The enumeration degrees: An overview

Mariya I. Soskova
University of Wisconsin–Madison

Higher Recursion Theory and Set Theory
Workshop on Recursion Theory, Set Theory and their interactions,
May 28 2019

Supported by the NSF Grant No. DMS-1762648

Outline

Enumeration reducibility captures a natural relationship between sets of natural numbers in which positive information about the first set is used to produce positive information about the second set.

By identifying sets that are reducible to each other we obtain an algebraic representation of this reducibility as a partial order: the structure of the enumeration degrees $\mathcal{D}_e$.

Motivation for the interest in this area comes from its nontrivial connections to the study of the Turing degrees. In particular, there is a natural structure preserving embedding of the Turing degrees in the enumeration degrees.

I. The first order theory of $\mathcal{D}_e$ and its fragments;
II. First order definability;
III. Automorphisms and automorphism bases.

The enumeration degrees

Friedberg and Rogers introduced enumeration reducibility in 1959.

Informally: $A \subseteq \omega$ is *enumeration reducible* to $B \subseteq \omega$ ($A \leqslant_e B$) if there is a uniform way to compute an enumeration of A from an enumeration of B. (Selman 1971 proved that the uniformity condition can be dropped.)

Definition

$A \leqslant_e B$ if there is a c.e. set W such that

$$A = \{n \colon (\exists e)\, \langle n, e \rangle \in W \text{ and } D_e \subseteq B\},$$

where D_e is the eth finite set in a canonical enumeration.

The degree structure $\mathcal{D}_e$ induced by $\leqslant_e$ is called the *enumeration degrees*. It is an upper semi-lattice with a least element (the degree of all c.e. sets).

Note! Enumeration reducibility is a definable relation in second order arithmetic Z_2. Thus Z_2 can interpret $\mathcal{D}_e$.

The main problem

The three parts of this talk address three aspects of the same problem:

Theorem (Slaman, Woodin 1986, S 2016)

The following are equivalent:

1. $\mathcal{D}_e$ is *biinterpretable* with second order arithmetic.
2. The definable relations in $\mathcal{D}_e$ are exactly the ones induced by degree invariant definable relations in second order arithmetic.
3. $\mathcal{D}_e$ is a rigid structure.

Problem

Are these statements true or false?

Part I: The first order theory of $\mathcal{D}_e$ and its fragments

The existential theory

To understand: *"What existential sentences in the language of partial order are true in $\mathcal{D}_e$?"*
we ask *"What finite partial orders can be embedded in $\mathcal{D}_e$?"*.

The answer is "all". All you need is an *independent sequence*: a sequence $\{A_i\}_{i<\omega}$ such that $A_i \nleqslant_e \bigoplus_{j \neq i} A_j$. The columns of a 1-generic set satisfy this.

And so the $\exists$-Th($\mathcal{D}_e$) is decidable.

Theorem (Slaman, Sorbi 2014)

Every countable partial order can be embedded below any non-zero element of $\mathcal{D}_e$.

And so the $\exists$-Th($\mathcal{D}_e$) is decidable.

Note! This generalizes:

Theorem (Gutteridge 1971)

The enumeration degrees are downwards dense and so $\mathcal{D}_T \not\equiv \mathcal{D}_e$.

The two quantifier theory

Problem

Is the ∀∃-theory of $\mathcal{D}_e$ decidable?

The problem of deciding the 2-quantifier theory is equivalent to the following:

Problem

We are given a finite lattice P and a partial orders $Q_0, \ldots Q_n \supseteq P$. Does every embedding of P extend to an embedding of one of the Q_i?

The algorithm for deciding $\exists\forall$-Th($\mathcal{D}_T$)

In $\mathcal{D}_T$ the problem is solved through the following:

Theorem (Lerman 1971)

Every finite lattice can be embedded into $\mathcal{D}_T$ as an initial segment.

- Suppose that P is a finite lattice and $Q \supseteq P$ is a partial order extending P.
- The initial segment embedding of P can be extended to an embedding of Q only if no element in $Q \setminus P$ is below any element of P.
- Q also needs to respect least upper bounds if $x \in Q \setminus P$ and $u, v \in P$ and $x \geqslant u, v$ then $x \geqslant u \vee v$.

