The e-verse

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Outline

Enumeration reducibility captures a natural relationship between sets of natural numbers in which positive information about the first set is used to produce positive information about the second set.

The induced structure of the enumeration degrees $\mathcal{D}_e$ can be viewed as an extension of the Turing degrees, as there is a natural way to embed $\mathcal{D}_T$ in $\mathcal{D}_e$ preserving structure.

In certain cases, the enumeration degrees can be used to capture the algorithmic content of mathematical objects, while the Turing degrees fail.

Certain open problems in degree theory present as more approachable in the extended context of the enumeration degrees, e.g. first order definability.

We have been working to develop a richer "e-verse": a system of classes of enumeration degrees with interesting properties and relationships, in order to better understand the enumeration degrees.

The enumeration degrees

Friedberg and Rogers introduced enumeration reducibility in 1959.

Informally: $A \subseteq \omega$ is *enumeration reducible* to $B \subseteq \omega$ ($A \leqslant_e B$) if there is a uniform way to compute an enumeration of A from an enumeration of B. (Selman 1971 proved that the uniformity condition can be dropped.)

Definition

$A \leqslant_e B$ if there is a c.e. set W such that

$$A = \{n \colon (\exists e)\, \langle n, e \rangle \in W \text{ and } D_e \subseteq B\},$$

where D_e is the eth finite set in a canonical enumeration.

The degree structure $\mathcal{D}_e$ induced by $\leqslant_e$ is called the *enumeration (e-) degrees*. It is an upper semi-lattice with a least element $\mathbf{0}_e$ (the degree of all c.e. sets).

Definition

Let $K_A = \bigoplus_e \Gamma_e(A)$. Note that $A \equiv_e K_A$. The *enumeration jump* of a set A is $A' = K_A \oplus \overline{K_A}$. The *jump* of a degree is $\deg_e(A)' = \deg_e(A')$.

The total enumeration degrees

Proposition. $A \leqslant_T B$ iff $A \oplus \overline{A}$ is B-c.e. iff $A \oplus \overline{A} \leqslant_e B \oplus \overline{B}$.

This suggests an embedding of the Turing degrees into the e-degrees.

Proposition. The embedding $\iota \colon \mathcal{D}_T \to \mathcal{D}_e$, defined by

$$\iota(d_T(A)) = d_e(A \oplus \overline{A}),$$

preserves the order and the least upper bound and even the jump.

Definition

A set A is *total* if $A \geqslant_e \overline{A}$ (or equivalently if $A \equiv_e A \oplus \overline{A}$). An enumeration degree is *total* if it contains a total set.

The image of the Turing degrees under the embedding ι is exactly the set of total enumeration degrees.

Proposition. Nontotal degrees exist. The enumeration degree of a 1-generic set is *quasiminimal*, i.e. it is nonzero and the only total degree below it is $\mathbf{0}_e$.

Kalimullin pairs

Jockusch (1968) introduced the *semicomputable sets* as left cuts in computable linear orderings on ω .

Arslanov, Cooper, and Kalimullin (2003) showed that if A is semicomputable then the degrees $\mathbf{a} = \deg_e(A)$ and $\bar{\mathbf{a}} = \deg_e(\overline{A})$ are a *robust minimal pair*: $(\forall \mathbf{x} \in \mathcal{D}_e)[(\mathbf{a} \vee \mathbf{x}) \wedge (\bar{\mathbf{a}} \vee \mathbf{x}) = \mathbf{x}]$.

Definition (Kalimullin 04)

A and B are a *$\mathcal{K}$-pair* relative to U if there is a set $W \leqslant_e U$ such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

If A is semi-computable and not c.e. or co-c.e. then $\{A, \overline{A}\}$ form a *maximal*, $\mathcal{K}$-pair (relative to $\varnothing$).

Theorem (Kalimullin 04)

A and B are a $\mathcal{K}$-pair relative to U if and only if their degrees $\mathbf{a}$ and $\mathbf{b}$ form a robust minimal pair relative to $\mathbf{u} = \deg_e(U)$:

$$(\forall \mathbf{x} \geqslant \mathbf{u})[(\mathbf{a} \vee \mathbf{u} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{u} \vee \mathbf{x}) = \mathbf{x}].$$

Natural definability

Theorem (Kalimullin 2003)

The enumeration jump is first order definable in $\mathcal{D}_e$: $\mathbf{u}'$ is the largest degree that can represented as the join of a triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that each pair $\{\mathbf{a}, \mathbf{b}\}$, $\{\mathbf{b}, \mathbf{c}\}$, and $\{\mathbf{a}, \mathbf{c}\}$ is a $\mathcal{K}$-pair relative to $\mathbf{u}$.

