The theory of the enumeration degrees and its fragments

Mariya I. Soskova
University of Wisconsin–Madison

CiE 2024, July 8

Supported by NSF Grant No. DMS-2053848

What is this talk about

Reducibilities between sets of natural numbers are used to compare their effective content:

1. $A \leqslant_m B$ means that there is a computable function f such that $x \in A$ if and only if $f(x) \in B$.
2. $A \leqslant_T B$ means that one can compute the members of A using an oracle Turing machine with oracle B.
3. $A \leqslant_e B$ means that one can (effectively) enumerate the members of A given any enumeration of the members of B.

We identify sets that are reducible to each other and get a *degree partial order*.

Question

How hard is it to tell whether a statement about partial orders is true for a particular degree structure?

Motivation for enumeration reducibility

When we define computable functions on $\mathbb{N}$, we naturally include partial functions.

When we define relative computation using Turing reducibility, there is mismatch: an oracle Turing machine is only well defined for total oracles, but produces partial functions.

The first straightforward attempt at extending oracle Turing machines so that they work with partial oracles causes problems: suppose we postulate that if during a computation the oracle is queried at an undefined value, the computation does not halt. This is called *partial reducibility* and studied by Sasso in the 1960s.

Example (Myhill)

We would not be able to show that an oracle B computes A if and only if it enumerates A and $\overline{A}$ unless A is computable.

A closer look at enumeration reducibility

Definition (Selman)

A is *enumeration reducible to* ($\leqslant_e$) B if and only if any Turing oracle that can enumerate B can also enumerate A.

We can now use enumeration reducibility on sets to define a reducibility on functions: $\psi \leqslant_e \varphi$ if and only if $G_\psi \leqslant_e G_\varphi$.

Definition (Friedberg and Rogers)

A is *enumeration reducible to* ($\leqslant_e$) B if and only if there is a c.e. table of axioms $\langle x, D \rangle$, so that $x \in A$ if and only if $D \subseteq B$.

Any c.e. set can be thought of as such a table. We call this an *enumeration operator*. Scott showed that enumeration operators give rise to a model of untyped λ-calculus.

The Turing degrees inside the e-degrees

Proposition (Post)

$A \leqslant_T B \iff A \oplus \overline{A} \leqslant_e B \oplus \overline{B}$.

We can embed the partial order of the Turing degrees $\mathcal{D}_T$ into the partial order of the enumeration degrees $\mathcal{D}_e$ by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$.

An enumeration degree that is the image of a Turing degree is called *total*.

So now we have two partial orders: $\mathcal{D}_T$ living inside $\mathcal{D}_e$. What statements in the language $L = \{\leqslant\}$ are true in either.

Theorem (Spector, Gutteridge)

$\mathcal{D}_T$ has minimal elements, while $\mathcal{D}_e$ is downwards dense.

$$(\exists \mathbf{x})(\forall \mathbf{y})[\mathbf{y} \leqslant \mathbf{x} \Rightarrow \mathbf{y} = \mathbf{x} \vee \mathbf{y} = \mathbf{0}].$$

The coding method

Second order arithmetic $\mathcal{Z}_2$ is the structure with to sorts of domains: the natural numbers and sets of natural numbers, along with standard operations $+$, $*$, and membership $\in$ for sets.

Classical results due to Kleene and Post show that the relations $\leqslant_T$, $\leqslant_e$ are definable in second order arithmetic.

Thus, every statement φ in the language of partial orders can be translated into a statements φ^*_e and φ^*_T in the language of arithmetic so that the first is true in $\mathcal{D}_e$ or $\mathcal{D}_T$ respectively if the corresponding translation is true in $\mathcal{Z}_2$.

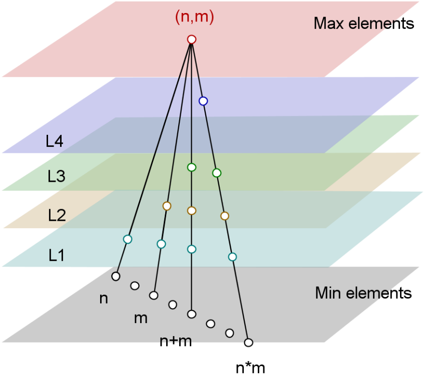
Definition

If $\mathcal{A}$ is a structure in a language $\mathcal{L}$ then $Th(\mathbf{A})$ is the *theory of* $\mathcal{A}$, consisting of all L-statements true in $\mathbf{A}$.

Corollary

The theories $Th(\mathcal{D}_T)$ and $Th(\mathcal{D}_e)$ are no more complicated than $Th(\mathcal{Z}_2)$.

The coding method



There is a way to represent a model of arithmetic in $\mathcal{D}_T$ and in $\mathcal{D}_e$.

Start by translating arithmetic into a partial order.

Prove that this partial order can be embedded into either degree structure.

Theorem (Slaman, Woodin 1986, 1997)

Every countable antichain $\mathcal{A}$ in $\mathcal{D}_T$ and in $\mathcal{D}_e$ can be coded by three parameters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$:

$\mathbf{x} \in \mathcal{A} \leftrightarrow \mathbf{x} \leqslant \mathbf{a} \ \wedge \ \mathbf{x} \neq (\mathbf{b} \vee \mathbf{x}) \wedge (\mathbf{c} \vee \mathbf{x}) \ \wedge \mathbf{x}$ is minimal with these properties.

Definability instead of coding

So even through $Th(\mathcal{D}_T)$ and $Th(\mathcal{D}_e)$ are different, they are both just as complicated as $Th(\mathcal{Z}_2)$.

Theorem (Simpson 70, Slaman–Woodin 86)

The theories $Th(\mathcal{D}_T)$, $Th(\mathcal{D}_e)$, and $Th(\mathcal{Z}_2)$ are all computably isomorphic.

There is a reason for that:

Theorem (Cai, Ganchev, Lempp, Miller, and Soskova 15)

The image of the Turing degrees is first order definable in $\mathcal{D}_e$.

The local structures

The Turing degree of the halting set K is a natural element to explore. It can be thought of as representing the complexity of one quantifier: indeed it can compute all existentially defined sets.

We denote the Turing degree of K by $\mathbf{0}'_T$ and its image in $\mathcal{D}_e$ by $\mathbf{0}'_e$.

Definition

The local structure of the Turing degrees $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$ and the local structure of the e-degrees $\mathcal{D}_e(\leqslant \mathbf{0}'_e)$ consist of the initial intervals below these two degrees.

The local structures are countable structures and the sets that occupy them have simple arithmetic definitions.

The local theories

In the local structures we can identify a degree with a natural number: say the least index of a Turing machine/enumeration operator that computes a member of the degree from the halting set.

Thus any statement in the language of partial orders has an arithmetical translation. The complexity of the theories of the local structures is no higher than that of arithmetic $\mathcal{N} = (\mathbb{N}, +, *)$.

Theorem (Shore 81, Ganchev–Soskova 12)

The theories $Th(\mathcal{D}_T(\leqslant \mathbf{0}'_T))$, $Th(\mathcal{D}_e(\leqslant \mathbf{0}'_e))$, and $Th(\mathcal{N})$ are all computably isomorphic.

Restricting to fragments

The existential fragment

Transfering undecidability

The Nies transfer lemma

Kent's result?

The extension of embeddings problem

What we know and what we don't

What's next globally

What's next locally

Changing the signature

The skip operator

The end

Bibliography