Cupping Δ_2 Enumeration Degrees to $0'$

Guohua Wu Mariya I. Soskova

School of Physical and Mathematical Sciences
Nanyang Technological University,
guohua@ntu.edu.sg

Department of Pure Mathematics
University of Leeds
mariya@maths.leeds.ac.uk

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Definitions

Definition

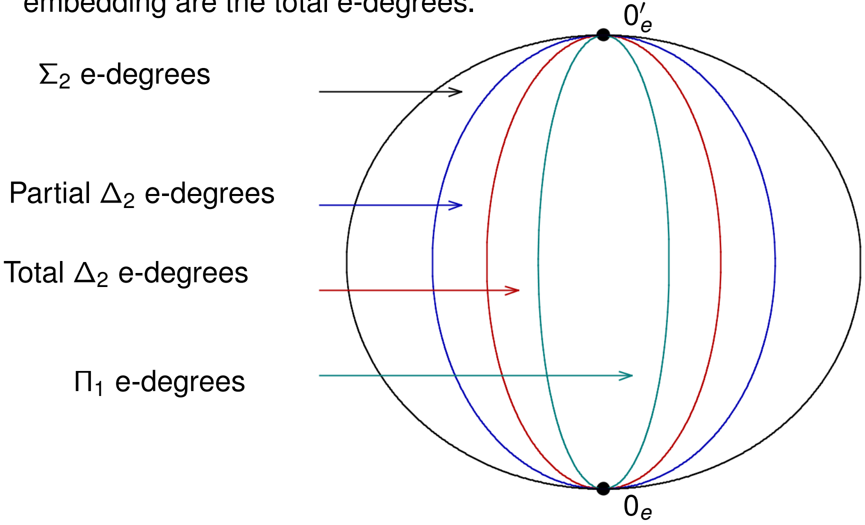
1. A set A is enumeration reducible to a set B ($A \leq_e B$), if there is a c.e. set Φ such that

$$n \in A \Leftrightarrow \exists D(\langle n, [D]\rangle \in \Phi \wedge D \subseteq B).$$

2. A is enumeration equivalent to B ($A \equiv_e B$) if $A \leq_e B$ and $B \leq_e A$.
3. Let $d_e(A) = \{B | A \equiv_e B\}$.
4. $(D_e, <, \cup, ', 0_e)$ is the semi-lattice of the enumeration degrees.

Local Degree Structure

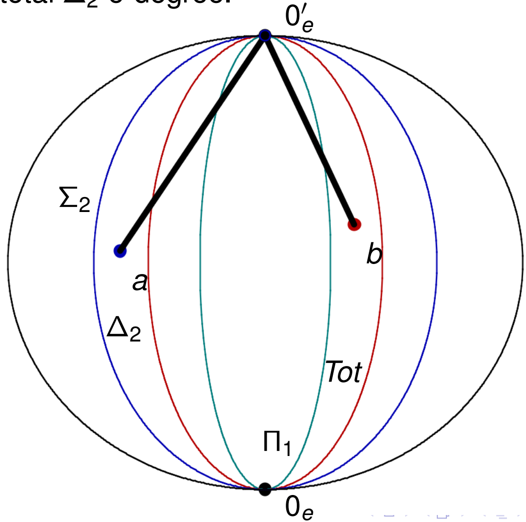
There is a natural embedding of the Turing degrees in the Enumeration degrees. Images of Turing degrees under this embedding are the total e-degrees.



Cupping

We say that a degree $\mathbf{a}$ is cuppable if there exists a degree $\mathbf{b} < \mathbf{0}'_{\mathbf{e}}$ such that $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'_{\mathbf{e}}$.

Cooper Sorbi and Yui proved that every nonzero Δ_2 e-degree is cuppable by a total Δ_2 e-degree.



Generic Sets

Definition

A set A is generic if for every c.e. set W there exists a finite string $\lambda \subset \chi_A$ such that:

$$\lambda \in W \vee (\forall \mu \supseteq \lambda)(\mu \notin W).$$

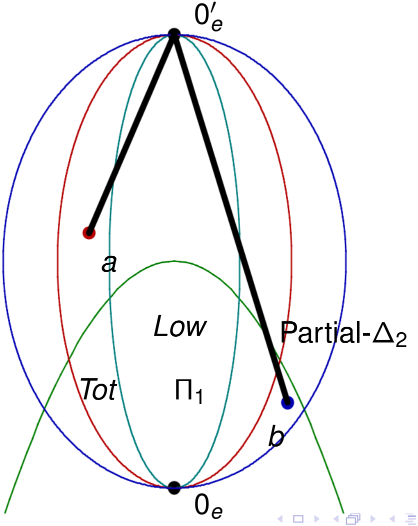
Degrees of generic sets are called generic degrees.

- Every generic enumeration degree **a** is quasiminimal, hence partial.
- Copestake proved that generic degrees are low if and only if they are Δ_2.

