains the complement of a maximal independent set for the graph $\omega^{<\omega}$.

Theorem (McCarthy 2018)

An enumeration degree is cototal if and only if it contains the complement of a maximal antichain in $\omega^{<\omega}$.

Cototal degrees and computable structure theory

Theorem (Montalbán 2018)

A degree spectrum of a structure is not the Turing-upward closure of an F_σ set of reals in ω^ω, unless it is an enumeration-cone.

In particular, it must be the cone above the enumeration degree of an e-pointed tree.

Definition

A tree $T \subseteq 2^{<\omega}$ is *e-pointed* if it has no dead ends and every infinite path $f \in [T]$ enumerates T.

Theorem (McCarthy 2018)

An enumeration degree is cototal if and only if it contains a (uniformly) e-pointed tree.

A degree spectrum is the Turing-upward closure of an F_σ set of reals in ω^ω if and only if it is the enumeration-cone of a cototal degree.

Characterizing the degrees of minimal subshifts

Recall

- If X is a minimal subshift, then its spectrum (i.e., set of Turing degrees of members of X) is the enumeration-cone above the enumeration degree of L_X, the language of X.
- L_X is a cototal set (Jeandel).

Theorem (McCarthy)

Every cototal enumeration degree is the degree of the language of a minimal subshift.

The cototal enumeration degrees as a substructure of the enumeration degrees

Definition (Lachlan, Shore 1992)

A uniformly computable sequence of finite sets $\{A_s\}_{s<\omega}$ is a *good approximation* to a set A if:

G1 $(\forall n)(\exists s)(A \upharpoonright n \subseteq A_s \subseteq A)$

G2 $(\forall n)(\exists s)(\forall t > s)(A_t \subseteq A \Rightarrow A \upharpoonright n \subseteq A_t)$.

An enumeration degree is *good* if it contains a set with a good approximation.

Theorem (Harris; Miller, S 2018)

The good enumeration degrees are exactly the cototal enumeration degrees.

Theorem (Miller, S 2018)

The cototal enumeration degrees are dense.

So the cototal enumeration degrees arose independently in symbolic dynamics, graph theory, computable structure theory, and in the enumeration degrees.

Part II: The Continuous Degrees

Computable metric spaces

Definition

A *computable metric space* is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n\in\omega}$ on which the metric is computable (as a function $\omega^2 \to \mathbb{R}$).

Example. The *Hilbert cube* is $[0,1]^\omega$ with the metric

$$d(\alpha, \beta) = \sum_{n\in\omega} |\alpha(n) - \beta(n)|/2^n.$$

Let $Q^{[0,1]^\omega}$ be the sequences of rationals in $[0,1]$ with finite support.

Other computable metric spaces include 2^ω, ω^ω, $\mathbb{R}$, and $\mathcal{C}[0,1]$.

Definition

$\lambda\colon \mathbb{Q}^+ \to \omega$ is a *name* of a point $x \in \mathcal{M}$ if for all rationals $\varepsilon > 0$ we have $d_{\mathcal{M}}(x, q^{\mathcal{M}}_{\lambda(\varepsilon)}) < \varepsilon$.

Points in computable metric spaces

As before, the complexity of a point in a metric space can be captured through the collection of Turing degrees of names of this point.

Question (Pour El and Lempp). Do elements of computable metric spaces have least Turing degree names?

Definition (Miller 2004)

If x and y are members of (possibly different) computable metric spaces, then $x \leqslant_r y$ if there is a uniform way to compute a name for x from a name for y.

This reducibility induces the *continuous degrees*.

Theorem (Miller 2004)

Every continuous degree contains a point from $[0,1]^\omega$ and a point from $C[0,1]$.

Embedding the continuous degrees into the e-degrees

For $\alpha \in [0,1]^\omega$, let

$$C_\alpha = \bigoplus_{i \in \omega} \{q \in \mathbb{Q} \colon q < \alpha(i)\} \oplus \{q \in \mathbb{Q} \colon q > \alpha(i)\}.$$

Observation. Enumerating C_α is exactly as hard as computing a name for α. So $\alpha \mapsto C_\alpha$ induces an embedding of the continuous degrees into the enumeration degrees.

- Elements of 2^ω, ω^ω, and $\mathbb{R}$ are mapped onto the *total* degree of their least Turing degree name (i.e., their Turing degree).
- A point $x \in \mathcal{M}$ has nontotal (enumeration) degree iff it has no least Turing degree name.
- Every continuous degree is mapped to a cototal enumeration degree: $q < \alpha(i)$ iff there is some $p \leqslant \alpha(i)$ such that $q < p$.

So the continuous enumeration degrees extend the total degrees and form a subclass of the cototal degrees.

Nontotal continuous degrees: quick proof

Theorem (Miller 2004)

There is a nontotal continuous degree.

Proof.

- If $x \in [0,1]^\omega$ has total degree, then there is a $y \in 2^\omega$ and Turing functionals Γ, Ψ that map (names of) x to (names of) y and back.
- The subspaces on which the functions induced by Γ and Ψ are inverses are homeomorphic (because computable functionals induce continuous functions).
- Subspaces of 2^ω are zero dimensional, so if $x \in [0,1]^\omega$ has total degree, then it is in one of *countably many* zero dimensional "patches".
- The Hilbert cube $[0,1]^\omega$ is strongly infinite dimensional, hence not a countable union of zero dimensional subspaces.
- So some $x \in [0,1]^\omega$ is not covered by one of these patches. □

Other proofs invoke Sperner's lemma or a variant of Brouwer's fixed point theorem to multivalued functions on an infinite dimensional space.

