Seven characterizations of the cototal enumeration degrees

Mariya I. Soskova[1]

Sofia University

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Motivation from symbolic dynamics

Definition

- A *subshift* is a nonempty closed set $X \subseteq 2^\omega$ such that if $a\alpha \in X$ then $\alpha \in X$.
- X is *minimal* if there is no $Y \subset X$, such that Y is a subshift.

Given a minimal subshift X, we would like to characterize the set of Turing degrees that compute members of X.

Definition

The *language* of subshift X is the set

$$L_X = \{\sigma \in 2^{<\omega} \mid \exists \alpha \in X(\sigma \text{ is a subword of } \alpha)\}.$$

1. If X is minimal and $\sigma \in L_X$ then for every $\alpha \in X$, σ is a subword of α. So every element of X can enumerate the set L_X.
2. If we can enumerate L_X then we can compute a member of X.

The enumeration degrees and cototal sets

Definition

$A \leq_e B$ if every enumeration of B can compute an enumeration of A.

The *enumeration-cone* of a set A is the set of Turing degrees that can enumerate A.

The enumeration-cone of L_X is the set of Turing degrees that compute members of X.

(Jaendel:) If we can enumerate the set of *forbidden words* $\overline{L_X}$ then we can enumerate L_X. Use compactness: if $\sigma \in L_X$ then there is a level n such that for every $\alpha \in X$, σ is a subword of $\alpha \upharpoonright n$.

So $L_X \leq_e \overline{L_X}$.

Definition

A set A is *cototal* if $A \leq_e \overline{A}$. An enumeration degree is cototal if it contains a cototal set.

Examples of cotelal enumeration degrees

Proposition

Every total e-degree is cototal.

$A \oplus \overline{A} \equiv_1 \overline{A \oplus \overline{A}} = \overline{A} \oplus A.$

Proposition

Every Σ_2^0 e-degree is cototal.

Let A be Σ_2^0. Consider the set $K_A = \bigoplus_{e<\omega} \Gamma_e(A)$. Then $A \equiv_e K_A$ and

$$\overline{K}_A = \bigoplus_{e<\omega} \overline{\Gamma_e(A)} \geq_e \overline{K} \geq_e A \equiv_e K_A.$$

Characterization I: The skip operator

Note, that $A \leq_e B$ if and only if $\overline{K_A} \leq_1 \overline{K_B}$.

Definition (AGKLMSS)

The *skip* of A is the set $A^\Diamond = \overline{K_A}$. The *skip* of a degree is $\mathbf{d}_e(A)^\Diamond = \mathbf{d}_e(A^\Diamond)$.

Recall, that the *enumeration jump* of A is defined by $A' = K_A \oplus \overline{K_A}$. So for every enumeration degree $\mathbf{a}$ we have that $\mathbf{a}' = \mathbf{a} \vee \mathbf{a}^\Diamond$.

Theorem (AGKLMSS)

For every $\mathbf{b} \geq \mathbf{0}'_e$ there is a degree $\mathbf{a}$ such that $\mathbf{a}^\Diamond = \mathbf{b}$.

Proposition (AGKLMSS)

A degree $\mathbf{a}$ is cototal if and only if $\mathbf{a} \leq \mathbf{a}^\Diamond$ (if and only if $\mathbf{a}^\Diamond = \mathbf{a}'$).

The continuous enumeration degrees

J. Miller introduced the *continuous degrees* $\mathcal{D}_r$ to compare the complexity of points in computable metric spaces.

A point x in a computable metric space can be described by a sequence of "rational" points that limit to it. For two points x, y we say that $x \leq_r y$ if every description of x computes a description of y.

The continuous degrees embed into $\mathcal{D}_e$.

In fact, $\mathcal{D}_T \subset \mathcal{D}_r \subset \mathcal{D}_e$.

Proposition (AGKLMSS)

Every continuous enumeration degree is cot、otal.

Characterization II: a topological perspective

Kihara and Pauly extend Miller's idea to points in arbitrary *represented* topological spaces. They define the *point degree spectrum* of a represented space.

