The Structure of the ω-enumeration degrees

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Background from enumeration reducibility

Definition (Friedberg and Rogers, 1959)

We say that $\Psi : 2^\omega \to 2^\omega$ is an *enumeration operator* (or e-operator) iff for some c.e. set W

$$\Psi(B) = \{x | (\exists u)[\langle x, u \rangle \in W \,\& D_u \subseteq B]\}$$

for each $B \subseteq \omega$.

A is *enumeration reducible to* B, written $A \leq_e B$, by $A = \Psi(B)$ for some e-operator Ψ.

Enumeration reducibility and the relation "c.e. in"

Theorem (Selman, 1971)

Let A and B be sets of natural numbers.

$$A \leq_e B \iff (\forall X)[B \text{ is c.e. in } X \Rightarrow A \text{ is c.e. in } X].$$

Theorem (Case, 1974)

For any sets A and B,

$$A \leq_e \emptyset^n \oplus B \iff (\forall X)[B \text{ is } \Sigma^X_{n+1} \Rightarrow A \text{ is } \Sigma^X_{n+1}].$$

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Ash's generalizations

In 1992 C. Ash defines a positive reducibility between sequences of sets:
Let $\mathcal{A} = \{A_n\}_{n<\omega}$ and $\mathcal{B} = \{B_n\}_{n<\omega}$ be two sequences of sets.

Definition

$\mathcal{A} \leq_\omega \mathcal{B}$ ($\mathcal{A}$ is *uniformly* reducible to $\mathcal{B}$) iff

$$(\forall X \subseteq 2^\omega)[(\forall n)(B_n \in \Sigma^X_{n+1} \text{ uniformly in } n) \Rightarrow (\forall n)(A_n \in \Sigma^X_{n+1} \text{ uniformly in } n)].$$

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Uniform enumeration reducibility

Let $\mathcal{S}$ be the set of all sequences of sets of natural numbers.

Definition

Let $\mathcal{A} = \{A_n\}_{n<\omega} \in \mathcal{S}$ and Ψ be an e-operator. $\Psi(\mathcal{A})$, is the sequence:

$$\{\Psi[n](A_n)\}_{n<\omega}.$$

We say that $\Psi(\mathcal{A})$ is enumeration reducible ($\leq_e$) to the sequence $\mathcal{A}$.

So $\mathcal{A} \leq_e \mathcal{B}$ is a combination of two notions:

- Enumeration reducibility: for every n we have that $A_n \leq_e B_n$ via, say, Ψ_n.
- Uniformity: the sequence $\{\Psi_n\}_{n<\omega}$ is uniform.

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The jump sequence

With every member $\mathcal{A} \in \mathcal{S}$ we connect a *jump sequence* $P(\mathcal{A})$.

Definition

The *jump sequence* of the sequence $\mathcal{A}$, denoted by $P(\mathcal{A})$ is the sequence $\{P_n(\mathcal{A})\}_{n<\omega}$ defined inductively as follows:

- $P_0(\mathcal{A}) = A_0$.
- $P_{n+1}(\mathcal{A}) = A_{n+1} \oplus P'_n(\mathcal{A})$, where $P'_n(\mathcal{A})$ denotes the enumeration jump of the set $P_n(\mathcal{A})$.

Example

Let $A \subseteq \mathbb{N}$. Consider the sequence $A \uparrow \omega = (A, \emptyset, \ldots, \emptyset, \ldots)$. Then

$$(P_0(\mathcal{A}), P_1(\mathcal{A}), \ldots, P_n(\mathcal{A}), \ldots) \equiv_e (A, A', \ldots, A^n, \ldots)$$

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Ash's generalizations in terms of e-reducibility

Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}$.

