Characterizing the strength of the local theory of the enumeration degrees

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SEALS

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Preliminiaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)\}$.
- $A \equiv_e B$ iff $A \leq_e B$ and $B \leq_e A$.
- $d_e(A) = [A]_{\equiv_e}$ and $D_e = \{d_e(A) \mid A \subseteq \mathbb{N}\}$.
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

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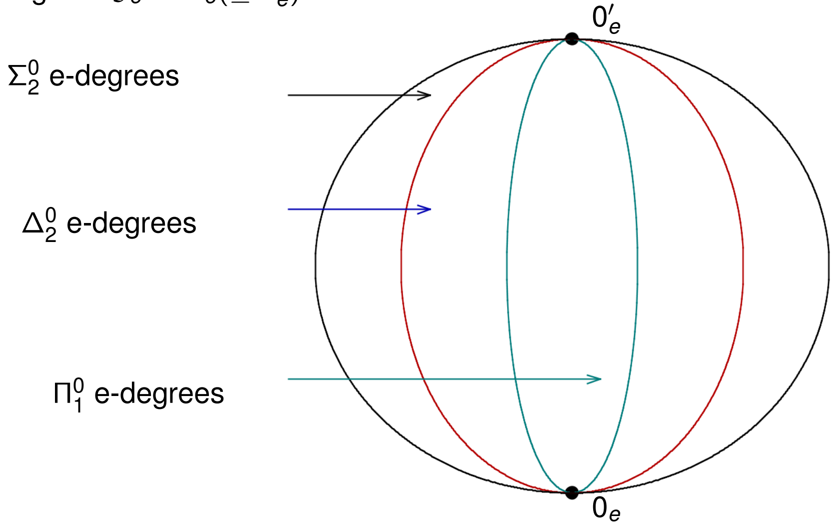
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

Preliminiaries: The local structure

The jump operation gives rise to the local structure of the enumeration degrees $\mathcal{G}_e = \mathcal{D}_e(\leq 0'_e)$.



Preliminaries: Previous results

Theorem (Slaman and Woodin)

The theory of $\mathcal{D}_e$ is comutably isomorphic to the theory of second order arithmetic.
The theory of $\mathcal{G}_e$ is undecidable.

Theorem (Kent)

The theory of the Δ^0_2 enumeration degrees is computably isomorphic to the theory of first order arithmetic.

Question

Is the theory of $\mathcal{G}_e$ computably isomorphic to first order arithmetic?

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The general plan: Coding standard models of arithmetic

Given a sentnece in the langauge of true arithemtic φ we want to be able to computably translate it into a sentence φ_e in the langaunge of the $\mathcal{G}_e$ so that:

$$\langle \mathbb{N}, +, * \rangle \models \varphi \text{ iff } \mathcal{G}_e \models \varphi_e$$

I Represent $\langle \mathbb{N}, +, * \rangle$ as a partial order (PO).

II Embed this partial order in $\mathcal{G}_e$ and code it with a finite number of parameters.

III Find a first order condition on the parameters, which ensures that they code a SMA.

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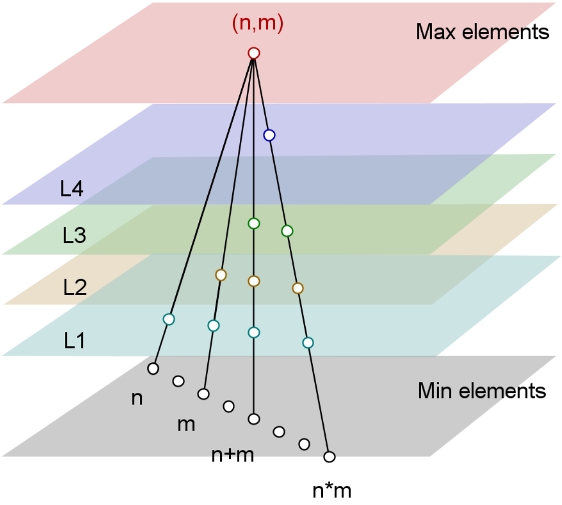
A special type of partial order

We can represent an SMA $\langle \mathbb{N}, +, * \rangle$ as follows:



