Fragments of the theory of the enumeration degrees

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The theory of a degree structure

Let $\mathcal{D}$ be a degree structure.

Question

- Is the theory of the structure in the language of partial orders decidable?
- How complicated is the theory?
- How many quantifiers does it take to break decidability?

Degree structure	Complexity of $Th(\mathcal{D})$	$\exists\forall\exists$-$Th(\mathcal{D})$	$\forall\exists$-$Th(\mathcal{D})$
$\mathcal{D}_T$	Simpson 77	Lerman-Schmerl 83	Shore 78; Lerman 83
$\mathcal{D}_T(\leqslant \mathbf{0}')$	Shore 81	Lerman-Schmerl 83	Lerman-Shore 88
$\mathcal{R}$	Slaman-Harrington 80s	Lempp-Nies-Slaman 98	Open
$\mathcal{D}_e$	Slaman-Woodin 97	Open	Open
$\mathcal{D}_e(\leqslant \mathbf{0}'_e)$	Ganchev-Soskova 12	Kent 06	Open

Related problems

- To understand what existential sentences are true $\mathcal{D}$ we need to understand what finite partial orders can be embedded into $\mathcal{D}$;
- At the next level of complexity is the *extension of embeddings problem*:

Problem

We are given a finite partial order P and a finite partial order $Q \supseteq P$. Does every embedding of P extend to an embedding of Q?

- To understand what $\forall\exists$-sentences are true in $\mathcal{D}$ we need to solve a slightly more complicated problem:

Problem

We are given a finite partial order P and finite partial orders $Q_0, \ldots Q_n \supseteq P$. Does every embedding of P extend to an embedding of one of the Q_i?

The Turing degrees and initial segment embeddings

Theorem (Lerman 71)

Every finite lattice can be embedded into $\mathcal{D}_T$ as an initial segment.

- Suppose that P is a finite partial order and $Q \supseteq P$ is a finite partial order extending P.
- We can extend P to a lattice by adding extra points for joins when necessary.
- The initial segment embedding of the lattice P can be extended to an embedding of Q only if new elements in $Q \smallsetminus P$ are compatible with joins in P:
  1. If $q \in Q \smallsetminus P$ is bounded by some element in P then q is one of the added joins.
  2. If $x \in Q \smallsetminus P$ and $u, v \in P$ and $x \geqslant u, v$ then $x \geqslant u \vee v$.

Theorem (Shore 78; Lerman 83)

That is the only obstacle.

A characterization

Let U be an upper semilattice.

Definition

We say that U *exhibits end-extensions* if for every pair of a finite lattice P and partial order $Q \supseteq P$ such that if $x \in Q \setminus P$ then x is never below any element of P and x respects least upper bounds, every embedding of P into U extends to an embedding of Q into U.

Theorem (Lempp, Slaman, Soskova)

Let φ be a Π_2-sentence in the language of partial orders. The sentence φ is true in $\mathcal{D}_T$ if and only if φ is true in every upper semilattice U with least element that exhibits end-extensions.

The theory of a degree structure

Lets take a look at the table again:

Question

- Both $\mathcal{R}$ and $\mathcal{D}_e(\leqslant \mathbf{0}')$ are dense structures.
- In fact, any countable partial order embeds into any nonempty interval.
- But what is the case of $\mathcal{D}_e$?

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The enumeration degrees

Theorem (Gutteridge 71)

The enumeration degrees are downwards dense.

A degree $\mathbf{b}$ is a *minimal cover* of a degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ and the interval $(\mathbf{a}, \mathbf{b})$ is empty.

Theorem (Slaman, Calhoun 96)

There are degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is a minimal cover of $\mathbf{a}$.

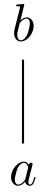
A degree $\mathbf{b}$ is a *strong minimal cover* of a degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ and for every degree $\mathbf{x} < \mathbf{b}$ we have that $\mathbf{x} \leqslant \mathbf{a}$.

Theorem (Kent, Lewis-Pye, Sorbi 12)

There are degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{b}$ is a strong minimal cover of $\mathbf{a}$

The simplest lattice

Consider the lattice $\mathcal{L} = \{a < b\}$. What properties should possible extensions $Q_0, Q_1 \dots Q_n$ have so that every embedding of $\mathcal{L}$ extends to Q_i for some i:



1. We can embed $\mathcal{L}$ as degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is a strong minimal cover of $\mathbf{a}$, blocking extensions to Q_i with new x in the interval $[a, b]$.
2. We can embed $\mathcal{L}$ as degrees $\mathbf{0}_e < \mathbf{b}$, blocking extensions to Q_i with new $x < a$.

Theorem (Slaman, Sorbi 14)

Every countable partial order can be embedded below any nonzero enumeration degree.

So these are the only obstacles.

A wild conjecture

Let U be an upper semilattice.

Definition

U exhibits strong downward density if every countable partial order can be embedded below any nonzero element of U.

Conjecture (Lempp, Slaman, Soskova)

A Π_2 sentence φ is true in $\mathcal{D}_e$ if and only if φ is true in every upper semilattice U with least element that exhibits end-extensions and strong downward density.