1. $Spec(2^\omega) = Spec(\omega^\omega) = \mathcal{D}_T$;
2. $Spec([0,1]^\omega) = Spec(C([0,1])) = \mathcal{D}_r$;
3. $Spec(S^\omega) = \mathcal{D}_e$, where $S = \{\emptyset, \{0\}, \{0,1\}\}$ is the Sierpinski space.

The point degree spectrum of any second countable T_0 space corresponds to a class of enumeration degrees.

If $\mathcal{X}$ is second countable T_0 and $\{N_i\}_{i<\omega}$ is a countable basis for $\mathcal{X}$ then we map a point x to the enumeration degree of $\{i \mid x \in N_i\}$.

Theorem (Kihara)

The cototal e-degrees are the elements of the point degree spectra of all sufficiently effective second countable G_δ spaces. (Every closed set is G_δ).

Characterization III and IV: Graph theory

Definition (Carl von Jaenisch)

Let $G = (V, E)$ be a graph. A set $M \subseteq V$ is *independent*, if no two members of M are edge related. M is *maximal* set, if every $v \in \overline{M}$ is edge related to a vertex in M.

$\overline{M} \leq_e M$ because $v \in \overline{M}$ iff there is a $w \in M$ such that w and v are edge related.

Theorem (AGKLMSS)

An enumeration degree is cototal if and only if it contains the complement of a maximal independent set for the graph $\omega^{<\omega}$.

Theorem (McCarthy)

An enumeration degree is cototal if and only if it contains the complement of a maximal antichain in $\omega^{<\omega}$.

Characterization V: E-pointed trees

Theorem (Montalbán)

A degree spectrum is never the Turing-upward closure of an F_σ set of reals in ω^ω, unless it is an enumeration-cone.

Definition (Montalbán)

A tree $T \subseteq 2^{<\omega}$ is *e-pointed* if it has no dead ends and every infinite path $f \in [T]$ enumerates T.

Theorem (McCarthy)

An e-degree is cototal if and only if it contains a (uniformly) e-pointed tree.

If T is e-pointed then $\sigma \in T$ iff for some level n all strings of length n are either in $\overline{T}$ or enumerate σ.

Corollary

A degree spectrum is the Turing-upward closure of an F_σ set of reals in ω^ω if and only if it is the enumeration-cone of a cototal e-degree.

Characterization VI: Minimal subshifts

The enumeration degree of the language L_X of a minimal subshift X characterizes the set of Turing degrees of members of X.

(Jaendel:) L_X is a cototal set.

Theorem (McCarthy)

Every cototal enumeration degree is the degree of the language of a minimal subshift.

Characterization VII: Good enumeration degrees

Definition (Lachlan, Shore)

A uniformly computable sequence of finite sets $\{A_s\}_{s<\omega}$ is a *good approximation* to a set A if:

G1 $(\forall n)(\exists s)(A \upharpoonright n \subseteq A_s \subseteq A)$

G2 $(\forall n)(\exists s)(\forall t > s)(A_t \subseteq A \Rightarrow A \upharpoonright n \subseteq A_t)$.

An enumeration degree is *good* if it contains a set with a good approximation.

1. Good e-degrees cannot be tops of empty intervals.
2. Total enumeration degrees and enumeration degrees of n-c.e.a. sets are good.

Characterization VII: Good enumeration degrees

Theorem (Harris; Miller, S)

The good enumeration degrees are exactly the cototal enumeration degrees.

If A has a good approximation then

$$A \leq_e \{\langle x, s\rangle \mid (\forall t > s)(A_t \subseteq A \Rightarrow x \in A\} \leq_e A^\Diamond.$$

Every uniformly e-pointed tree has a good approximation.

Theorem (Miller, S)

The cototal enumeration degrees are dense.

If $V <_e U$ are cototal and U has a good approximation we can build Θ such that $\Theta(U)$ is the complement of a maximal independent set and

$$V <_e \Theta(U) \oplus V <_e U.$$

The end

Thank you!