Theorem (Shore 1978; Lerman 1983)

That is the only obstacle.

The downward density of $\mathcal{D}_e$ makes this approach not applicable.

Towards a solution

Theorem (Slaman, Calhoun 1996)

There are e-degrees $\mathbf{a} < \mathbf{b}$ such that the interval $(\mathbf{a}, \mathbf{b})$ is empty.

Theorem (Kent, Lewis-Pye, Sorbi 2012)

There are e-degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is a *strong minimal cover* of $\mathbf{a}$: if $\mathbf{x} < \mathbf{b}$ then $\mathbf{x} \leqslant \mathbf{a}$.

Theorem (Lempp, Slaman, S.)

Every finite distributive lattice can be embedded as an interval $[\mathbf{a}, \mathbf{b}]$ so that if $\mathbf{x} \leqslant \mathbf{b}$ then $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ or $\mathbf{x} \leqslant \mathbf{a}$.

Corollary

The $\exists\forall\exists$-theory of $\mathcal{D}_e$ is undecidable.

The complexity of the full theory of $\mathcal{D}_e$

Theorem (Slaman, Woodin 1997)

The theory of $\mathcal{D}_e$ is computably isomorphic to the theory of second order arithmetic Z_2: there are algorithms that translate a formula φ in the language of partial orders to a formula ψ in the language of second order arithmetic and vise versa so that:

$$\mathcal{D}_e \models \varphi \text{ if and only if } Z_2 \models \psi.$$

To translate ψ into φ they use their Coding Theorem:

Theorem

For every n, every countable n-ary relation on $\mathcal{D}_e$ can be uniformly defined from finitely many parameters.

and prove that it is a definable property of finitely many parameters $\vec{\mathbf{p}}$ that they code a model of $(\mathbb{N}, +, \times, <, C)$ where C is a unary predicate on $\mathbb{N}$.

The biinterpretability conjecture

Conjecture

The relation Bi, where $Bi(\vec{\mathbf{p}}, \mathbf{c})$ holds when $\vec{\mathbf{p}}$ codes a model of $(\mathbb{N}, +, \times, <, C)$ and $\deg_e(C) = \mathbf{c}$, is first order definable in $\mathcal{D}_e$.

Theorem (Slaman and Woodin 1987, S 2016)

There is a parameter $\mathbf{g}$ such that relation Bi is first order definable in $\mathcal{D}_e$ with parameter $\mathbf{g}$.

Equivalently,

Corollary

If R is an n-ary relation invariant under $\equiv_e$ and definable in Z_2 then $\mathcal{R} = \{(\deg(A_1), \ldots \deg_e(A_n)) \mid Z_2 \models R(A_1, \ldots A_n)\}$ is definable in $\mathcal{D}_e$ with one parameter $\mathbf{g}$.

Part II: First order definability

The enumeration jump

Theorem (Shore, Slaman 1999)

The Turing jump is first order definable.

The proof relies on Slaman and Woodin's work on the biinterpretability conjecture for the Turing degrees.

Let $K_A = \bigoplus_e \Gamma_e(A)$. Note that $A \equiv_e K_A$.

Definition

The *enumeration jump* of a set A is $A' = K_A \oplus \overline{K_A}$. The *jump* of a degree is $\deg_e(A)' = \deg_e(A')$.

Is the enumeration jump first order definable?

The total enumeration degrees

Proposition. $A \leqslant_T B$ iff $A \oplus \overline{A}$ is B-c.e. iff $A \oplus \overline{A} \leqslant_e B \oplus \overline{B}$.

This suggests a natural embedding of the Turing degrees into the enumeration degrees.

Proposition. The embedding $\iota \colon \mathcal{D}_T \to \mathcal{D}_e$, defined by

$$\iota(d_T(A)) = d_e(A \oplus \overline{A}),$$

preserves the order and the least upper bound and even the jump.

Definition

A set A is *total* if $A \geqslant_e \overline{A}$ (or equivalently if $A \equiv_e A \oplus \overline{A}$). An enumeration degree is *total* if it contains a total set.

The image of the Turing degrees under the embedding ι is exactly the set of total enumeration degrees.

Are the total degrees first order definable?