Theorem (Soskov, Kovachev)

Let $\mathcal{A}$ and $\mathcal{B}$ be sequences of sets.

$$\mathcal{A} \leq_\omega \mathcal{B} \iff (\forall n)[A_n \leq_e P_n(\mathcal{B}) \text{ uniformly in } n],$$

$$\iff \mathcal{A} \leq_e P(\mathcal{B}).$$

Remark: Uses a technique called Uniform Regular Enumerations.

Easy observations

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1. $\mathcal{A} \leq_e P(\mathcal{A})$.
2. If $\mathcal{B} \leq_e P(\mathcal{C})$ then $P(\mathcal{B}) \leq_e P(\mathcal{C})$.
3. If $\mathcal{A} \leq_e P(\mathcal{B})$ and $\mathcal{B} \leq_e P(\mathcal{C})$ then $\mathcal{A} \leq_e P(\mathcal{C})$.

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Let $\mathcal{A}$ and $\mathcal{B}$ be sequences of sets of natural numbers.

- ω-enumeration reducibility: $\mathcal{A} \leq_\omega \mathcal{B}$, if $\mathcal{A} \leq_e P(\mathcal{B})$.
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- ω-enumeration degrees: $d_\omega(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_\omega \mathcal{B}\}$.
- The structure of the ω-enumeration degrees: $\mathcal{D}_\omega = \langle \{d_\omega(\mathcal{A}) \mid \mathcal{A} \in \mathcal{S}\}, \leq_\omega \rangle$, where $d_\omega(\mathcal{A}) \leq_\omega d_\omega(\mathcal{B})$ if $\mathcal{A} \leq_\omega \mathcal{B}$.
- The least ω-enumeration degree: $\mathbf{0}_\omega = d_\omega((\emptyset, \emptyset, \emptyset, \dots))$ or equivalently $d_\omega((\emptyset, \emptyset', \emptyset'', \dots))$.

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Jump classes and least upper bounds

Let $\mathcal{A} = \{A_n\}_{n<\omega}$ be given. We define the *jump class* $\mathcal{J}_\mathcal{A}$ of $\mathcal{A}$ by

$$\begin{aligned}\mathcal{J}_\mathcal{A} &= \{d_T(X) | (\forall n)(A_n \text{ is c.e. in } X^n \text{ uniformly in } n)\}\\ &= \{d_T(X) | (\forall n)(A_n \in \Sigma^X_{n+1} \text{ uniformly in } n)\}.\end{aligned}$$

An alternative way to phrase the definition of $\leq_\omega$ is:

$$\mathcal{A} \leq_\omega \mathcal{B} \iff \mathcal{J}_\mathcal{B} \subseteq \mathcal{J}_\mathcal{A}.$$

Definition

Let $\mathcal{A} \oplus \mathcal{B} = \{A_n \oplus B_n\}_{k<\omega}$.

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We would like to define a jump operation $'$ so that:

$$\mathcal{J}_{\mathcal{A}'} = \{d_T(X)' \mid d_T(X) \in \mathcal{J}_{\mathcal{A}}\}.$$

Definition

Let $\mathcal{A}' = \{P_{n+1}(\mathcal{A})\}_{n<\omega}$.

$\mathcal{D}_\omega$ is an upper semi-lattice with least element and jump operation.

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$\mathcal{D}_\omega$ is an upper semi-lattice with least element and jump operation.

Where does $\mathcal{D}_\omega$ stand?

$\langle \mathcal{D}_e, \leq_e, \vee, ' \rangle$ can be embedded in $\langle \mathcal{D}_\omega, \leq_\omega, \vee, ' \rangle$ via the embedding κ defined as follows:

$$\kappa(d_e(A)) = d_\omega((A \uparrow \omega) = d_\omega((A, \emptyset, \emptyset, \dots)) = d_\omega((A, A', A'', \dots)).$$

$\langle \mathcal{D}_\omega, \leq_\omega \rangle$ can be embedded in the Medvedev degrees $\langle \mathcal{D}_s, \leq_s \rangle$, via the mapping:

