An automorphism analysis for the Δ^0_2 Turing degrees

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joint work with Theodore Slaman

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2015 Spring Eastern Sectional Meeting, Georgetown University

[1] Supported by a Marie Curie International Outgoing Fellowship STRIDE (298471).

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2. Understanding the definable relations in the structure of the Turing degrees.
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Automorphism bases

Definition

Let $\mathcal{A}$ be a structure. A set $B \subseteq |\mathcal{A}|$ is an automorphism base for $\mathcal{A}$ if whenever f and g are automorphisms of $\mathcal{A}$ such that $(\forall x \in B)(f(x) = g(x))$, then $f = g$.

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There is an element $\mathbf{g} \leq \mathbf{0}^{(5)}$ *such that* $\{\mathbf{g}\}$ *is an automorphism base for the structure of the Turing degrees* $\mathcal{D}_T$.

$Aut(\mathcal{D}_T)$ *is countable and every member has an arithmetically definable presentation.*

Every relation induced by a degree invariant definable relation in Second order arithmetic is definable with parameters.

Local structure of the Turing degrees

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Question

Can we show that $\mathcal{D}_T(\leq \mathbf{0}')$ relates to first order arithmetic in the same way that $\mathcal{D}_T$ relates to second order arithmetic?

The local coding theorem

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A set of degrees $\mathcal{Z}$ contained in $\mathcal{D}_T(\leq \mathbf{0}')$ is *uniformly low* if it is bounded by a low degree and there is a sequence $\{Z_i\}_{i<\omega}$, representing the degrees in $\mathcal{Z}$, and a computable function f such that $\{f(i)\}^{\emptyset'}$ is the Turing jump of $\bigoplus_{j<i} Z_j$.

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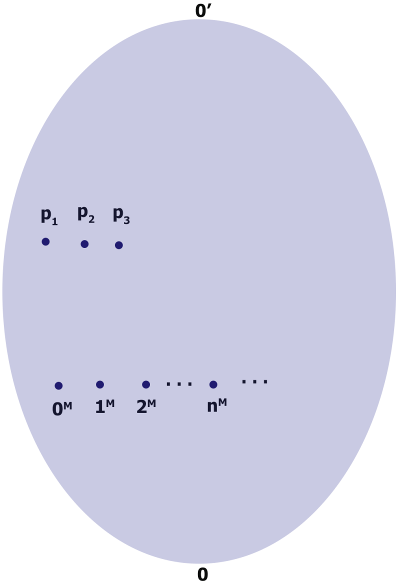
Theorem (Slaman and Woodin)

If $\mathcal{Z}$ is a uniformly low subset of $\mathcal{D}_T(\leq \mathbf{0}')$ then $\mathcal{Z}$ is definable from finitely many parameters in $\mathcal{D}_T(\leq \mathbf{0}')$.

Applications of the coding theorem



Using parameters we can code a model of arithmetic $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, 0^{\mathcal{M}}, s^{\mathcal{M}}, +^{\mathcal{M}}, \times^{\mathcal{M}}, \leq^{\mathcal{M}})$.

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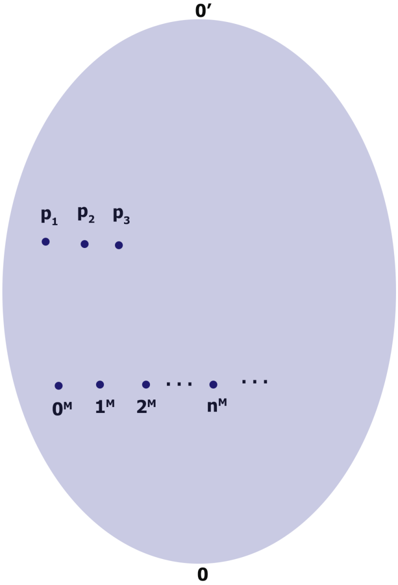

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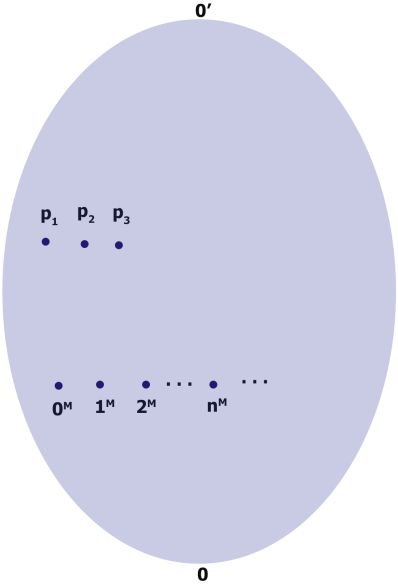