Semicomputable sets and natural definability

Jockusch introduced the *semicomputable sets* as left cuts in computable linear orderings on ω .

Theorem (Jockusch 1968)

Every Turing degree contains a semicomputable set that is not c.e. or co-c.e., so every total degree can be represented as $\deg_e(A) \vee \deg_e(\overline{A})$ for such a set A.

Theorem (Arslanov, Cooper, Kalimullin 2003)

If A is semicomputable and not c.e. or co-c.e. then the degrees $\mathbf{a} = \deg_e(A)$ and $\bar{\mathbf{a}} = \deg_e(\overline{A})$ are a *robust minimal pair*:

$$(\forall \mathbf{x} \in \mathcal{D}_e)[(\mathbf{a} \vee \mathbf{x}) \wedge (\bar{\mathbf{a}} \vee \mathbf{x}) = \mathbf{x}].$$

Definability of the enumeration jump

Kalimullin studied pairs of enumeration degrees that form *relative* robust minimal pairs.

Theorem (Kalimullin 2003)

The enumeration jump is first order definable.

A definition with unrelativized robust minimal pairs is as follows:

Theorem (Ganchev, S 2012)

$\mathbf{x}'$ is the largest degree above $\mathbf{x}$ which can be represented as $\mathbf{a} \vee \mathbf{b}$, where $\{\mathbf{a}, \mathbf{b}\}$ is a robust minimal pair with $\mathbf{a} \leqslant \mathbf{x}$.

A definable copy of the Turing degrees

Theorem (Cai, Ganchev, Lempp, Miller, S 2016)

The pairs of degrees of a semicomputable set and its complement are first order definable in $\mathcal{D}_e$. They are *maximal* robust minimal pairs.

The total enumeration degrees are first order definable

Note! The characterization of the complexity of $\mathrm{Th}(\mathcal{D}_e)$ and Biinterpretability with parameters for $\mathcal{D}_e$ now follow from the corresponding theorems for $\mathcal{D}_T$.

Theorem (Cai, Ganchev, Lempp, Miller, S 2016)

For total degrees $\mathbf{a}$ and $\mathbf{x}$, $\mathbf{a}$ is c.e. in $\mathbf{x}$ if and only if $\mathbf{a}$ is the join of a semicomputable pair with one side bounded by $\mathbf{x}$.

The image of the relation "c.e. in" on Turing degrees is first order definable in $\mathcal{D}_e$.

The continuous degrees

Definition (Lacombe 1957)

A *computable metric space* is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n\in\omega}$ on which the metric is computable, i.e. there is a computable function that maps a pairs of indices i, j and a precision $\varepsilon \in \mathbb{Q}^+$ to a rational that is within ε of $d_{\mathcal{M}}(q_i, q_j)$.

Examples:. 2^ω, ω^ω, $\mathbb{R}$, $\mathcal{C}[0,1]$, and *Hilbert cube* $[0,1]^\omega$.

Definition

$\lambda\colon \mathbb{Q}^+ \to \omega$ is a *name* of a point $x \in \mathcal{M}$ if for all rationals $\varepsilon > 0$ we have $d_{\mathcal{M}}(x, q^{\mathcal{M}}_{\lambda(\varepsilon)}) < \varepsilon$.

Definition (Miller 2004)

If x and y are members of (possibly different) computable metric spaces, then $x \leqslant_r y$ if there is a uniform way to compute a name for x from a name for y.

This reducibility induces the *continuous degrees*.

The continuous degrees

Theorem (Miller 2004)

Every continuous degree contains a point from $[0,1]^\omega$ and a point from $C[0,1]$.

For $\alpha \in [0,1]^\omega$, let

$$C_\alpha = \bigoplus_{i\in\omega} \{q \in \mathbb{Q} \mid q < \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q > \alpha(i)\}.$$

Observation. Enumerating C_α is exactly as hard as computing a name for α. So $\alpha \mapsto C_\alpha$ induces an embedding of the continuous degrees into the enumeration degrees.

Elements of 2^ω, ω^ω, and $\mathbb{R}$ are mapped onto the *total* degree of their least Turing degree name (i.e., their Turing degree).

Theorem (Miller 2004)

There is a nontotal continuous degree.

