Semi-recursive sets and definability in the enumeration degrees

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Preliminaries: Enumeration reducibility

$A \leq_T B$ iff χ_A is computable with oracle B .

$A \leq_T B$ iff $A \oplus \overline{A}$ is c.e. in B.

$A \leq_T B$ iff there is a c.e. set W such that
$x \in A \oplus \overline{A}$ if and only if there are finite sets D_B and $D_{\overline{B}}$ such that
$\langle x, D_B \oplus D_{\overline{B}} \rangle \in W$ and $D_B \oplus D_{\overline{B}} \subseteq B \oplus \overline{B}$.

Definition

$A \leq_e B$ if and only if there is a c.e. set W, such that
$A = W(B) = \{x \mid \exists u(\langle x, u \rangle \in W \wedge D_u \subseteq B)\}$.

Note that $A \leq_T B$ if and only if $A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

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The structure of the enumeration degrees

- $A \equiv_e B$ if $A \leq_e B$ and $B \leq_e A$.
- $d_e(A) = \{B \mid A \equiv_e B\}$.
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W$ is c.e. $\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, \mathbf{0}_e \rangle$ is an upper semi-lattice with least element.

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The total degrees

Proposition

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$*, defined by* $\iota(d_T(A)) = d_e(A \oplus \overline{A})$*, preserves the order and the least upper bound.*

The substructure of the total e-degrees is defined as $\mathcal{TOT} = \iota(D_T)$.

$$(\mathcal{D}_T, \leq, \vee, \mathbf{0}_T) \cong (\mathcal{TOT}, \leq, \vee, \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq, \vee, \mathbf{0}_e)$$

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The enumeration jump

- Let $K_A = \{x \mid x \in W_x(A)\}$. Note that $K_A \equiv_e A$.
- The jump of A is $A' = K_A \oplus \overline{K_A}$. Then $d_e(A)' = d_e(A')$.
- The embedding ι preserves the jump operation.

$$(\mathcal{D}_T, \leq, \vee, \mathbf{0}_T, ') \cong (\mathcal{TOT}, \leq, \vee, \mathbf{0}_e, ') \subseteq (\mathcal{D}_e, \leq, \vee, \mathbf{0}_e, ')$$

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Semi-recursive sets

Definition (Jockusch)

A set of natural numbers A is semi-recursive if there is a total computable selector function s_A, such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$.

- Let A be a set of natural numbers. Let $L_A = \{\sigma \in 2^{<\omega} \mid \sigma \leq \chi_A\}$.
- L_A is a semi-recursive set:

$$s_{L_A}(\sigma, \tau) = \begin{cases} \sigma, & \sigma \leq \tau; \\ \tau, & \textit{otherwise}. \end{cases}$$

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The enumeration degrees of semi-recursive sets

Denote by R_A the set $\overline{L_A}$. For every set of natural numbers A the following holds.

1. $L_A \oplus R_A \equiv_e A \oplus \overline{A}$;
2. $L_A \leq_e A$; (Mainly because if $\{x \mid \sigma(x) = 1\} \subseteq A$ then $\sigma \leq A$.)
3. $R_A \leq_e \overline{A}$;
4. A is semi-recursive if and only if $A \leq_1 L_A$.

Theorem (Jockusch)

A nonzero enumeration degree $d_e(T)$ is total if and only if there is a semi-recursive set A, which is not c.e. and not co-c.e. such that:

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Semi-recursive sets as effective minimal pairs

Theorem (Arslanov, Cooper, Kalimullin)

If A is a semi-recursive set then for every X:

$$(d_e(X) \vee d_e(A)) \wedge (d_e(X) \vee d_e(\overline{A})) = d_e(X).$$

Proof: Suppose that $\Gamma(X \oplus A) = \Lambda(X \oplus \overline{A}) = Y$.
Suppose $\langle y, F_1 \oplus D_1 \rangle$ is in Γ and $\langle y, F_2 \oplus D_2 \rangle$ is in Λ.
Check:

- $F_1 \cup F_2 \subseteq X$.
- It is the case that both $D_1 \not\subseteq A$ and $D_2 \not\subseteq \overline{A}$.
- Equivalently there is no pair $\langle \overline{a}, a \rangle \in D_1 \times D_2$ such that $s_A(\overline{a}, a) = a$.

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A generalization of semi-recursive sets

Definition (Kalimullin)

A pair of sets $\{A, B\}$ is a $\mathcal{K}$-pair if there is a c.e. set W, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

- Trivial $\mathcal{K}$-pairs: For every A and c.e. set U, $\{A, U\}$ is a $\mathcal{K}$-pair, witnessed by $\mathbb{N} \times U$.
- For every semi-recursive set A, $\{A, \overline{A}\}$ is a $\mathcal{K}$-pair witnessed by $\{\langle x, y, \rangle \mid s_A(x, y) = x\}$.
- If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $\overline{A} \leq_e B \oplus \emptyset'$.