First tool: Coding antichains

$$\varphi_{SW}(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}) \iff \boldsymbol{x} \leq \boldsymbol{a} \text{ is a minimal solution to}$$

$$\boldsymbol{x} \neq (\boldsymbol{x} \vee \boldsymbol{p}) \wedge (\boldsymbol{x} \vee \boldsymbol{q}).$$

Theorem (Slaman, Woodin)

Let $\{X_i \mid i \in \mathbb{N}\}$ be a system of incomparable sets uniformly enumeration reducible to a low set A with degree $\mathbf{a}$. There are Σ^0_2 e-degrees $\mathbf{p}$ and $\mathbf{q}$, such that for arbitrary Σ^0_2 degree $\mathbf{x}$

$$\mathcal{G}_e \models \varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}, \mathbf{q}) \iff \exists i[X_i \in \mathbf{x}].$$

Goal: Embed the PO so that each level is *well presented*.

Second tool: $\mathcal{K}$-pairs

Iskander Kalimullin: Definability of the jump operator in the enumeration degrees
Journal of Mathematical Logic (2003)

Definition

Let A and B be non-c.e. sets of a natural numbers. The pair (A, B) is a $\mathcal{K}$-pair (e-ideal) if there exists a c.e. set W, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

Theorem (Kalimullin)

(A, B) is a $\mathcal{K}$-pair if and only if the degrees $\mathbf{a} = d_e(A)$ and $\mathbf{b} = d_e(B)$ have the following property:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons \mathbf{a}, \mathbf{b} > \mathbf{0}_e \&$$

$$(\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

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Properties of $\mathcal{K}$-pairs

Theorem (Kallimulin)

1. *If* $(\mathbf{a}, \mathbf{b})$ *are a* Σ^0_2 $\mathcal{K}$*-pair then* $\mathbf{a}$ *and* $\mathbf{b}$ *are low.*
2. *Every* $\mathcal{K}$*-pair is a minimal pair.*
3. *Every nonzero* Δ^0_2 *enumeration degree bounds a* $\mathcal{K}$*-pair.*
4. *The set of degrees* $\mathbf{b}$ *which form a* $\mathcal{K}$*-pair with a fixed degree* $\mathbf{a}$ *is an ideal.*

$\mathcal{K}$-systems

Definition

We shall say that a system of degrees $\{\mathbf{a}_i \mid i \in I\}$ ($|I| \geq 2$) is a $\mathcal{K}$-system, if $\mathcal{K}(\mathbf{a}_i, \mathbf{a}_j)$ for each $i, j \in I$, such that $i \neq j$.

- Every $\mathcal{K}$-system is an antichain.
- If $\{\mathbf{a}_i \mid i \in I\}$ is a $\mathcal{K}$-system and $i_1 \neq i_2 \in I$ then $\{\mathbf{a}_{i_1} \vee \mathbf{a}_{i_2}\} \cup \{\mathbf{a}_i \mid i \in I, i \neq i_1, i_2\}$ is a $\mathcal{K}$-system .

Theorem

Let A be a Δ^0_2 non-c.e. set. There is a sequence $\{A_i\}_{i<\omega}$ uniformly enumeration reducible to A such that $\{d_e(A_i)\}_{i<\omega}$ is a $\mathcal{K}$-system.

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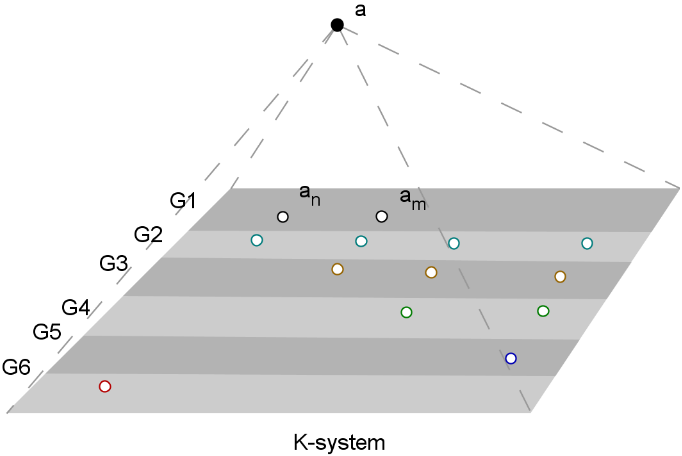
Coding an SMA below any half of a $\mathcal{K}$-pair

Construction:
Let $\mathbf{a} = d_e(A)$ be half of a $\mathcal{K}$-pair. (Hence a low nonzero Δ_2^0 enumeration degree.)
Let $\{A_i\}_{i<\omega}$ be the uniformly reducible to A sequence whose degrees $\{\mathbf{a}_i\}_{i<\omega}$ form a $\mathcal{K}$-system. This is a *well presented system*.
We computably divide the system $\{\mathbf{a}_i\}_{i<\omega}$ into six infinite groups.