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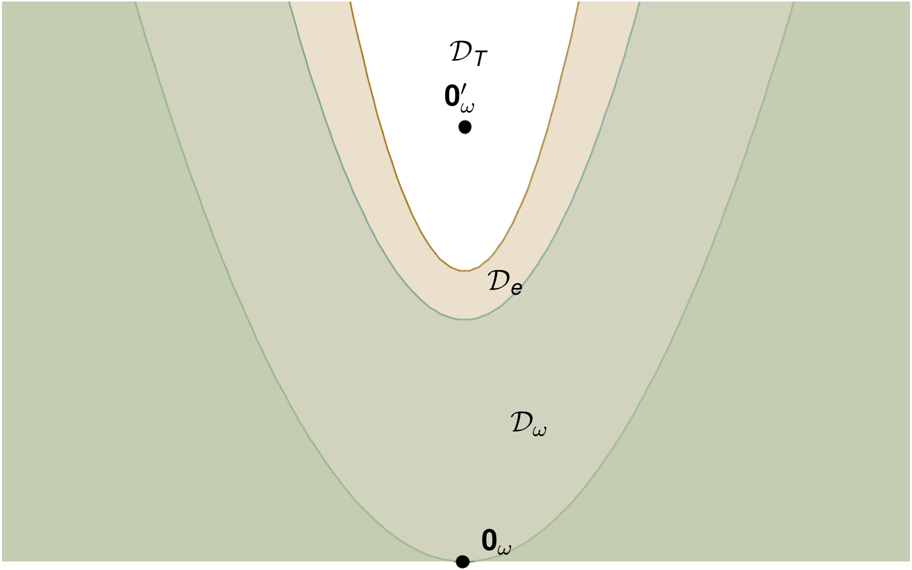
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Let $\mathcal{D}_1$ be the structure $\kappa(\mathcal{D}_e)$, where;

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$\mathcal{D}_1$ is the analog of $\mathcal{TOT} = \iota(\mathcal{D}_T)$, where:

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Conclusion: $\mathcal{D}_1$ is an automorphism base for $\mathcal{D}_\omega$.

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Conclusion: $\mathcal{D}_1$ is an automorphism base for $\mathcal{D}_\omega$.

The structure $\mathcal{D}_1$.

Let $\mathcal{D}_1$ be the structure $\kappa(\mathcal{D}_e)$, where;

$$\kappa(d_e(A)) = d_\omega(A \uparrow \omega) = d_\omega((A, A', A'', \dots)).$$

$\mathcal{D}_1$ is the analog of $\mathcal{TOT} = \iota(\mathcal{D}_T)$, where:

$$\iota(d_T(A)) = d_e(A \oplus \overline{A}).$$

$$\mathcal{J}_{\mathcal{A}} = \{d_T(X) \mid (\forall n)(A_n \text{ is c.e. in } X^n \text{ uniformly in } n)\}$$

$$\mathcal{J}^e_{\mathcal{A}} = \{\iota(d_T(X)) \mid (\forall n)(A_n \leq_e (X \oplus \overline{X})^n \text{ uniformly in } n)\}$$

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In 1992 Lachlan and Shore introduce the notion "good approximation" and use it to prove the density of the n-c.e.a enumeration degrees.

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Theorem (Soskov, S)

If $\mathbf{a} <_\omega \mathbf{b}$ *and* $\mathbf{b}$ *has a member with a good approximation, then every countable partial ordering can be embedded in the interval* $(\mathbf{a}, \mathbf{b})$.