- This would imply a decision procedure for the two quantifier theory of $\mathcal{D}_e$
- This would imply that we can extend the existence of strong minimal covers significantly:

Strong interval embeddings

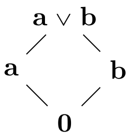
Definition

Let $\mathcal{L}$ be a lattice. We say that $\mathcal{L}$ *strongly embeds as an interval* in $\mathcal{D}_e$ if there are degrees $\mathbf{a} < \mathbf{b}$ and a bijection $f : \mathcal{L} \to [\mathbf{a}, \mathbf{b}]$ such that for every $\mathbf{x} \leqslant \mathbf{b}$ we have that $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ or else $\mathbf{x} < \mathbf{a}$.

- A strong minimal cover induces a strong interval embedding of the 2-element lattice.
- The conjecture implies that every finite lattice has a strong interval embedding in $\mathcal{D}_e$.
- In fact, it would imply much more—for instance, the following statement:

There are degrees $\mathbf{a}$ *and* $\mathbf{b}$ *such that:*

1. $\mathbf{a}$ *and* $\mathbf{b}$ *are a minimal pair.*
2. *if* $\mathbf{x} < \mathbf{a} \vee \mathbf{b}$ *then* $\mathbf{x} \leqslant \mathbf{a}$ *or* $\mathbf{x} \leqslant \mathbf{b}$.



A small victory

Theorem (Lempp, Slaman, Soskova)

Every finite distributive lattice has a strong interval embedding.

Applying Nies' Transfer Lemma we get:

Corollary

The $\exists\forall\exists$-theory of $\mathcal{D}_e$ is undecidable.

Degree structure	Complexity of $Th(\mathcal{D})$	$\exists\forall\exists$-$Th(\mathcal{D})$	$\forall\exists$-$Th(\mathcal{D})$
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An additional application

Theorem (Lempp, Slaman, Soskova)

The extension of embeddings problem in $\mathcal{D}_e$ is decidable.

Proof sketch:
Given finite orders $P \subseteq Q$, if $q \in Q \setminus P$ is a point that violates the conditions of the usual algorithm (the one for $\mathcal{D}_T$) then we build a specific embedding that blocks q.

The common fragment of the theories of $\mathcal{D}_T$ and $\mathcal{D}_e$

Note that the theories of $\mathcal{D}_e$ and $\mathcal{D}_T$ differ at a Σ_2 sentence φ:

$$(\exists \mathbf{a})[\mathbf{a} \neq \mathbf{0} \wedge \forall \mathbf{x}[\mathbf{x} < \mathbf{a} \rightarrow \mathbf{x} = \mathbf{0}]]$$

Theorem

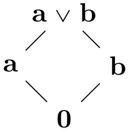
Let E denote the set of Π_2-sentences in the language of a partial orders that formalize an instance of the extension of embeddings problem. Then $E \cap Th(\mathcal{D}_e) = E \cap Th(\mathcal{D}_T)$.

Proof sketch:

- One direction uses our characterization of the two quantifier theory of $\mathcal{D}_T$ and the fact that $\mathcal{D}_e$ is an upper semilattice that exhibits end extensions.
- The reverse direction follows from the proof of the extension of embedding theorem.

An unexpected defeat

Recall that our conjecture implies that there are degrees $\mathbf{a}$ and $\mathbf{b}$ such that: $\mathbf{a}$ and $\mathbf{b}$ are a minimal pair and if $\mathbf{x} < \mathbf{a} \vee \mathbf{b}$ then $\mathbf{x} \leqslant \mathbf{a}$ or $\mathbf{x} < \mathbf{b}$.



This is an instance of a *super minimal pair*: a minimal pair $\{\mathbf{a}, \mathbf{b}\}$ such that every degree $\mathbf{x} < \mathbf{a}$ joins $\mathbf{b}$ above $\mathbf{a}$ and every degree $\mathbf{x} < \mathbf{b}$ joins $\mathbf{a}$ above $\mathbf{b}$

Theorem (Jacobsen-Grocott, Soskova)

If $\mathbf{a}$ and $\mathbf{b}$ are enumeration degrees such that every degree $\mathbf{x} \leqslant \mathbf{a} \vee \mathbf{b}$ is bounded by $\mathbf{a}$ or bounded by $\mathbf{b}$, then $\{\mathbf{a}, \mathbf{b}\}$ is not a minimal pair.

However!

Theorem (Jacobsen-Grocott)

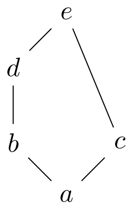
There are degrees $\mathbf{a}$ and $\mathbf{b}$ that form a minimal pair and every degree $\mathbf{x} < \mathbf{a}$ joins $\mathbf{b}$ above $\mathbf{a}$.

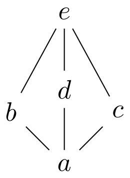
Questions

Question

Can we embed all finite lattices in $\mathcal{D}_e$ as strong intervals?

Important test cases are N_5 and M_3:





Question

Are there super minimal pairs in $\mathcal{D}_e$?

Question

What property characterizes the two quantifier theory of $\mathcal{D}_e$?

Thank you!