The structure of nontotal continuous degrees

Definition

A Turing degree **a** is *PA*

- if **a** computes a complete extension of Peano Arithmetic, or equivalently
- if **a** computes a path in every infinite computable tree.

The degree **a** is *PA above* **b** if **a** computes a path in every infinite **b**-computable tree.

Theorem (Miller 2004)

The total degrees below a nontotal continuous enumeration degree form a *Scott set*, an ideal closed under the relation “PA above”.

The total (Turing) degree **a** is *PA above* the total (Turing) degree **b** if and only if there is a nontotal continuous degree **c** such that $\mathbf{b} < \mathbf{c} < \mathbf{a}$.

Characterizing the continuous enumeration degrees

As it turns out, the contiunuous enumeration degrees have a very simple characterization inside the enumeration degrees.

Definition

An enumeration degree $\mathbf{a}$ is *almost total* if whenever $\mathbf{b} \nleq \mathbf{a}$ is total, $\mathbf{a} \vee \mathbf{b}$ is also total.

In other words, an enumeration degree is almost total if adding any new total information takes it to a total degree.

Note. The join of any two total degrees is total, so total degrees are almost total.

Are there nontotal almost total degrees?

Continuous degrees are almost total

Are there nontotal almost total degrees? Yes!

Note. If $\beta \in [0,1]^\omega$ contains no dyadic rationals, then β is equivalent the join of the binary expansions of its coordinates, which has total degree.

Fact (Cai, Lempp, Miller, S. 2014 (unpublished)). Continuous enumeration degrees are almost total.

Proof. Take $\alpha \in [0,1]^\omega$ and $x \in [0,1]$ such that $x \not\leqslant_r \alpha$. Define $\beta \in [0,1]^\omega$ by $\beta(n) = (\alpha(n) + x)/2$. Note that

- No component of β is rational, so β has total degree.
- $\alpha \oplus x \equiv_r \beta \oplus x$, hence it is also total. □

There are nontotal continuous degrees, so there are nontotal almost total degrees. This is the only way we know how to produce nontotal almost total degrees. (In particular, we have no "direct" construction.)

Almost total degrees are continuous

Theorem (Andrews, Igusa, Miller, S.)

Almost total degrees are continuous.

We used a series of implications:

$$\begin{aligned}\text{Almost total} &\implies \text{Uniformly codable} \\ &\implies \text{Contains a holistic set} \\ &\implies \text{Continuous.}\end{aligned}$$

- All known constructions of nontotal continuous degrees involve a nontrivial topological component.
  - Conversely, the fact that the Hilbert cube is not a countable union of subspaces of Cantor space follows easily from the fact that there is a nontotal continuous degrees in every cone.

So a purely topological fact is reflected in the structure of the enumeration degrees.

Uniform Codability and Holistic sets

Definition. Let $A \subseteq \omega$. Call $U \subseteq 2^\omega$ a *$\Sigma^0_1\langle A\rangle$ class* if there is a set of strings $W \leqslant_e A$, such that

$$U = [W] = \{X \in 2^\omega : (\exists \sigma \in W)\ X \succeq \sigma\}.$$

A *$\Pi^0_1\langle A\rangle$ class* is the complement of a $\Sigma^0_1\langle A\rangle$ class.

Definition. $A \subseteq \omega$ is *codable* if there is a nonempty $\Pi^0_1\langle A\rangle$ class P such that every $X \in P$ enumerates A. If there is a c.e. operator W such that $A = W^X$ for every $X \in P$, then A is *uniformly codable*.

Definition. $S \subseteq \omega^{<\omega}$ is *holistic* if for every $\sigma \in \omega^{<\omega}$,

1. $(\forall n)\ \sigma^\frown(2n)$ and $\sigma^\frown(2n+1)$ are not both in S,
2. If $\sigma \in S$, then $(\exists n)\ \sigma^\frown(2n+1) \in S$.
3. If $\sigma \notin S$, then $(\forall n)\ \sigma^\frown(2n) \in S$,

Definability in the enumeration degrees

Theorem (Cai, Ganchev, Lempp, Miller, and S. 2016). The total degrees are first order definable in the enumeration degrees (as a partial order).

The definition is "natural". It builds on work of Kalimullin (2003) and Ganchev and S. (2015).

Corollary (AIMS). The continuous degrees are first order definable in the enumeration degrees.

Corollary (AIMS). The relation "PA above" between total degrees is first order definable in the enumeration degrees.

This relation is not known to be definable in the Turing degrees.

To finish: a unifying example

Kihara and Pauly assigned (enumeration) degrees to points in any second countable T_0 topological space.

Definition (Kihara and Pauly)

If $\mathcal{X}$ has countable basis $\{N_i\}_{i\in\omega}$, then the degree of $x \in \mathcal{X}$ is the enumeration degree of $\{i : x \in N_i\}$.

The point degree spectrum of a topological space $\mathcal{X}$ (i.e., the degrees of points $\mathcal{X}$) is a subclass of the enumeration degrees.

1. $\mathrm{Spec}(2^\omega) = \mathrm{Spec}(\omega^\omega) = \mathrm{Spec}(\mathbb{R}) = \mathcal{D}_T$;
2. $\mathrm{Spec}([0,1]^\omega) = \mathrm{Spec}(C[0,1]) = \mathcal{D}_r$;
3. $\mathrm{Spec}(S^\omega) = \mathcal{D}_e$, where $S = \{\varnothing, \{0\}, \{0,1\}\}$ is the Sierpinski space.

Theorem (Kihara)

If $\mathcal{X}$ is a sufficiently effective, second countable G_δ space (i.e., every closed set is G_δ), then every point in $\mathcal{X}$ has cototal degree. Conversely, all cototal degrees arise in this way.

Thank you!