Theorem (Soskov)

There is no minimal ω*-enumeration degree.*

Suppose $(P_0(\mathcal{A}), P_1(\mathcal{A}) \dots) <_\omega (P_0(\mathcal{B}), P_1(\mathcal{B}), \dots)$.
Case 1. For some n, $P_n(\mathcal{A}) <_e P_n(\mathcal{B})$: then the theorem follows from the corresponding one for $\mathcal{D}_e$.
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The relation "Almost"

Definition

Let $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\omega$. Then $\mathbf{a}$ is *almost* $\mathbf{b}$ if there are members $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$, such that

$$(\forall n)(P_n(\mathcal{A}) \equiv_n P_n(\mathcal{B})).$$

- If $\mathbf{a}$ is almost- then for every $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$,

$$(\forall n)(P_n(\mathcal{A}) \equiv_n P_n(\mathcal{B})).$$

- The class of almost-$\mathbf{b}$ degrees is closed under least upper bound.
- If $\mathbf{b} \in \mathcal{D}_1$ then $\mathbf{b}$ is the least almost-$\mathbf{b}$ ω-enumeration degree.

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Are there almost degrees?

Theorem (Soskov,S)

Let $\mathbf{b} <_\omega \mathbf{a}$ *be two* ω*-enumeration degrees, such that* $\mathbf{a}$ *contains a member with a good approximation. There exists an almost-*$\mathbf{b}$ *degree* $\mathbf{z}$ *such that* $\mathbf{b} <_\omega \mathbf{z} \leq_\omega \mathbf{a}$ *if and only if:*

$$(\exists)(\forall m \geq n)[\mathbf{b}^m <_\omega \mathbf{a}^m].$$

In particular when $\mathbf{b} = \mathbf{0}_\omega$ and $\mathbf{a} = \mathbf{0}'_\omega$: there exists a nontrivial *almost-zero* degree below $\mathbf{0}'_\omega$.

Let $L = \{\mathbf{a} \mid \exists n(\mathbf{0}^n = \mathbf{a}^n)\} = Low_1 \cup Low_2 \cdots \cup Low_n \cup \dots$.

Then $\mathbf{a}$ is in L if and only if it bounds no nonzero *almost-zero* degree.

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Inverting the jump operation

- $(P_0(\mathcal{A}), P_1(\mathcal{A}), \dots, P_n(\mathcal{A}) \dots)' = (P_1(\mathcal{A}), \dots, P_n(\mathcal{A}) \dots)$.
- $(P_0(\mathcal{A}), P_1(\mathcal{A}), \dots, P_n(\mathcal{A}) \dots)' = (\emptyset, P_1(\mathcal{A}), \dots, P_n(\mathcal{A}) \dots)'$.
- The degree of $(\emptyset, P_1(\mathcal{A}), ; P_n(\mathcal{A}) \dots)$ is the least degree, whose jump is equal to $d_\omega(\mathcal{A}')$.

Definition

Let $n \in \mathbb{N}$ and $\mathcal{A} \in \mathcal{S}$.

$$I_n(\mathcal{A}) = (\underbrace{\emptyset, \dots, \emptyset}_{n}, P_0(\mathcal{A}), P_1(\mathcal{A}), \dots)$$

If $\mathcal{A} \in \mathbf{a}$, then $I_n(\mathbf{a}) = d_\omega(I_n(\mathcal{A}))$.
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Relativized jump inversion

Definition

Let $n \in \mathbb{N}$ and $\mathcal{A}, \mathcal{B} \in \mathcal{S}$.

$$I_n^{\mathcal{B}}(\mathcal{A}) = (B_0, B_1, \dots, B_{n-1}, P_0(\mathcal{A}), P_1(\mathcal{A}), \dots)$$

If $\mathcal{A} \in \mathbf{a}$ and $\mathbf{b} \in \mathcal{B}$, then $I_n^{\mathbf{b}}(\mathbf{a}) = d_\omega(I_n^{\mathcal{B}}(\mathcal{A}))$.

If $\mathbf{b}^n \leq_\omega \mathbf{a}$ then $I_n^{\mathbf{b}}(\mathbf{a})$ is the least degree $\mathbf{x}$, such that $\mathbf{b} \leq_\omega \mathbf{x}$ and $\mathbf{x}^{(n)} = \mathbf{a}$.

