The Strongest Non-splitting Theorem

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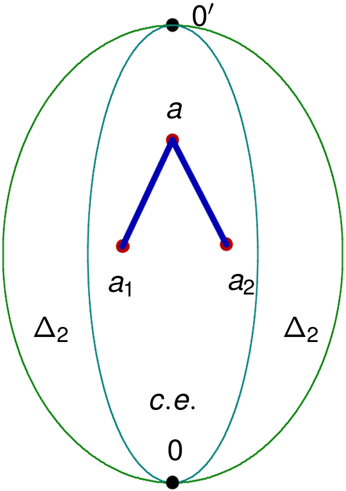
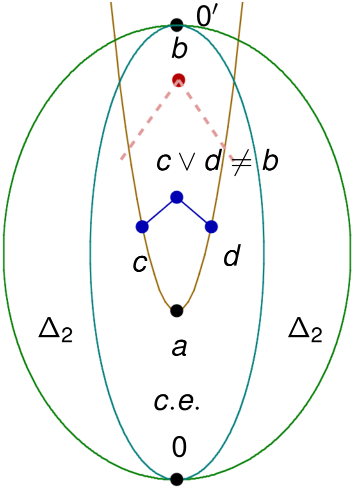
Definitions

We will say that a pair of degrees $\mathbf{a_1}$ and $\mathbf{a_2}$ form a splitting of $\mathbf{a}$ if $\mathbf{a_1} < \mathbf{a}$ and $\mathbf{a_2} < \mathbf{a}$ but $\mathbf{a_1} \cup \mathbf{a_2} = \mathbf{a}$.



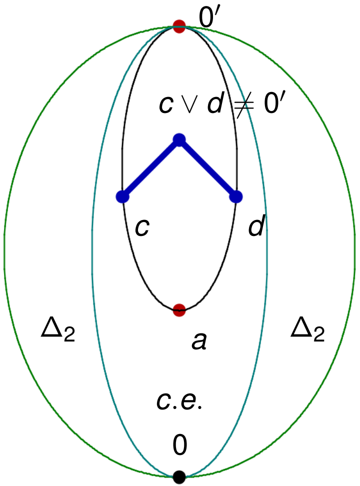
Lachlan's non-splitting theorem

There exist c.e. degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ can not be split in the c.e. degrees above $\mathbf{a}$.



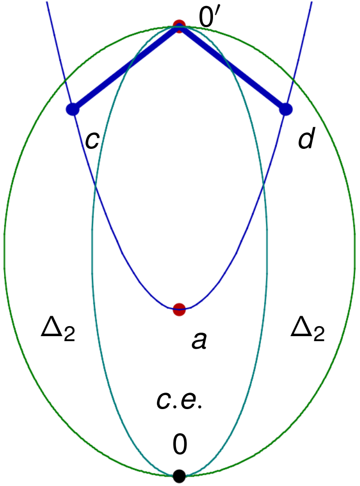
Harrington's non-splitting theorem

There exists a c.e. degree $\mathbf{a} < \mathbf{0}'$ such that $\mathbf{0}'$ can not be split in the c.e. degrees above $\mathbf{a}$.



Arslanov’s splitting theorem

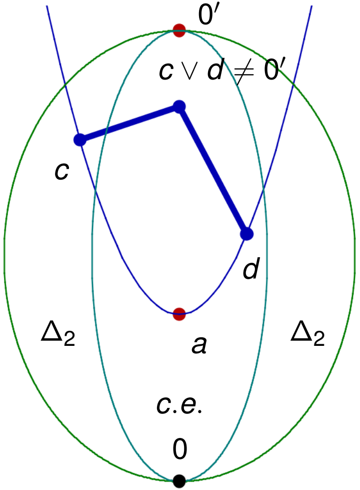
There is a d.c.e. splitting of $\mathbf{0}'$ above each c.e. degree $\mathbf{a} < \mathbf{0}'$.