Theorem (Cai, Ganchev, Lempp, Miller, S 2016)

The total enumeration degrees are first order definable in $\mathcal{D}_e$: a nonzero degree is total if and only if it is the join of a maximal $\mathcal{K}$-pair.

Theorem (Cai, Ganchev, Lempp, Miller, S 2016)

The image of the relation "c.e. in" on Turing degrees is first order definable in $\mathcal{D}_e$: for total degrees $\mathbf{a}$ and $\mathbf{x}$, $\mathbf{a}$ is c.e. in $\mathbf{x}$ if and only if $\mathbf{a}$ is the join of a maximal $\mathcal{K}$-pair with one side bounded by $\mathbf{x}$.

The automorphism problem

Problem (Rogers 1969)

Are there nontrivial automorphisms of $\mathcal{D}_e$ or $\mathcal{D}_T$?

Recall by Selman's theorem an enumeration degree is determined by the total degrees above it.

Corollary

The total enumeration degrees form a definable automorphism base for $\mathcal{D}_e$.

- Every nontrivial automorphism of $\mathcal{D}_e$ gives rise to a unique non-trivial automorphism of $\mathcal{D}_T$.
- $\mathcal{D}_e$ has at most countably many automorphisms.

Local and global structural interactions

Theorem (Slaman, S 2017)

There is a finite set of Δ^0_2 enumeration degrees that is an automorphism base for $\mathcal{D}_e$.

Theorem (Slaman, S 2017)

If $\mathcal{D}_e$ has a nontrivial automorphism then so does:

- The local structure $\mathcal{D}_e(\leqslant \mathbf{0}'_e)$.
- The structure of the Δ^0_2 Turing degrees $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$.
- The structure of the c.e. Turing degrees.

Insight. We need to identify more subclasses of e-degrees, understand how they interact with each other and with the local structure, to help us understand what's going on.

1. Turn to effective mathematics, focusing on cases where enumeration degrees provide a better tool for capturing algorithmic content.
2. Extend properties/relations from Turing oracles to enumeration oracles.

Computable metric spaces

Definition (Lacombe 1957)

A *computable metric space* is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n\in\omega}$ on which the metric is computable, i.e. there is a computable function that maps a pairs of indices i, j and a precision $\varepsilon \in \mathbb{Q}^+$ to a rational that is within ε of $d_{\mathcal{M}}(q_i, q_j)$.

Examples. 2^ω, ω^ω, $\mathbb{R}$, $\mathcal{C}[0,1]$, and *Hilbert cube* $[0,1]^\omega$.

Definition

$\lambda\colon \mathbb{Q}^+ \to \omega$ is a *name* of a point $x \in \mathcal{M}$ if for all rationals $\varepsilon > 0$ we have $d_{\mathcal{M}}(x, q^{\mathcal{M}}_{\lambda(\varepsilon)}) < \varepsilon$.

Question. How can we assign a measure of algorithmic complexity to a point in a computable metric space?

Points in $\mathbb{R}$ have a name of least Turing degree. Do points in all metric spaces have such a name?

The continuous degrees

Definition (Miller 2004)

If x and y are members of (possibly different) computable metric spaces, then $x \leqslant_r y$ if there is a uniform way to compute a name for x from a name for y.

This reducibility induces the *continuous degrees*.

Theorem (Miller 2004)

Every continuous degree contains a point from $[0,1]^\omega$ and a point from $C[0,1]$.

For $\alpha \in [0,1]^\omega$, let $C_\alpha = \bigoplus_{i\in\omega} \{q \in \mathbb{Q} \mid q < \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q > \alpha(i)\}$. Enumerating C_α is exactly as hard as computing a name for α. So $\alpha \mapsto C_\alpha$ induces an embedding of the continuous degrees into the enumeration degrees.

Elements of 2^ω, ω^ω, and $\mathbb{R}$ are mapped to the total degrees.

Topology realized as a structural property

Theorem (Miller 2004)

There is a nontotal continuous degree.

Points in such degrees do not have a name of least Turing degree.
Every known proof invokes nontrivial topological theorems, e.g. Brouwer's fixed point theorem for multi-valued functions.

Theorem (Andrews, Igusa, Miller, S.)

The continuous degrees are definable in $\mathcal{D}_e$: $\mathbf{a}$ is continuous if and only if it is *almost total*: if $\mathbf{x} \nleq \mathbf{a}$ and $\mathbf{x}$ is total then $\mathbf{a} \vee \mathbf{x}$ is total.

Theorem (Ganchev, Kalimullin, Miller, S.)