First order definability of $\mathcal{D}_1$ in $\mathcal{D}'_\omega$

Main idea: Consider a sequence $\mathcal{A}$ of the form

$$(A, \emptyset, \emptyset, \dots).$$

Out of all sequences $\mathcal{X}$ with $X_0 \equiv_e A$, the sequence $\mathcal{A}$ is the least one.

Let $\mathcal{I}_{\mathbf{a}} = \{ I^1_{\mathbf{a}}(\mathbf{x}) \mid \mathbf{a}' \leq_\omega \mathbf{x} \}$.

Theorem (Ganchev, Soskov)

Let $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\omega$. *Then:*
$\mathcal{I}_{\mathbf{b}} \subseteq \mathcal{I}_{\mathbf{a}} \iff \mathbf{a} \leq_\omega \mathbf{b} \ \&\ A_0 \equiv_e B_0$.

So $\mathbf{a} \in D_1$ if and only if for every $\mathbf{b}$:

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Some consequences

Corollary

The structure $\mathcal{D}_1$ is first order definable in $\mathcal{D}'_\omega$, the structure of the ω-enumeration degrees, augmented by the jump operation.

Corollary

The theory of $\mathcal{D}'_\omega$ is computably isomorphic to second order arithmetic.

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The reverse direction

Every automorphism of $\mathcal{D}_e$ is the identity on the cone above $\mathbf{0}_e^{(4)}$.

Consider an automorphism φ of $\mathcal{D}_e$ and define Φ working as follows: $\Phi(d_\omega(\mathcal{A})) = d_\omega(\mathcal{B})$, where $\mathcal{B}$ is defined as

$B_0 \in \varphi(d_e(A_0)), \ldots, B_3 \in \varphi(d_e(A_3))$ and for $n \geq 4$, $B_n = P_n(\mathcal{A})$.

Theorem (Soskov, Ganchev)

$$Aut(\mathcal{D}'_\omega) \cong Aut(\mathcal{D}_e).$$

Question

Is the ω-enumeration jump first order definable in $\mathcal{D}_\omega$?

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The Local structure $\mathcal{G}_\omega$

Consider the structure $\mathcal{G}_\omega$ consisting of all degrees reducible to $\mathbf{0}'_\omega = d_\omega((\emptyset', \emptyset'', \emptyset''', \dots))$ also called the Σ^0_2 ω-enumeration degrees.

Definition

Let $\mathbf{a} \in \mathcal{G}_\omega$.

1. $\mathbf{a}$ is low_n if $\mathbf{a}^n = \mathbf{0}^n_\omega$. The class of all low_n degrees is denoted by L_n. $\bigcup_n L_n = L$
2. $\mathbf{a}$ is $high_n$ if $\mathbf{a}^n = \mathbf{0}^{n+1}_\omega$. The class of all $high_n$ degrees is denoted by H_n. $\bigcup_n H_n = H$

There is a copy of the Σ^0_2 enumeration degrees $\mathcal{G}_1 = D_1 \cap \mathcal{G}_\omega$.

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There is a copy of the Σ^0_2 enumeration degrees $\mathcal{G}_1 = D_1 \cap \mathcal{G}_\omega$.

The Local structure $\mathcal{G}_\omega$

Consider the structure $\mathcal{G}_\omega$ consisting of all degrees reducible to $\mathbf{0}'_\omega = d_\omega((\emptyset', \emptyset'', \emptyset''', \dots))$ also called the Σ^0_2 ω-enumeration degrees.

Definition

Let $\mathbf{a} \in \mathcal{G}_\omega$.