The strongest non-splitting theorem

Theorem

There exists a c.e. degree $\mathbf{a} < \mathbf{0}'$ *such that there exists no nontrivial cuppings of c.e. degrees in the* Δ_2 *degrees above* $\mathbf{a}$*.*



The semi-lattice of the enumeration degrees

Definition

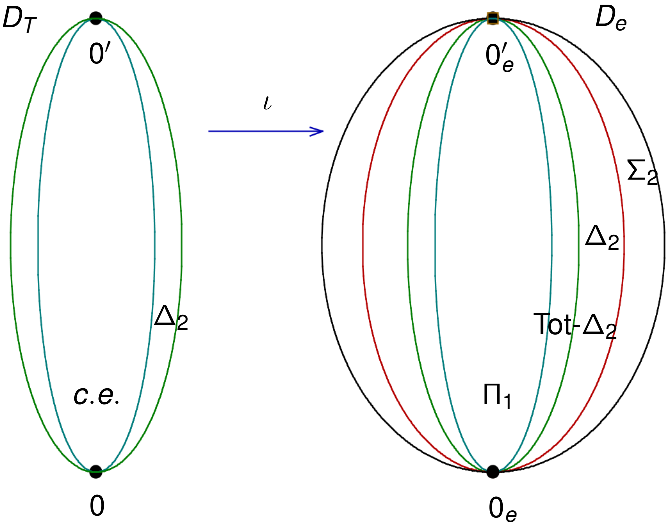
1. A set A is enumeration reducible to a set B ($A \leq_e B$), if there is a c.e. set Φ such that

$$n \in A \Leftrightarrow \exists D(\langle n, [D] \rangle \in \Phi \wedge D \subseteq B).$$

2. A is enumeration equivalent to B ($A \equiv_e B$) if $A \leq_e B$ and $B \leq_e A$.
3. Let $d_e(A) = \{B | A \equiv_e B\}$.
4. $(D_e, <, \cup, ', 0_e)$ is the semi-lattice of the enumeration degrees.

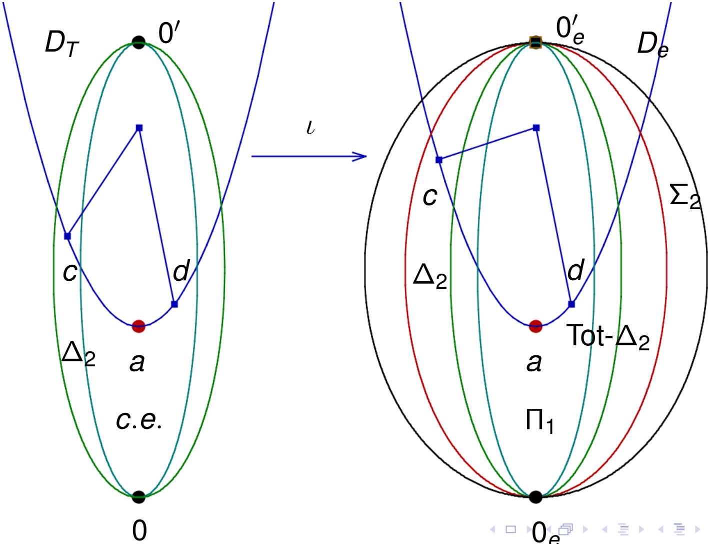
Embedding the Turing degrees into the enumeration degrees

There exists an order theoretic embedding $\iota : D_T \to D_e$.



Main result

There exists a Π_1 e-degree $\mathbf{a} < 0'_e$ such that there exist no nontrivial cuppings of Π_1 e-degrees in the Δ_2 e-degrees above $\mathbf{a}$.