Every known proof of this result uses nontrivial topological facts: Brouwer's fixed point theorem for multivalued functions on an infinite dimensional space, or Sperner's lemma, or results from topological dimension theory.

Topology realized as a structural property

Theorem (Andrews, Igusa, Miller, S.)

An enumeration degree $\mathbf{a}$ is continuous if and only if it is *almost total*: if $\mathbf{x} \nleqslant \mathbf{a}$ and $\mathbf{x}$ is total then $\mathbf{a} \vee \mathbf{x}$ is total.

The continuous degrees are definable in $\mathcal{D}_e$.

Definition

A Turing degree $\mathbf{a}$ is *PA*

- if $\mathbf{a}$ computes a complete extension of Peano Arithmetic, or equivalently
- if $\mathbf{a}$ computes a path in every infinite computable tree.

The degree $\mathbf{a}$ is *PA above* $\mathbf{b}$ if $\mathbf{a}$ computes a path in every infinite $\mathbf{b}$-computable tree.

Theorem (Miller 2004)

For total degrees $\mathbf{a}$ is *PA above* $\mathbf{b}$ if and only if there is a nontotal continuous degree $\mathbf{c}$ such that $\mathbf{b} < \mathbf{c} < \mathbf{a}$.

The image of the relation "PA above" is first order definable in $\mathcal{D}_e$.

The cototal enumeration degrees

Definition

A set A is *cototal* if $A \leqslant_e \overline{A}$. A degree is *cototal* if it contains a cototal set.

The cototal degrees contain the continuous degrees. Not every e-degree is cototal.

The cototal enumeration degrees are characterized as:

1. The degrees of complements of maximal independent sets in computable graphs by AGKLMSS 2019.
2. The degrees of complements of maximal antichains in $\omega^{<\omega}$ by McCarthy 2018.
3. The degrees of languages of minimal subshifts by McCarthy 2018.
4. The degrees of sets with good approximations by Miller and S 2018.
5. The degrees of points in computable G_δ topological spaces by Kihara, Ng, and Pauly 2019.

Problem

Are the cototal degrees first order definable in $\mathcal{D}_e$?

Topological classification of classes of e-degrees

Definition (Kihara, Pauly 2018)

A *represented space* is a pair of a second countable topological space X and listing of an open basis $B^X = \{B_i\}_{i<\omega}$.

A name for a point $x \in X$ is an enumeration of the set $N_x = \{i \mid x \in B_i\}$.

For $x, y \in X$, say that $x \leqslant y$ if every name for y (uniformly) computes a name for x.

Thus a represented space X gives rise to a class of e-degrees $\mathcal{D}_X \subset \mathcal{D}_e$.

Examples:.

- $\mathcal{D}_{2^\omega} = \mathcal{D}_{\mathbb{R}}$ is the total enumeration degrees.
- $\mathcal{D}_{[0,1]^\omega}$ is the continuous degrees.
- $\mathcal{D}_{S^\infty} = \mathcal{D}_e$, where S is the Sierpinski topology $\{\varnothing, \{1\}, \{0,1\}\}$.
- $\mathcal{D}_{\mathbb{R}^<}$, where $\mathbb{R}^<$ is the real line with topology generated by $\{(q,\infty)\}_{q\in\mathbb{Q}}$, is exactly the semicomputable degrees.

Kihara, Ng, and Pauly 2019 investigate $\mathcal{D}_X$, where X is the ω-power of the: cofinite topology on ω, telophase space, double origin space, quasi-Polish Roy space, irregular lattice space.

Part III: Automorphisms and automorphism bases

Slaman and Woodin's automorphism analysis

Theorem (Slaman, Woodin 1986)

The Turing degrees have at most countably many automorphisms.

There is a single degree $\mathbf{g} \leqslant \mathbf{0}^{(5)}$ that is an *automorphism base* for $\mathcal{D}_T$: if π is an automorphism such that $\pi(\mathbf{g}) = \mathbf{g}$ then $\pi = \mathrm{id}$.

Relations on $\mathcal{D}_T$ induced by definable relations in Z_2 are first order definable in $\mathcal{D}_T$ with such a parameter $\mathbf{g}$.