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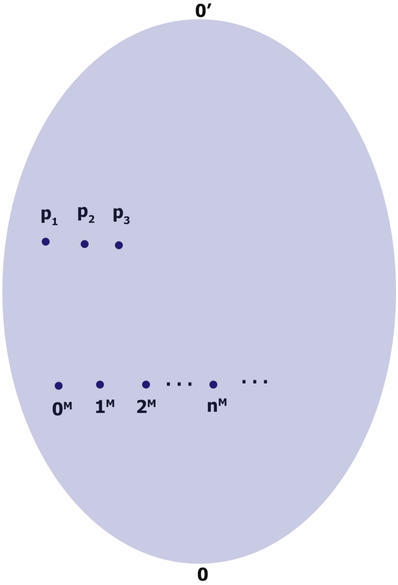

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2. The graphs of s, $+$, $\times$ and the relation $\leq$ are definable with parameters $\vec{\mathbf{p}}$.
3. $\mathbb{N} \models \varphi$ iff $\mathcal{D}_T(\leq \mathbf{0}') \models \varphi_T(\vec{\mathbf{p}})$

Applications of the coding theorem

If $\mathcal{Z} \subseteq \mathcal{D}_T(\leq \mathbf{0}')$ is uniformly low and represented by the sequence $\{Z_i\}_{i<\omega}$ then there are parameters that code a model of arithmetic $\mathcal{M}$ and a function $\varphi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq \mathbf{0}')$ such that $\varphi(i^{\mathcal{M}}) = d_T(Z_i)$.

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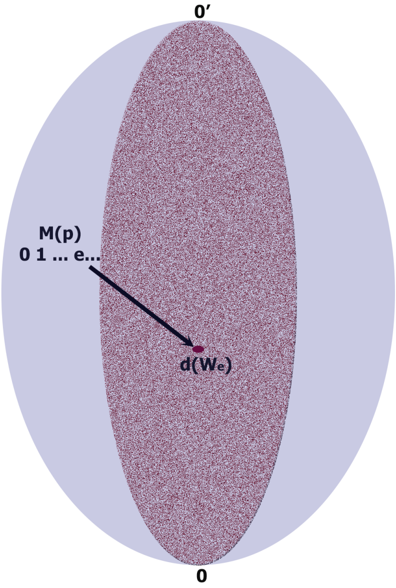
Theorem (Slaman and Woodin)

There are finitely many Δ^0_2 parameters which code a model of arithmetic $\mathcal{M}$ and an indexing of the c.e. degrees: a function $\psi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq \mathbf{0}')$ such that $\psi(e^{\mathcal{M}}) = d_T(W_e)$.

An indexing of the c.e. degrees



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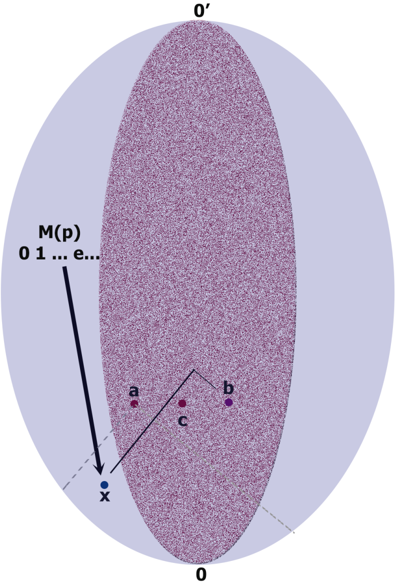

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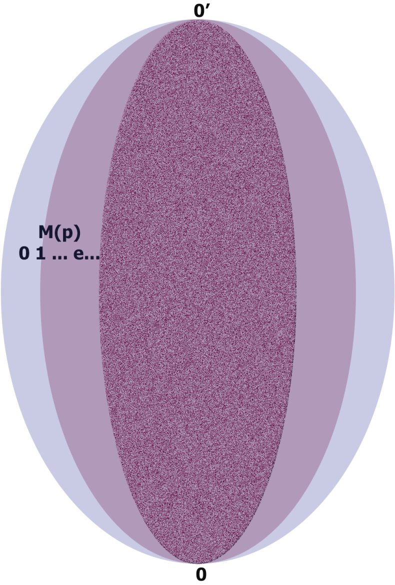