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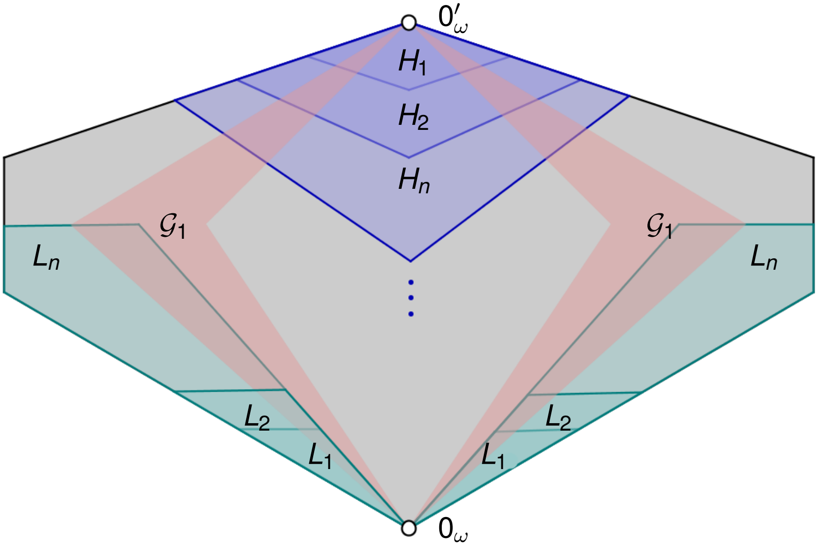
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H_2
H_n
$\mathcal{G}_1$
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L_n
L_n
L_2
L_1
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L_1
0_ω

The $\mathbf{o}_n$ degrees

Definition

For every n let $\mathbf{o}_n = I^n(\mathbf{0}_\omega^{n+1}) = d_\omega((\underbrace{\emptyset, \dots, \emptyset}_{n}, \emptyset^{n+1}, \emptyset^{n+2}, \dots))$.

Proposition

Let $\mathbf{a} \in \mathcal{G}_\omega$.

1. $\mathbf{a} \in H_n$ *if and only if* $\mathbf{a} \geq \mathbf{o}_n$
2. $\mathbf{a} \in L_n$ *if and only if* $\mathbf{a} \wedge \mathbf{o}_n = \mathbf{0}_\omega$

Proof: 1. $\mathbf{o}_n$ is the least n-th jump invert of $\emptyset^{n+1}$, hence $\mathbf{a}^n \geq \mathbf{o}_n^n = \emptyset^{n+1}$ if and only if $\mathbf{a} \in H_n$.

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Proof: 2. For every $\mathcal{A} \equiv_e P(\mathcal{A}) \in \mathbf{a}$:

$$(A_0, A_1 \dots, A_n, A_{n+1}, \dots) \wedge (\underbrace{\emptyset, \dots, \emptyset}_{n}, \emptyset^{n+1}, \emptyset^{n+2}, \dots)$$

$$= (\underbrace{\emptyset, \dots, \emptyset}_{n}, A_n, A_{n+1}, \dots) = I^n(\mathcal{A}^n)$$

Hence $\mathbf{a} \wedge \mathbf{o}_n = \mathbf{0}_\omega$ if and only if $I^n(\mathbf{a}^n) = \mathbf{0}_\omega$, if and only if $\mathbf{a}^n = \mathbf{0}_\omega^n$.

The almost zero degrees

Definition

An ω-enumeration degree **a** called almost zero (a.z.) if for $\mathcal{A} \in \mathbf{a}$ and every n, $P_n(\mathcal{A}) \equiv_e \emptyset^n$.

- The a.z. degrees form an ideal.
- If $\mathbf{a} \in \mathcal{G}_\omega$ then **a** is a.z. if and only if $\mathbf{a} < \mathbf{o}_n$ for all n.
- Nonzero a.z. degrees exist.