There is a structural dichotomy in the enumeration degrees:

1. An e-degree $\mathbf{a}$ is either continuous, hence almost total;
2. or half of a nontrivial relativized $\mathcal{K}$-pair, hence there is a total degree $\mathbf{x}$ such that $\mathbf{a} \vee \mathbf{x}$ is *strong quasi-minimal cover* of $\mathbf{x}$.

The relation “relatively PA”

The continuous degrees interact nicely with the relation “relatively PA”.

Definition

For $W \subseteq 2^{<\omega}$ let $[W] = \{X \in 2^\omega \mid \exists\sigma \in W(\sigma \preceq X)\}$.

P is a Π^0_1 class is there is a c.e. set $W \subseteq 2^{<\omega}$ such that $P = 2^\omega \setminus [W]$.

P is a $\Pi^0_1(A)$ class is there is a c.e. in A set $W \subseteq 2^{<\omega}$ such that $P = 2^\omega \setminus [W]$.

For Turing oracles A and B we say that B *is PA relative to* A if B computes a member of every nonempty $\Pi^0_1(A)$ class

Theorem (Miller 2004)

For Turing degrees $\mathbf{b}$ is PA relative to $\mathbf{a}$ if and only if there is a nontotal continuous degree $\mathbf{c}$ such that $\iota(\mathbf{a}) < \mathbf{c} < \iota(\mathbf{b})$.

And so, the image of the relation “PA above” is first order definable in $\mathcal{D}_e$.

Relative to an enumeration oracle

We can relativize to an enumeration oracle by replacing "c.e. in A" by "$\leqslant_e A$".

Definition

P is a $\Pi^0_1\langle A\rangle$ class if there is some $W \leqslant_e A$ such that $P = 2^\omega \setminus [W]$.

Note that a $\Pi^0_1\langle A \oplus A^c\rangle$ class is just a $\Pi^0_1(A)$-class.

Definition

$\langle B\rangle$ is PA relative to $\langle A\rangle$ if B enumerates a member of every nonempty $\Pi^0_1\langle A\rangle$ class.

Thus, B is PA above A if and only if $\langle B \oplus B^c\rangle$ is PA above $\langle A \oplus A^c\rangle$.

We have extended the image of the relation "PA above" from total degrees to all enumeration degrees.

Preserving or violating properties

Transferring theorems from the Turing degrees to total oracles we have:

1. No total degree **a** is *⟨self⟩-PA*: **a** is not PA relative to **a**.
2. If **b** is PA relative to **a** and **a** is total then $\mathbf{b} \geqslant \mathbf{a}$. We say that **a** is *PA-bounded*.
3. If **a** is total then it has a *universal class*: a $\Pi^0_1\langle \mathbf{a} \rangle$ class P such that $\langle \mathbf{b} \rangle$ is PA relative to $\langle \mathbf{a} \rangle$ if and only if **b** enumerates a member of P.

Theorem (Franklin, Lempp, Miller, Schweber, and S 2019)

The continuous degrees are exactly the PA bounded degrees. They cannot be ⟨self⟩-PA and they have universal classes.

Theorem (Miller, S. 2014)

There are ⟨self⟩-PA oracles. They cannot have universal oracles.

Investigating the oracles with a universal class lead to the introduction and exploration of many other classes.

The skip operator

Recall that $A' = K_A \oplus \overline{K}_A$, where $K_A = \bigoplus_e \Gamma_e(A)$.

Definition (AGKLMSS 2019)

The *skip* of A is the set $A^\Diamond = \overline{K}_A$. The *skip* of a degree is $\deg_e(A)^\Diamond = \deg_e(A^\Diamond)$.

In some ways the skip is a more natural analog of the Turing jump operator:

- $A \leqslant_e B$ if and only if $A^\Diamond \leqslant_1 B^\Diamond$;
- If $S \geqslant_e \varnothing'$ then there is some X such that $X^\Diamond \equiv_e S$;
- If $\mathbf{a}$ is total then $\mathbf{a}' = \mathbf{a}^\Diamond$, and so the skip is an extension of the Turing jump operator to the enumeration degrees.

There are degrees $\mathbf{a}$ such that $\mathbf{a} \nleqslant \mathbf{a}^\Diamond$:

- AGKLMSS show that there are *skip* 2-*cycles*: degrees $\mathbf{a}$ and $\mathbf{b}$ with $\mathbf{a}^\Diamond = \mathbf{b}$ and $\mathbf{b}^\Diamond = \mathbf{a}$. Such degrees must be above all hyperarithmetical enumeration degrees.
- Goh proved that an e-degree is hyperarithmetical if and only if it is bounded by every skip 2-cycle.