Coding an SMA below any half of a $\mathcal{K}$-pair

To every pair of elements from G1 we assign 4 unique elements of G2, 3 of G3, 2 of G4 and 1 of each G5 and G6.



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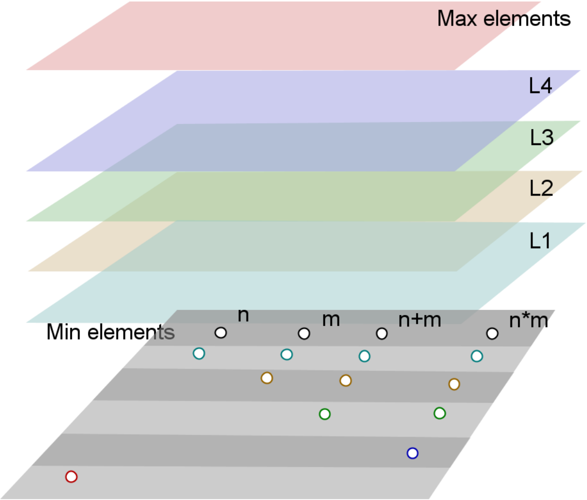

The elements of G1 will represent the natural numbers. There are parameters $\mathbf{p}_0$ and $\mathbf{q}_0$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_0, \mathbf{q}_0)$ defines them.

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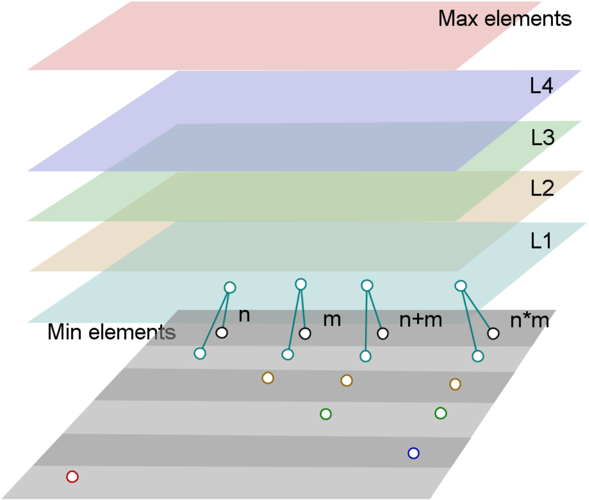

L1 is constructed from lub's of elements from G1 and G2. There are parameters $\mathbf{p}_1$ and $\mathbf{q}_1$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_1, \mathbf{q}_1)$ defines them.

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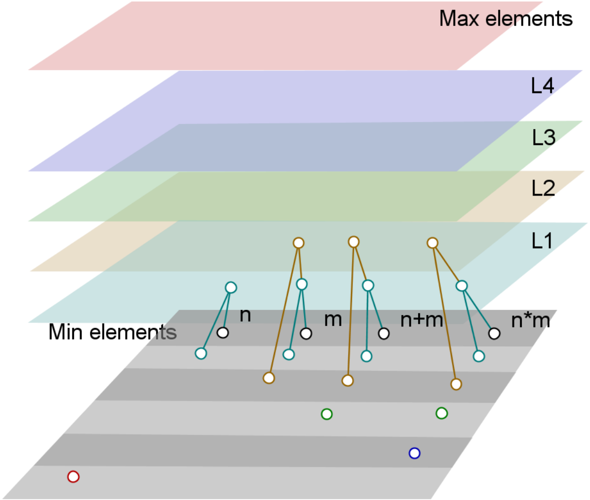

L2 is constructed from lub's of elements from L1 and G3. There are parameters $\mathbf{p}_2$ and $\mathbf{q}_2$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_2, \mathbf{q}_2)$ defines them.