Theorem (Ganchev, Soskov)

1. $\mathbf{a} \in L = \bigcup_n L_n$ *if and only if* **a** *does not bound any nonzero a.z degree.*
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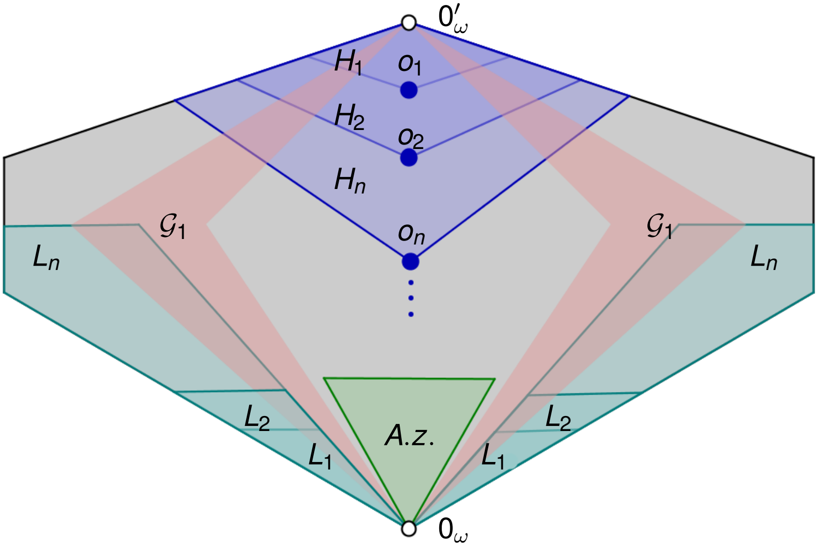

$0'_\omega$
H_1
o_1
H_2
o_2
H_n
o_n
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L_n
L_n
L_2
L_2
L_1
L_1
$A.z.$
0_ω

$\mathcal{K}$-pairs in $\mathcal{G}_\omega$

Definition

A pair of degrees $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\omega$ is called a $\mathcal{K}$-pair if

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{D}_\omega)[(\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}]$$

Theorem (Ganchev)

If $d_\omega(\mathcal{A})$ and $d_\omega(\mathcal{B})$ form a nontrivial $\mathcal{K}$-pair in $\mathcal{G}_\omega$ then both $\mathcal{A}$ and $\mathcal{B}$ are a.z. or for some n there exists a $\mathcal{K}$-pair over $\emptyset^n$ in $\mathcal{D}_e$ $\{A, B\}$, such that $\emptyset^n < A, B, A' = B' = \emptyset^{n+1}$ and:

$$\mathcal{A} \equiv_\omega \{\underbrace{\emptyset, \ldots, \emptyset}_{n}, A, \emptyset, \ldots, \emptyset, \ldots\} \text{ and}$$

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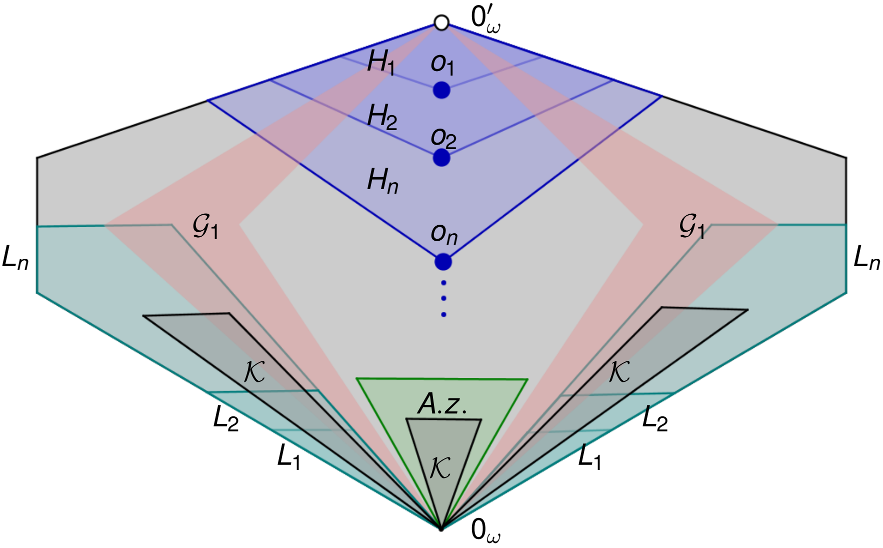
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Distinguishing $\mathcal{K}$-pairs at different levels