The requirements

We will construct Π_1 sets A and E such that:

- For all enumeration operators Ψ:

$$\mathcal{N}_\Psi : E \neq \Psi^A$$

- For each pair of a Δ_2 set U and a Π_1 set $\overline{W}$ and each enumeration operator Θ:

$$\mathcal{P}_{\Theta,U,W} : E = \Theta^{U,\overline{W}} \Rightarrow (\exists \Gamma, \Lambda)[\overline{K} = \Gamma^{U,A} \vee \overline{K} = \Lambda^{\overline{W},A}]$$

The $\mathcal{P}$-strategy

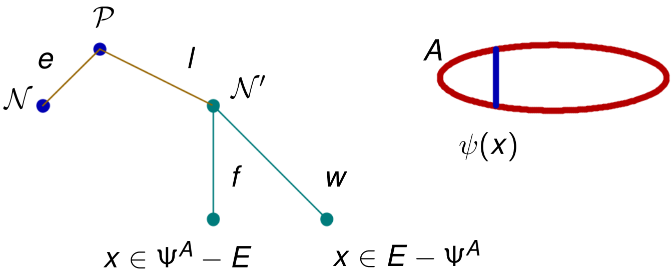
$$\mathcal{P}_{\Theta,U,W} : E = \Theta^{U,\overline{W}} \Rightarrow (\exists \Gamma, \Lambda)[\overline{K} = \Gamma^{U,A} \vee \overline{K} = \Lambda^{\overline{W},A}]$$

- We monitor the length of agreement $l(E, \Theta^{U,\overline{W}})$ and act only on expansionary stages.
- Construct an e-operator Γ so that $n \in \overline{K} \leftrightarrow \langle n, (U \oplus A) \upharpoonright \gamma_n \rangle \in \Gamma$.
- We define a good approximations to the sets U, $\overline{W}$ and $U \oplus \overline{W}$ with following properties:
  - For sufficiently large stages $U \upharpoonright n = U_s \upharpoonright n$.
  - Stability: Changes in $\overline{W}$ are permanent.
  - If $\Theta^{U,\overline{W}} = E$, then there are infinitely many expansionary stages.

The $\mathcal{N}$-strategies

$$\mathcal{N}_\Psi : E \neq \Psi^A$$

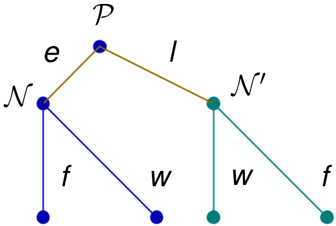
Below the l-outcome $\mathcal{N}'$ can follow a simple Friedberg-Muchnik strategy:

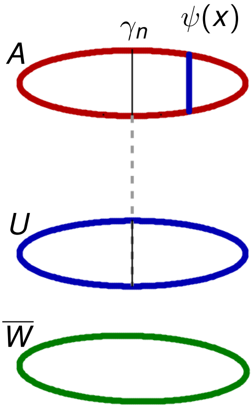


The $\mathcal{N}$-strategies

$$\mathcal{N}_\Psi : E \neq \Psi^A$$

There is a conflict between $\mathcal{P}$ and $\mathcal{N}$ below outcome *e*.