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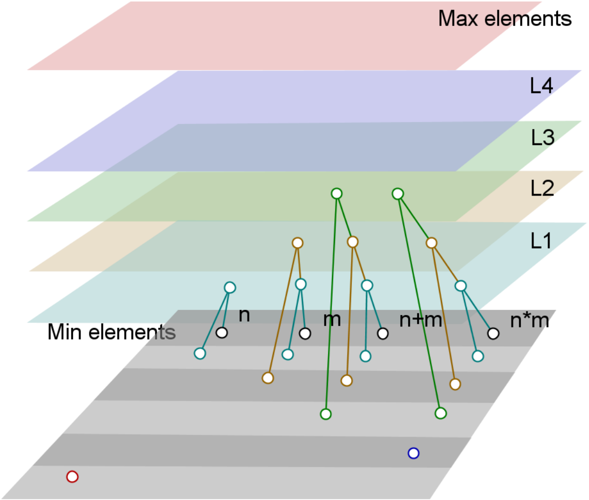

L3 is constructed from lub's of elements from L2 and G4. There are parameters $\mathbf{p}_3$ and $\mathbf{q}_3$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_3, \mathbf{q}_3)$ defines them.

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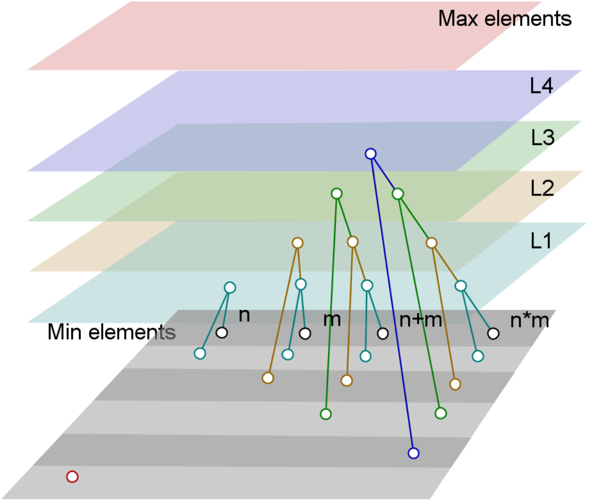
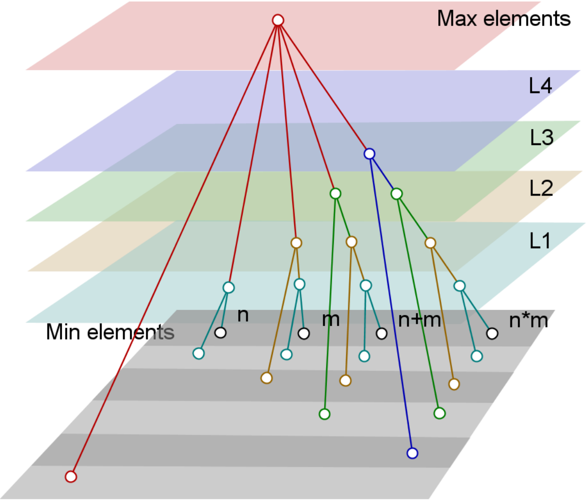


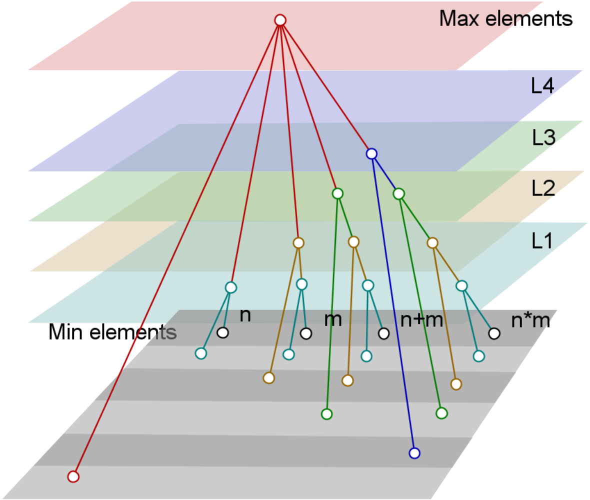
L4 is constructed from lub's of elements from L3 and G5. There are parameters $\mathbf{p}_4$ and $\mathbf{q}_4$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_4, \mathbf{q}_4)$ defines them.