Relations on $\mathcal{D}_T$ induced by definable relations in Z_2 that are furthermore invariant under automorphism are first order definable in $\mathcal{D}_T$ (without parameters).

Theorem (Selman 1971)

For enumeration degrees $\mathbf{a}, \mathbf{b}$: $\mathbf{a} \leqslant \mathbf{b}$ if and only if every total degree above $\mathbf{b}$ is above $\mathbf{a}$.

Implications for the e-degrees

Corollary

The total enumeration degrees form a definable automorphism base for $\mathcal{D}_e$.

- Every nontrivial automorphism of $\mathcal{D}_e$ gives rise to a unique non-trivial automorphism of $\mathcal{D}_T$.
- This automorphism preserves the relations "c.e. in" and "PA above".
- $\mathcal{D}_e$ has at most countably many automorphisms.
- A single total degree below $\mathbf{0}_e^{(5)}$ is an automorphism base of $\mathcal{D}_e$.

Problem

Does every automorphism of $\mathcal{D}_T$ extend to an automorphism of $\mathcal{D}_e$?

A positive answer would imply the first order definability (without parameters) of the relations "c.e. in" and "PA above" in $\mathcal{D}_T$.

Local structure

Definition

The *local structure of the enumeration degrees* is the interval $\mathcal{D}_e(\leqslant \mathbf{0}'_e)$ consisting of all Σ^0_2 enumeration degrees.

1. Cooper 1984 proved that this is a dense structure.
2. Bianchini 2000 proved that you can embed every countable partial order in any nonempty interval.
3. Kent 2005 showed that the 3-quantifier theory is undecidable.

Problem

Is the 2-quantifier theory of $\mathcal{D}_e(\leqslant \mathbf{0}'_e)$ decidable?

4. Ganchev, S 2012 showed that $\mathrm{Th}(\mathcal{D}_e(\leqslant \mathbf{0}'_e))$ is computably isomorphic to the theory of first order arithmetic.
5. Ganchev, S 2012, 2018 showed that many classes if Σ^0_2 degrees are definable in $\mathcal{D}_e(\leqslant \mathbf{0}'_e)$ including the total degrees, all levels of the jump hierarchy: the low$_n$ and high$_n$ degrees for $n \geqslant 1$.

Biinterpretability for the local structure

Fix an effective listing of all Σ^0_2 sets $\{U_e\}_{e<\omega}$.

Problem

The Biinterpretability conjecture for the local structure is that in $\mathcal{D}_e$ there is a definable coded model of first order arithmetic $\mathcal{M} = (\mathbb{N}^M, 0^{\mathcal{M}}, +, \times, <)$ and a definable function $\varphi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}(\leqslant \mathbf{0}'_e)$ such that $\varphi(e^{\mathcal{M}}) = \deg_e(U_e)$.

Local structure determines global structure

Theorem (Slaman, Woodin 1986)

There is an indexing of the c.e. Turing degrees that is definable from Δ^0_2 parameters in the local structure $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$.

Slaman and S start with the result above (transferred to $\mathcal{D}_e$ via the embedding ι).

- Every indexing of the image of the c.e. Turing degrees can be extended to the image of the Δ^0_2 Turing degrees. (Uses the local definability of the total and of the low enumeration degrees).
- Every indexing of the image of the Δ^0_2 Turing degrees can be extended to an indexing of the image of the degrees that are c.e. in and above some Δ^0_2 Turing degree. (Uses the definability of the enumeration jump and the relation "c.e. in").
- Every such indexing can be extended to an indexing of the image of the Turing degrees below $\mathbf{0}''_e$.
- And now we can iterate...

Local structure determines global structure

Theorem (Slaman, S 2017)

There is a finite set of Δ^0_2 enumeration degrees that is an automorphism base for $\mathcal{D}_e$.

Theorem (Slaman, S 2017)

If $\mathcal{D}_e$ has a nontrivial automorphism then so does:

- The local structure $\mathcal{D}_e(\leqslant \mathbf{0}'_e)$.
- The structure of the Δ^0_2 Turing degrees $\mathcal{D}_T(\leqslant \mathbf{0}'_T)$.
- The structure of the c.e. Turing degrees.

Problem

Does an automorphism of any of these structures extend to an automorphism of $\mathcal{D}_e$?

Thank you!