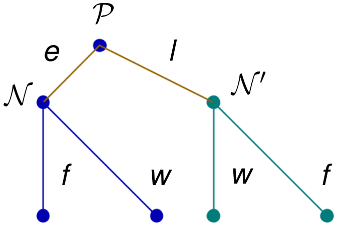


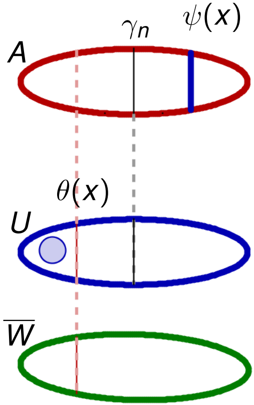


The $\mathcal{N}$-strategies

$$\mathcal{N}_\Psi : E \neq \Psi^A$$

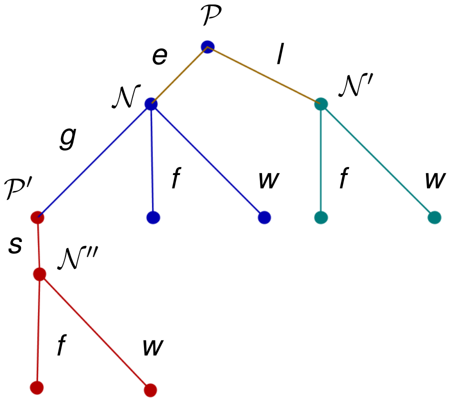
We have $x < l(\Theta^{U,\overline{W}}, E)$ so $x \in \Theta^{U,\overline{W}}$ before the attack with axiom $\langle x, U \oplus \overline{W} \upharpoonright \theta(x)\rangle$:

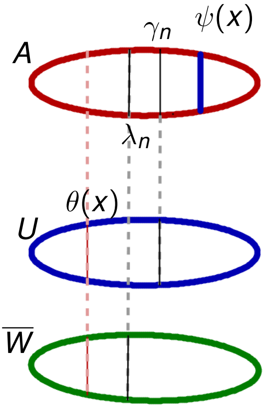




The backup strategies

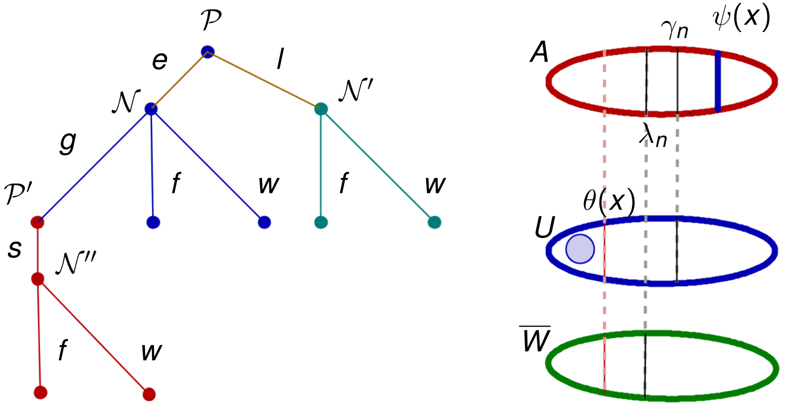
A backup strategy $\mathcal{P}'$ constructs Λ so that
$n \in \overline{K} \leftrightarrow \langle n, (\overline{W} \oplus A) \restriction \lambda_n \rangle \in \Lambda$:





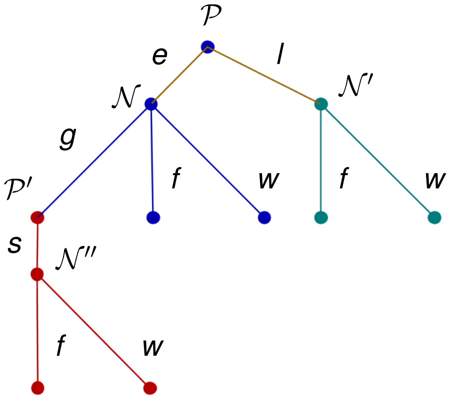
The $\mathcal{N}$-strategies

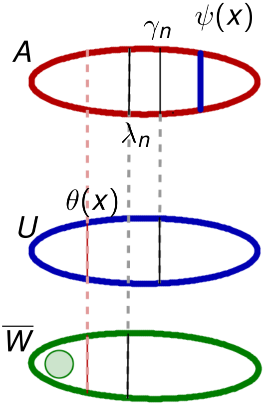
$\mathcal{N}$ and $\mathcal{N}''$ attack simultaneously. No change in $\overline{W}$:



The $\mathcal{N}$-strategies

A change in $\overline{W}$, $\mathcal{N}''$ is satisfied, $\mathcal{N}$ starts from the beginning.





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