Coding an SMA below any half of a $\mathcal{K}$-pair



Finally the maximal elements are constructed from lub's of elements from L1, L2, L3, L4 and G6. $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_5, \mathbf{q}_5)$ defines them.

Coding an SMA below any half of a $\mathcal{K}$-pair



So the parameters $\mathbf{a}$, $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$ $\mathbf{p}_3$, $\mathbf{p}_4$, $\mathbf{p}_5$, $\mathbf{q}_0$, $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_4$, $\mathbf{q}_5$ code a partial order, which represents a standard model of arithmetic $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$.

The other direction

Given parameters $\mathbf{a}$, $\mathbf{p_0}$, $\mathbf{p_1}$, $\mathbf{p_2}$ $\mathbf{p_3}$, $\mathbf{p_4}$, $\mathbf{p_5}$, $\mathbf{q_0}$, $\mathbf{q_1}$, $\mathbf{q_2}$, $\mathbf{q_3}$, $\mathbf{q_4}$, $\mathbf{q_5}$, let $PO = \{\mathbf{x} \mid \varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_i, \mathbf{q}_i) \text{ for some } i = 0, 1, 2, 3, 4, 5\}$.
We can define a first order condition $ST_0(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ so that the partial order $(PO, \leq)$ satisfies:

(M1) Every element is either minimal, maximal or in an interval with endpoints a minimal and a maximal element.

(M2) For every pair of minimal elements there exists a unique maximal element at distance 1 from the first and distance 2 from the second.

(M3) For every maximal element m there exists a unique quadruple of minimal elements below it such that the first one is at distance 1 from m, the second is at distance 2, the third at distance 3 and the fourth at distance 4 from m.

The other direction

Given parameters $\mathbf{a}$, $\mathbf{p_0}$, $\mathbf{p_1}$, $\mathbf{p_2}$ $\mathbf{p_3}$, $\mathbf{p_4}$, $\mathbf{p_5}$, $\mathbf{q_0}$, $\mathbf{q_1}$, $\mathbf{q_2}$, $\mathbf{q_3}$, $\mathbf{q_4}$, $\mathbf{q_5}$, let $PO = \{\mathbf{x} \mid \varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_i, \mathbf{q}_i)$ for some $i = 0, 1, 2, 3, 4, 5\}$.
We can define a first order condition $ST_0(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ on the parameters $\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}}$ so that the partial order $(PO, \leq)$ satisfies:

R_+ The relation $R_+(x, y, z) =_{def} min(x) \& \min(y) \& \min(z) \& \exists m(\max(m) \& x <_1 m \;\&\; y <_2 m \;\&\; z <_3 m)$ defines an operation $+$;

R_* The relation $R_*(x, y, z) =_{def} min(x) \& \min(y) \& \min(z) \& \exists m(\max(m) \& x <_1 m \;\&\; y <_2 m \;\&\; z <_4 m)$ defines an operation $*$;

PA^- The structure $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}}) = \langle \{x \in PO \mid \min(x)\}, +, * \rangle$ is a model of arithmetic which contains a standard part.

Isolating parameters which code SMA'a

We will ask that $ST_0(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ ensures also:

- $\mathbf{a}$ is half of $\mathcal{K}$-pair
- The minimal elements in *PO* form a $\mathcal{K}$-system.

Let **b** be such that **a** and **b** are a $\mathcal{K}$-pair. There are parameters $\overline{\mathbf{p}'}$ and $\overline{\mathbf{q}'}$ such that $ST_0(\mathbf{b}, \overline{\mathbf{p}'}, \overline{\mathbf{q}}')$ and $\mathfrak{A}(\mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')$ is a standard model of arithmetic.

It will be enough to require that $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ can be embedded into $\mathfrak{A}(\mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')$.