Theorem 1

Theorem

Every nonzero Δ_2 enumeration degree **a** *can be cupped by a Δ_2 generic e-degree* **b**, *hence by a partial low degree.*



Requirements

Given a nonzero Δ_2 set A we will construct a Δ_2-set B such that:

$$S : \Gamma^{A,B} = \overline{K}.$$

$$G_i : (\exists \lambda \subset B)(\lambda \in W_i \vee \forall \mu \supseteq \lambda[\mu \notin W_i]).$$

The S-strategy

$$S : \Gamma^{A,B} = \overline{K}.$$

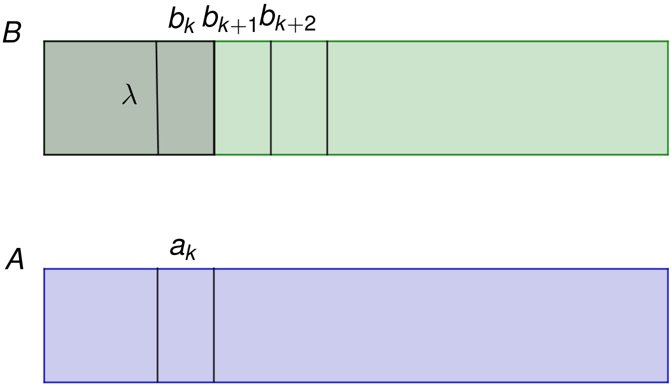
The S-strategy runs at the beginning of every stage and constructs an e-operator Γ such that:

- For every $n \in \overline{K}$ there is a valid axiom $\langle n, A \upharpoonright a_n, B \upharpoonright b_n \rangle$.
- If n exits $\overline{K}$ we correct Γ by extracting b_n from B.

The G -strategy

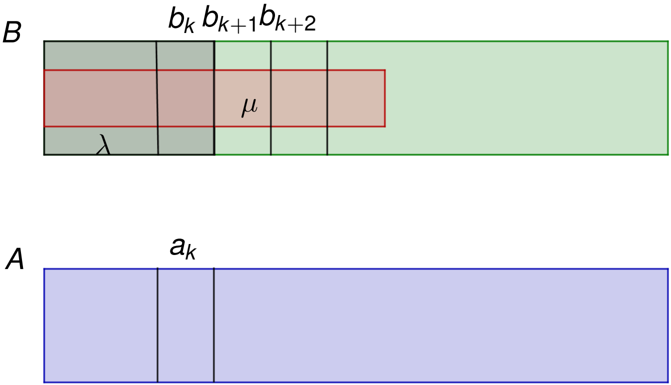
$$G_i : (\exists \lambda \subset B)(\lambda \in W_i \vee \forall \mu \supseteq \lambda[\mu \notin W_i]).$$

The G-strategy will select a threshold k. Choose a witness $\lambda = B \restriction b_k$. Wait for $\mu \supseteq \lambda$ to enter W.



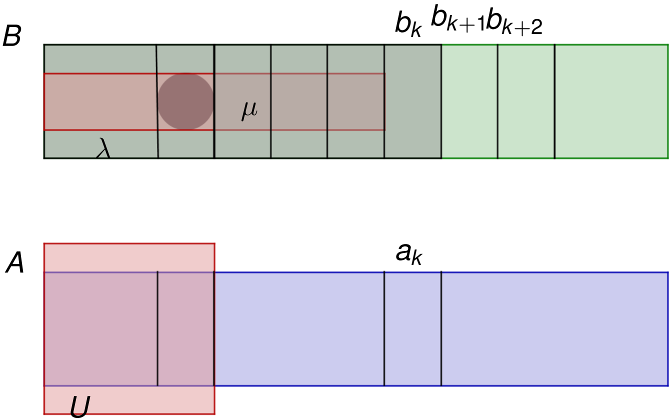
Conflict

G would like to preserve μ as an initial segment of B, meanwhile S might like to change B to rectify Γ.



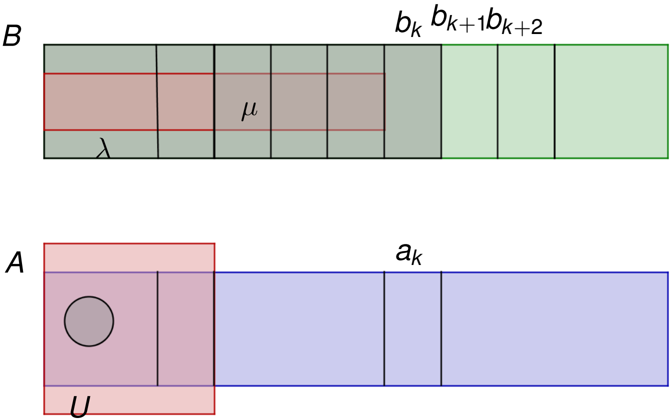
Solution

Extract the marker b_k to prevent S from injuring the restraint. Approximate A up to a_k threatening to prove that it is c.e. Start a new cycle.