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- Trivial $\mathcal{K}$-pairs: For every A and c.e. set U, $\{A, U\}$ is a $\mathcal{K}$-pair, witnessed by $\mathbb{N} \times U$.
- For every semi-recursive set A, $\{A, \overline{A}\}$ is a $\mathcal{K}$-pair witnessed by $\{\langle x, y, \ \rangle | \ s_A(x, y) = x\}$.
- If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $\overline{A} \leq_e B \oplus \emptyset'$.

Definability of $\mathcal{K}$-pairs

Theorem (Kalimullin)

$\{A, B\}$ is a $\mathcal{K}$-pair if and only if the degrees $\mathbf{a} = d_e(A)$, $\mathbf{b} = d_e(B)$ have the following property:

$$\mathcal{K}(\mathbf{a},\mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{D}_e)((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}).$$

- If $\mathcal{K}(\mathbf{a},\mathbf{b})$ and $\mathbf{c} \leq \mathbf{b}$ then $\mathcal{K}(\mathbf{a},\mathbf{c})$.
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Definability of the enumeration jump

Theorem (Kalimullin)

The enumeration jump is first order definable in $\mathcal{D}_e$.

- $\mathbf{0}'_e$ is the largest e-degree such that there are e-degrees $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, such that $\mathbf{a} \vee \mathbf{b} \vee \mathbf{c} = \mathbf{0}'_e$ and $\mathcal{K}(\mathbf{a}, \mathbf{b})$, $\mathcal{K}(\mathbf{b}, \mathbf{c})$, $\mathcal{K}(\mathbf{a}, \mathbf{c})$.
- $\mathcal{K}$-pairs can be relativized.

An alternative definition of the enumeration jump

Theorem (Ganchev, S)

For every nonzero enumeration degree $\mathbf{u} \in \mathcal{D}_e$, $\mathbf{u}'$ *is the largest among all least upper bounds* $\mathbf{a} \vee \mathbf{b}$ *of nontrivial* $\mathcal{K}$*-pairs* $\{\mathbf{a}, \mathbf{b}\}$, *such that* $\mathbf{a} \leq \mathbf{u}$.

Proof.

- If $\mathcal{K}(A, B)$ and $A \leq_e U$ then $B \leq_e \overline{A} \leq_e A'$, so $A \oplus B \leq_e A' \leq_e U'$.
- Consider the $\mathcal{K}$-pair $\{L_{K_U}, R_{K_U}\}$: then $L_{K_U} \leq_e K_U \equiv_e U$ and $L_{K_U} \oplus R_{K_U} \equiv_e K_U \oplus \overline{K_U} = U'$.

What if $\{L_{K_U}, R_{K_U}\}$ *is a trivial* $\mathcal{K}$*-pair, i.e. U is low* $(U' = \emptyset')$?

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Definability in the local structure of the enumeration degrees

Initial motivation: Prove that the theory of first order arithmetic is interpretable in $\mathcal{D}_e(\leq \mathbf{0}_e')$.

Theorem (Slaman and Woodin)

A uniformly low antichain can be coded by parameters in $\mathcal{D}_e(\leq \mathbf{0}_e')$.

1. Non-trivial Σ_2^0 $\mathcal{K}$-pairs are low.
2. A $\mathcal{K}$-system is a sequence of $\{\mathbf{a}_i\}_{i \in I}$ of e-degrees such that if $i \neq j$ then $\mathcal{K}(\mathbf{a}_i, \mathbf{a}_j)$.
3. $\mathcal{K}$ systems are antichains.
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$\mathcal{K}(\mathbf{a},\mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{a}\vee\mathbf{x})\wedge(\mathbf{b}\vee\mathbf{x}) = \mathbf{x})$

Is it enough to require that this formula is satisfied by all Σ^0_2 e-degrees?

Theorem (Ganchev, S)

There is a first order formula $\mathcal{LK}(x,y)$, which defines the $\mathcal{K}$-pairs in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

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Theorem (Ganchev, S)

$Th(\mathcal{D}_e(\leq \mathbf{0}'_e)) \equiv_1 Th(\mathbb{N})$.

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- Recall that an enumeration degree is total if an only if it is the least upper bound of $d_e(A) \oplus d_e(\overline{A})$ for some semi-recursive set $A \notin \Sigma^0_1 \cup \Pi^0_1$.
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We know that:

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Question

Is $\mathcal{TOT}$ first order definable in $\mathcal{D}_e$?

Every enumeration degree is the greatest lower bound of two total degrees. The total degrees are an automorphism base for $\mathcal{D}_e$.

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A positive answer would connect the problems of the existence of a non-trivial automorphism in both structures.

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One step further in the dream world

Theorem

For every nonzero enumeration degree $\mathbf{u} \in \mathcal{D}_e$*,* $\mathbf{u}'$ *is the largest among all least upper bounds* $\mathbf{a} \vee \mathbf{b}$ *of nontrivial* $\mathcal{K}$*-pairs* $\{\mathbf{a}, \mathbf{b}\}$*, such that* $\mathbf{a} \leq \mathbf{u}$*.*

- Say that the Turing degree $\mathbf{x}$ is c.e. in $\mathbf{u}$ if there are sets $X \in \mathbf{x}$ and $U \in \mathbf{u}$, such that X is c.e. in U.
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The end

Thank you!