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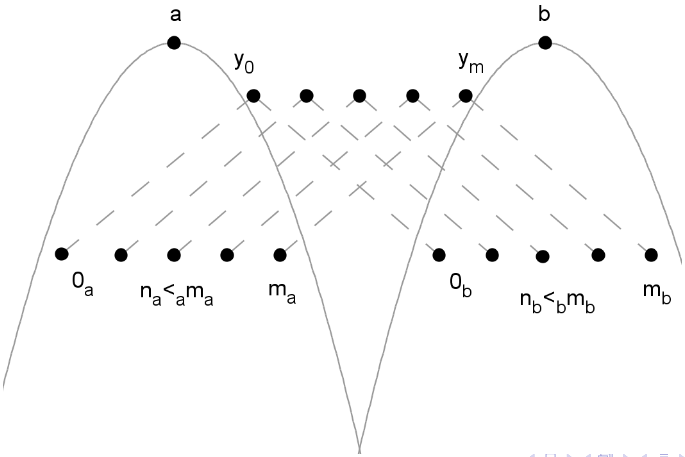
Comparison maps

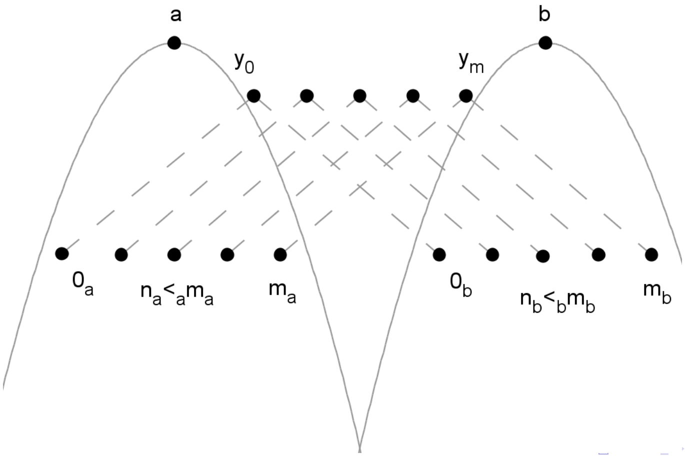
We will additionally ask that for every element $m_a \in \mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ there is an element $m_b \in \mathfrak{A}(\mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')$ and an antichain $(y_0, y_1, \ldots y_m)$ coded by parameters $\mathbf{c}$, $\mathbf{p}''$ and $\mathbf{q}''$ such that:



Comparison maps

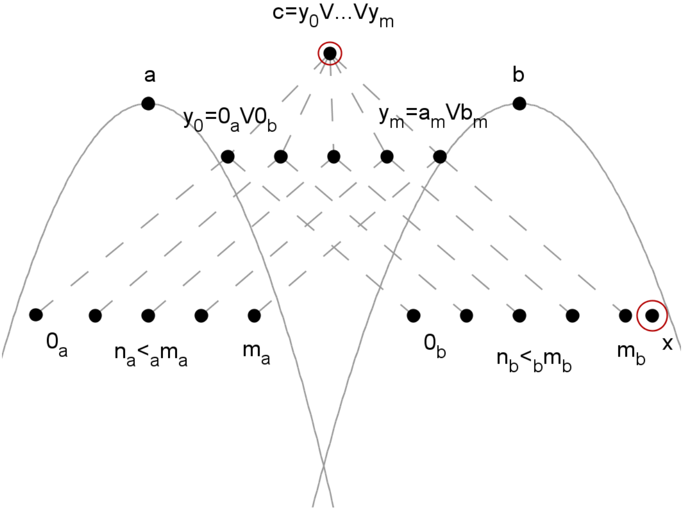
Denote this requirement by $\mathcal{M}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}}, \mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')$

- If for all $\overline{\mathbf{p}}', \overline{\mathbf{q}}'$ such that $ST_0(\mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')$ we have $\mathcal{M}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}}, \mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')$ then $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ is an SMA.



Comparison maps

- If $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ is an SMA then this condition is true.



SMA condition

If **a**, $\overline{\mathbf{p}}$, $\overline{\mathbf{q}}$ satisfy:

SMA

$$ST_0(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$$

and

$$\exists \mathbf{b}(\mathcal{K}(\mathbf{a}, \mathbf{b})) \ \& \ \forall \overline{\mathbf{p}'}, \forall \overline{\mathbf{q}'}[ST_0(\mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}') \Longrightarrow \mathcal{M}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}}, \mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')]$$

then $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ is a standard model of arithmetic.

Of course, all this relies on the assumption that being a $\mathcal{K}$-pair is a property definable in $\mathcal{G}_e$!