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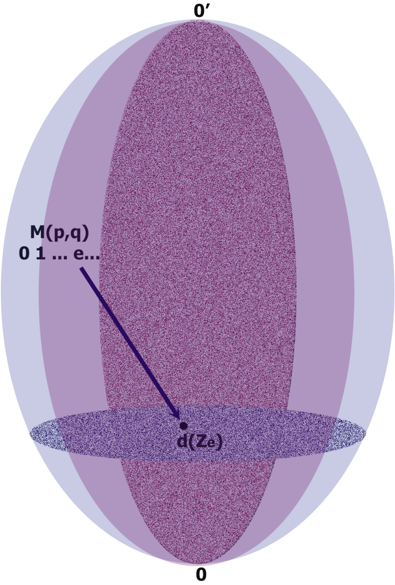

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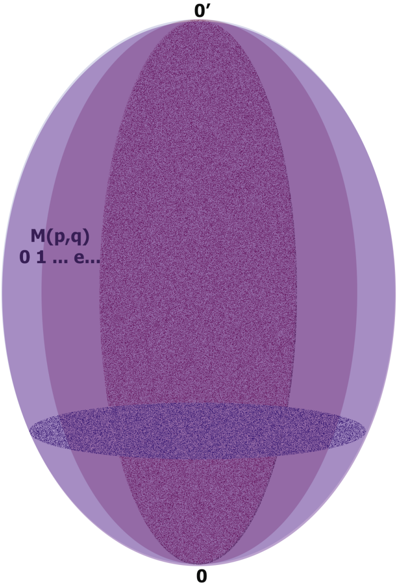

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   If $\mathbf{x}, \mathbf{y} \leq \mathbf{0}'$, $\mathbf{x}' = \mathbf{0}'$ and $\mathbf{y} \not\leq \mathbf{x}$ then there are $\mathbf{g}_i \leq \mathbf{0}'$, c.e. degrees $\mathbf{a}_i$ and Δ^0_2 degrees $\mathbf{c}_i, \mathbf{b}_i \in \mathcal{Z}$ for $i = 1, 2$ such that:

   1. $\mathbf{g}_i$ is the least element below $\mathbf{a}_i$ which joins $\mathbf{b}_i$ above $\mathbf{c}_i$.
   2. $\mathbf{x} \leq \mathbf{g}_1 \vee \mathbf{g}_2$.
   3. $\mathbf{y} \not\leq \mathbf{g}_1 \vee \mathbf{g}_2$.

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Then π fixes every degree that represents a natural number in $\mathcal{M}$.

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Hence π fixes every Δ^0_2 Turing degree.

Corollary

The automorphism group of $\mathcal{D}_T(\leq \mathbf{0}')$ is countable.

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The automorphism group of $\mathcal{D}_T(\leq \mathbf{0}')$ is countable.
Every automorphism of $\mathcal{D}_T(\leq \mathbf{0}')$ has an arithmetic presentation.

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The formula for the tuple $(\mathbf{x}_1 \dots \mathbf{x}_n)$ expresses this relationship: whenever $\vec{\mathbf{p}}$ are parameters that define a standard model of arithmetic and a bijection θ that respects the constraints of an indexing, θ maps $e_i^{\mathcal{M}}$ to $\mathbf{x}_i$.

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Proof: Let $\mathcal{M}$ be a model of arithmetic and φ an indexing of the Δ^0_2 degrees.
We define the set of all $(\varphi(e_1^{\mathcal{M}}), \dots, \varphi(e_n^{\mathcal{M}}))$ where $\mathcal{M} \models R(e_1, \dots e_n)$.

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We use the same trick as in the previous application to define $\mathcal{R}$ in the second case.

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Proof: $\mathcal{D}_T(\leq \mathbf{0}')$ is biinterpretable with first order arithmetic if there are a definable model of arithmetic and indexing of the Δ^0_2 degrees.

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If $\mathcal{D}_T(\leq \mathbf{0}')$ is rigid then the tuple of the finitely many indexing parameters is an example of a relation $\mathcal{R}$ that is induced by an arithmetically definable degree invariant relation R and $\mathcal{R}$ is invariant under automorphisms.

The End

Thank you!