The cotototal enumeration degrees

What are the degrees that preserve the property of total degrees that $\mathbf{a} \leqslant \mathbf{a}^{\diamondsuit}$?

Definition

A set A is *cotototal* if $A \leqslant_e \overline{A}$. A degree is *cotototal* if it contains a cotototal set.

The cotototal degrees contain the continuous degrees. Not every e-degree is cotototal.

The cotototal enumeration degrees are characterized as:

1. The degrees which are bounded by their skip.
2. The degrees of complements of maximal independent sets in computable graphs by AGKLMSS 2019.
3. The degrees of languages of minimal subshifts by Jeandel and McCarthy 2018.

The graph cototal enumeration degrees

An e-degree is total if and only if it contains the graph of a total function.

Definition

An enumeration degree is *graph cototal* if it contains the complement of the graph of a total function.

It is easy to see that graph cototal degrees are cototal, but are they the same class?

Theorem (AGKLMSS 2019)

There is a cototal degree that is not graph cototal.

Problem

Are all continuous degrees graph cototal?

Topological classification of classes of e-degrees

Definition (Kihara, Pauly 2018)

A *represented space* is a pair of a second countable T_0 topological space X and listing of an open basis $B^X = \{B_i\}_{i<\omega}$.

A name for a point $x \in X$ is an enumeration of the set $N_x = \{i \mid x \in B_i\}$.

For $x, y \in X$, say that $x \leqslant y$ if every name for y (uniformly) computes a name for x.

Thus a represented space X gives rise to a class of e-degrees $\mathcal{D}_X \subseteq \mathcal{D}_e$.

Examples:.

- $\mathcal{D}_{S^\omega} = \mathcal{D}_e$, where S is the Sierpinski topology $\{\varnothing, \{1\}, \{0,1\}\}$.
- $\mathcal{D}_{2^\omega} = \mathcal{D}_{\mathbb{R}}$ is the total enumeration degrees.
- $\mathcal{D}_{[0,1]^\omega}$ is the continuous degrees.
- $\mathcal{D}_{\mathbb{N}^\omega_{cof}}$ is the graph cototal degrees.

The problem "Is every continuous degree graph cototal?" can be restated as

> *Can you cover the Hilbert cube with countably many homeomorphic copies of subspaces of* $\mathbb{N}^\omega_{cof}$*?*

New classes arising from topology

Theorem(Kihara, Ng, and Pauly). The following are subclasses of the graph co-total degrees.

- The cocylinder topology on ω^ω gives rise to the cylinder cototal degrees.
- The Telophase space gives rise to the telograph co-total degrees.
- The double origin space gives rise to the doubled co-d.c.e.a degrees.
- The product quasi-Polish Arens space gives rise to the Arens co-d.c.e.a degrees
- The product of the quasi-Polish Roy space gives rise to the Roy halfgraph-above degrees .

Yvette Ren gives alternative proofs characterizing the relative position of the cylinder cototal degrees and telograph co-total degrees with respect to known classes.

Goh and I are currently studying these two classes further within the local structure of the enumeration degrees.

Jacobsen-Grocott is investigating properties of the last three classes.

Separation axioms

Recall that:

$$T_0 \supset T_1 \supset T_2 \supset T_{2.5} \supset \text{submetrizable} \supset \text{metrizable}$$

Theorem (Kihara, Ng, Pauly)

Degrees of points in computable metrizable spaces are exactly the continuous degrees. Every e-degree is the degree of a point in some computable submetrizable space.

How can we distinguish the degree-theoretic properties of spaces satisfying different separation axioms?

Definition

Let T be a class of represented spaces. A class of degrees C is *not-T* if for every $Y \in T$ there is a degree $\mathbf{a} \in C$ such that $\mathbf{a} \notin \mathcal{D}_Y$.

We can separate two classes S and T by producing an S-space X such that $\mathcal{D}_X$ is not-T.

Degree separations of separation axioms

Theorem (Kihara, Ng, Pauly)

- There is a computable T_0-space X such that $\mathcal{D}_X$ is not T_1: $\mathcal{D}_X = \mathcal{D}_e$.
- There is a computable T_1-space X such that $\mathcal{D}_X$ is not T_2: $\mathcal{D}_X$ is the cylinder cototal degrees.
- There is a computable T_2 space X such that $\mathcal{D}_X$ is not $T_{2.5}$.

Theorem (Jacobsen-Grocott)

There is a computable $T_{2.5}$ space X such that $\mathcal{D}_X$ is not submetrizable: $\mathcal{D}_X$ an be either the Arens co-d-c.e.a. degrees or the Roy halfgraph above degreess.

Thank you!