Theorem (Ganchev, S)

Let $\mathbf{a}, \mathbf{b} \in \mathcal{G}_\omega$ form a nontrivial $\mathcal{K}$-pair. Then for every natural number n

$$\forall \mathbf{x} \lneq_\omega \mathbf{o}_n [\mathbf{a} \vee \mathbf{x} \lneq_\omega \mathbf{o}_n] \iff \mathbf{a}, \mathbf{b} \leq_\omega \mathbf{o}_{n+1}.$$

Proof Sketch:

Suppose that $\mathbf{a}, \mathbf{b} \leq_\omega \mathbf{o}_n$ and $\mathbf{a}, \mathbf{b} \nleq_\omega \mathbf{o}_{n+1}$. Then

$$\mathbf{a} = d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \emptyset, \dots, \emptyset, \dots\})$$

and $\emptyset^n < A < \emptyset^{n+1}$ and $A' = \emptyset^{n+1}$.

Theorem (S, Wu)

For every non-c.e. Δ^0_2 set A there is a low$_1$ X, such that $A \oplus X \equiv_e \emptyset'$.

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Proof Sketch:
Relativizing it follows that there is an X, $\emptyset^n <_e X <_e \emptyset^{n+1}$, such that $X' = \emptyset^{n+1}$ and $X \vee A \equiv_e \emptyset^{n+1}$, hence

$$d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \dots\}) \vee d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, X, \emptyset, \dots\}) = \mathbf{o}_n.$$

Definability of $\mathbf{o}_n$

Theorem (Ganchev, S)

For every n $\{\mathbf{o}_n\}$ is first order definable in $\mathcal{G}_\omega$.

Proof Sketch: Fix $n \geq 0$. Then $\mathbf{o}_{n+1}$ is the greatest degree which is the least upper bound of a nontrivial $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ in $\mathcal{G}_\omega$, such that

$$\forall \mathbf{x} \lneq \mathbf{o}_n[\mathbf{a} \vee \mathbf{x} \lneq_\omega \mathbf{o}_n].$$

Corollary

For all n the classes H_n and L_n are first order definable in $\mathcal{G}_\omega$.

Corollary

Every automorphism of $\mathcal{D}'_\omega$ is the identity on the ideal below $\mathbf{o}_4$.

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For every sequence $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$ we have that:

$$d_\omega(\mathcal{A}) \vee \mathbf{o}_1 = d_\omega(\{A_0, \emptyset'', \emptyset''', \dots\}).$$

If $\mathcal{A}^e \in \mathcal{G}_1$ and $\mathcal{A}^e = \{A_0, \emptyset, \emptyset, \dots\}$ then $d_\omega(\mathcal{A}^e) \vee \mathbf{o}_1 = d_\omega(\mathcal{A}) \vee \mathbf{o}_1$ and $d_\omega(\mathcal{A}^e) \leq_\omega d_\omega(\mathcal{A})$.

Theorem (Ganchev, S)

$\mathcal{G}_1$ is first order definable in $\mathcal{G}_\omega$ by:

$$\mathbf{a} \in \mathcal{G}_1 \leftrightarrow \forall \mathbf{y}(\mathbf{a} \vee \mathbf{o}_1 = \mathbf{y} \vee \mathbf{o}_1 \Rightarrow \mathbf{a} \leq \mathbf{y}).$$

Corollary

The theory of first order arithmetic can be interpreted in $\mathcal{G}_\omega$.

Question

What is the strength of the theory of $\mathcal{G}_\omega$?

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The end

Thank you!