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An order theoretic characterization of $\mathcal{K}$-pairs

Theorem (Kalimullin)

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Is it enough to check that:

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for all $\mathbf{x} \in \mathcal{G}_e$?

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Definability of $\mathcal{K}$-pairs

Theorem (Kalimullin)

If (A, B) is not a $\mathcal{K}$-pair then there is a witness C computable from $A \oplus B \oplus K$ such that:

$$(d_e(A) \vee d_e(C)) \wedge (d_e(B) \vee d_e(C)) \neq d_e(C)$$

- If $\mathbf{a}$ and $\mathbf{b}$ are Δ_2^0 then C is also Δ_2^0 and $\mathcal{K}(\mathbf{a}, \mathbf{b})$ ensures "$\mathbf{a}$ and $\mathbf{b}$ are a true $\mathcal{K}$-pair".
- If $\mathbf{a}$ and $\mathbf{b}$ are properly Σ_2^0 then C is Δ_3^0. So it is possible that there is a fake $\mathcal{K}$-pair $\mathbf{a}$ and $\mathbf{b}$ such that

$$\mathcal{G}_e \models \mathcal{K}(\mathbf{a}, \mathbf{b}), \text{ but } \mathcal{D}_e \models \neg\mathcal{K}(\mathbf{a}, \mathbf{b})$$

Definability of $\mathcal{K}$-pairs

Theorem (Kalimullin)

If (A, B) is not a $\mathcal{K}$-pair then there is a witness C computable from $A \oplus B \oplus K$ such that:

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- If $\mathbf{a}$ and $\mathbf{b}$ are Δ^0_2 then C is also Δ^0_2 and $\mathcal{K}(\mathbf{a}, \mathbf{b})$ ensures "**a** and **b** are a true $\mathcal{K}$-pair".
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Cupping properties

Definition

A Σ^0_2 enumeration degree $\mathbf{a}$ is called *cuppable* if there is an incomplete Σ^0_2 e-degree $\mathbf{b}$, such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.
If furthermore $\mathbf{b}$ is low, then $\mathbf{a}$ will be called *low cuppable*.

Theorem (S,Wu)

Every nonzero Δ^0_2 enumeration degree $\mathbf{a}$ is low cuppable, i.e. there is a low $\mathbf{b}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.

Theorem (Cooper, Sorbi, Yi)

There are non-cuppable nonzero Σ^0_2 enumeration degrees.

Question

Are all cuppable degrees also low cuppable?

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Are all cuppable degrees also low cuppable?

A locally definable set of low degrees

Theorem

If **u** *and* **v** *are* Σ^0_2 *enumeration degrees such that* $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ *then* **u** *is low cuppable or* **v** *is low cuppable.*

Proof:

Uses a construction very similar to the construction of a non-splitting enumeration degree.

A locally definable set of low degrees

Corollary

The formula $\mathcal{L}(\mathbf{a}) \Leftrightarrow (\exists \mathbf{b})(\mathcal{K}(\mathbf{a}, \mathbf{b}) \& (\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e))$ *defines in* $\mathcal{G}_e$ *a nonempty set of true halves of* $\mathcal{K}$*-pairs.*

Proof:
Kallimulin has proved that there is a Δ^0_2 $\mathcal{K}$-pair which splits $\mathbf{0}'_e$ so the set $\{\mathbf{a} \mid \mathcal{G}_e \models \mathcal{L}(\mathbf{a})\} \neq \emptyset$.

Let $\mathbf{a}$ be a Σ^0_2 degree such that $\mathcal{G}_e \models \mathcal{L}(\mathbf{a})$. Let $\mathbf{b}$ a witness such that $\mathcal{K}(\mathbf{a}, \mathbf{b}) \wedge (\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e)$.

Then $\mathbf{a}$ is low cuppable or $\mathbf{b}$ is low cuppable.