A-retreat

If there is an A-change, restore B. Now $\mu \subseteq B$.
A is nonzero and Δ_2 hence there will be a permanent change in A eventually.



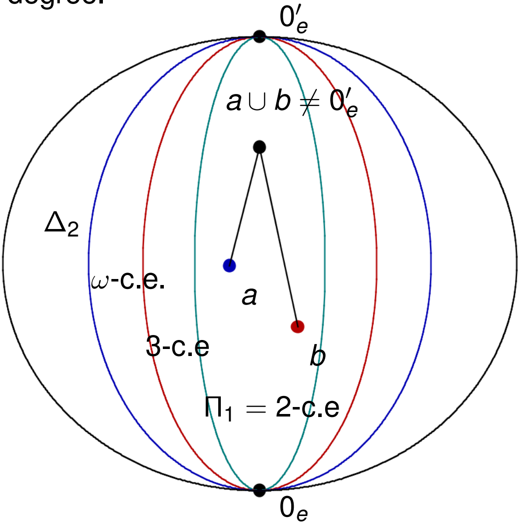
Definitions

Definition

1. A set A is n-c.e. if there is a computable function f such that for each x, $f(x,0)=0$, $|\{s+1 \mid f(x,s) \neq f(x,s+1)\}| \leq n$ and $A(x)=\lim_s f(x,s)$.
2. A is ω-c.e. if there are two computable functions $f(x,s), g(x)$ such that for all x, $f(x,0)=0$, $|\{s+1 \mid f(x,s) \neq f(x,s+1)\}| \leq g(x)$ and $\lim_s f(x,s) \downarrow = A(x)$.
3. A degree **a** is n-c.e.(ω-c.e.) if it contains a n-c.e.(ω-c.e.) set.

A noncuppable c.e. degree

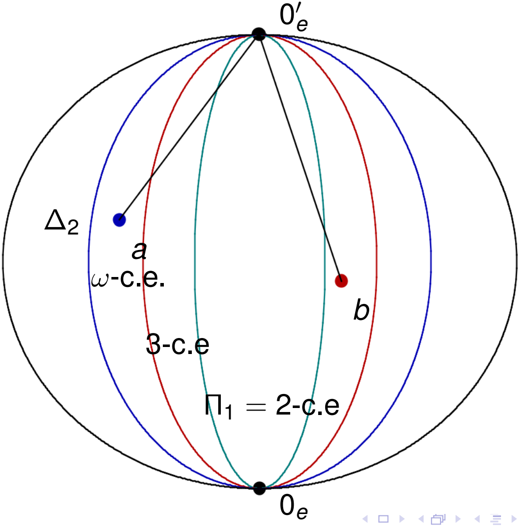
Cooper and Yates proved that there is a noncuppable c.e. Turing degree. Hence a 2-c.e. e-degree that cannot be cupped by any 2-c.e. e-degree.



Theorem 2

Theorem

Given a nonzero ω-c.e. e-degree $\mathbf{a}$*, there is a* 3*-c.e. e-degree* $\mathbf{b}$ *such that* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'_\mathbf{e}$*.*



Requirements

Given a nonzero ω-c.e set A we will construct a 3-c.e. set B and an extra Π_1 set E such that:

$$S : \Gamma^{A,B} = \overline{K}.$$

$$N_i : E \neq \Psi_i^B.$$

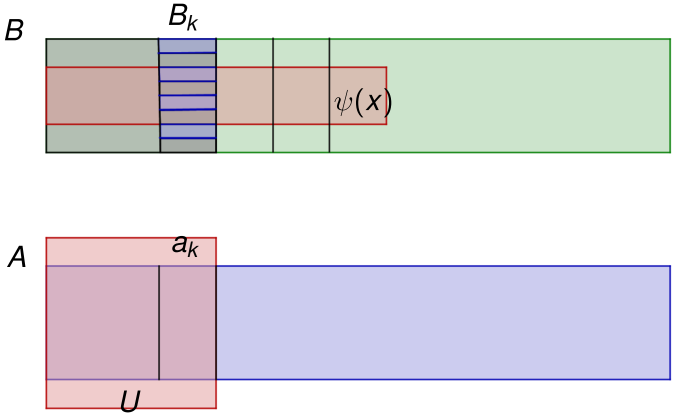
The N-strategy

$$N_i : E \neq \Psi_i^B.$$

- Choose a threshold k and a witness $x > k$.
- Wait for x to enter Ψ^B.
- Approximate $A \upharpoonright a_k$ and extract b_k. Start a new cycle.
- If there is an A-change re-enumerate b_k to restore the initial segment of B.

New tricks

- Sets of markers - if n has A-marker a_n then it has a B-marker B_n a set of size $\sum_{x<a_n} g(x)$.
- Make the approximations of the set A monotone and always restore the last computation.



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