A locally definable set of low degrees

Corollary

The formula $\mathcal{L}(\mathbf{a}) \Leftrightarrow (\exists \mathbf{b})(\mathcal{K}(\mathbf{a},\mathbf{b}) \& (\mathbf{a} \vee \mathbf{b} = \mathbf{0}_e'))$ *defines in* $\mathcal{G}_e$ *a nonempty set of true halves of* $\mathcal{K}$*-pairs.*

Proof:

If $\mathbf{b}$ is low cuppable then let $\mathbf{c}$ be a low Δ^0_2 e-degree which cups $\mathbf{b}$.

$$(\mathbf{a} \vee \mathbf{c}) \wedge \underbrace{(\mathbf{b} \vee \mathbf{c})} = \mathbf{c}$$

$$(\mathbf{a} \vee \mathbf{c}) \wedge \quad \mathbf{0}_e' \quad = \mathbf{c}$$

$$(\mathbf{a} \vee \mathbf{c}) \qquad = \mathbf{c}$$

So $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{a}' \leq \mathbf{c}' = \mathbf{0}_e'$ and $\mathbf{a}$ is low, hence Δ^0_2 and hence low-cuppable .

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The formula $\mathcal{L}(\mathbf{a}) \Leftrightarrow (\exists \mathbf{b})(\mathcal{K}(\mathbf{a}, \mathbf{b}) \& (\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e))$ *defines in* $\mathcal{G}_e$ *a nonempty set of true halves of* $\mathcal{K}$*-pairs.*

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The formula $\mathcal{L}(\mathbf{a}) \Leftrightarrow (\exists \mathbf{b})(\mathcal{K}(\mathbf{a}, \mathbf{b}) \& (\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e))$ *defines in* $\mathcal{G}_e$ *a nonempty set of true halves of* $\mathcal{K}$*-pairs.*

Proof:
If **b** is low cuppable then let **c** be a low Δ^0_2 e-degree which cups **b**.

$$(\mathbf{a} \vee \mathbf{c}) \wedge \underbrace{(\mathbf{b} \vee \mathbf{c})} = \mathbf{c}$$

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The formula $\mathcal{L}(\mathbf{a}) \Leftrightarrow (\exists \mathbf{b})(\mathcal{K}(\mathbf{a}, \mathbf{b}) \& (\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e))$ *defines in* $\mathcal{G}_e$ *a nonempty set of true halves of* $\mathcal{K}$*-pairs.*

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The formula $\mathcal{L}(\mathbf{a}) \Leftrightarrow (\exists \mathbf{b})(\mathcal{K}(\mathbf{a},\mathbf{b})\&(\mathbf{a} \vee \mathbf{b} = \mathbf{0}_e'))$ *defines in* $\mathcal{G}_e$ *a nonempty set of true halves of* $\mathcal{K}$*-pairs.*

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If $\mathbf{b}$ is low cuppable then let $\mathbf{c}$ be a low Δ_2^0 e-degree which cups $\mathbf{b}$.

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If **a** is low cuppable then let **d** be a low Δ_2^0 e-degree which cups **a**.

$$\underbrace{(\mathbf{a} \vee \mathbf{d})} \wedge (\mathbf{b} \vee \mathbf{d}) = \mathbf{d}$$

$$\mathbf{0}_e' \wedge (\mathbf{b} \vee \mathbf{d}) = \mathbf{d}$$

$$(\mathbf{b} \vee \mathbf{d}) = \mathbf{d}$$

So $\mathbf{b} \leq \mathbf{d}$ and hence **b** is low, Δ_2^0 and low cuppable.

In either case both **a** and **b** are Δ_2^0 and hence $\mathcal{K}(\mathbf{a},\mathbf{b})$ ensures that they form a true $\mathcal{K}$-pair.

A locally definable set of low degrees

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The formula $\mathcal{L}(\mathbf{a}) \Leftrightarrow (\exists \mathbf{b})(\mathcal{K}(\mathbf{a}, \mathbf{b}) \& (\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e))$ *defines in* $\mathcal{G}_e$ *a nonempty set of true halves of* $\mathcal{K}$*-pairs.*

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The final condition SMA

Finally if **a**, $\overline{\mathbf{p}}$, $\overline{\mathbf{q}}$ satisfy:

SMA

$$ST_0(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}}) \& \mathcal{L}(\mathbf{a})$$

and

$$\exists \mathbf{b}(\mathcal{K}(\mathbf{a}, \mathbf{b})) \ \& \ \forall \overline{\mathbf{p}'}, \forall \overline{\mathbf{q}'}[ST_0(\mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}') \Longrightarrow \mathcal{M}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}}, \mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')]$$

then $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ is a standard model of arithmetic.

The end

Thank you!