The enumeration degrees:
Local and global structural interactions

Mariya I. Soskova[1]

Sofia University

10th Panhellenic Logic Symposium

[1]Supported by a Marie Curie International Outgoing Fellowship STRIDE (298471) and Sofia University Science Fund project 81/2015.

The spectrum of relative definability

If a set of natural numbers A can be *defined* using as parameter a set of natural numbers B, then A is reducible to B.

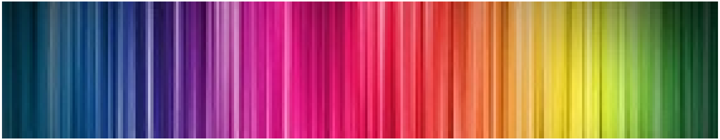
1. There is a total computable function f, such that $x \in A$ if and only if $f(x) \in B$: many-one reducibility ($A \leq_m B$).
2. There is an algorithm to determine whether $x \in A$ using finitely many facts about membership in B: Turing reducibility ($A \leq_T B$).
3. There is an algorithm that allows us to enumerate A using any enumeration of B: enumeration reducibility ($A \leq_e B$).
4. There is an arithmetical formula with parameter B that determines whether $x \in A$: arithmetical reducibility ($A \leq_a B$).
5. B can compute a complete description of A in terms of the Borel hierarchy: hyperarithmetical reducibility ($A \leq_h B$).

Degree structures

Definition

- $A \equiv B$ if $A \leq B$ and $B \leq A$.
- $d(A) = \{B \mid A \equiv B\}$.
- $d(A) \leq d(B)$ if and only if $A \leq B$.
- There is a least upper bound operation $\vee$.
- There is a jump operation $'$.

$\mathcal{D}_m$ $\mathcal{D}_T$ $\mathcal{D}_e$ $\mathcal{D}_a$ $\mathcal{D}_h$



The many-one degrees

Theorem (Ershov, Paliutin)

The partial ordering of the many-one degrees is the unique partial order P such that the following conditions hold.

1. P is a distributive upper-semi-lattice with least element.
2. Every element of P has at most countably many predecessors.
3. P has cardinality the continuum.
4. Given any distributive upper-semi-lattice L with least element and of cardinality less than the continuum with the countable predecessor property and given an isomorphism π between an ideal I in L and an ideal $\pi(I)$ in P , there is an extension π^* of π to an isomorphism between L and $\pi^*(L)$ such that $\pi^*(L)$ is an ideal in P.

The automorphism group of $\mathcal{D}_m$ has cardinality 2^{2^ω} and every element of $\mathcal{D}_m$ other than its least one, $\mathbf{0}_m$, has a nontrivial orbit.

The hyperarithmetical degrees

Theorem (Slaman and Woodin: Biinterpretability)

The partial ordering of the hyperarithmetical degrees is *biinterpretable* with the structure of second-order arithmetic. There is a way within the ordering $\mathcal{D}_h$ to represent the standard model of arithmetic $\langle \mathbb{N}, +, *, <, 0, 1 \rangle$ and each set of natural numbers X so that the relation

$\vec{\mathbf{p}}$ *represents the set* X *and* $\mathbf{x}$ *is the hyper-arithmetical degree of* X.

can be defined in $\mathcal{D}_h$ as a property of $\vec{\mathbf{p}}$ and $\mathbf{x}$.

- There are no nontrivial automorphisms of $\mathcal{D}_h$.
- A relation on degrees is definable in $\mathcal{D}_h$ if and only if the corresponding relation on sets is definable in second order arithmetic.

Understanding the middle of the spectrum

Theorem (Simpson)

The first order theory of $\mathcal{D}_T$ is computably isomorphic to the theory of second order arithmetic.

Theorem (Slaman, Woodin: Biinterpretability with parameters)

There is a way within $\mathcal{D}_T$ to represent the standard model of arithmetic $\langle \mathbb{N}, +, *, <, 0, 1 \rangle$ and each set of natural numbers X so that the relation

$$\vec{\mathbf{p}} \textit{ represents the set } X \textit{ and } \mathbf{x} \textit{ is the Turing degree of } X.$$

can be defined using a parameter $\mathbf{g}$ in $\mathcal{D}_T$ as a property of $\vec{\mathbf{p}}$ and $\mathbf{x}$.

- There are at most countably many automorphisms of $\mathcal{D}_T$.
- Relations on degrees induced by a relations on sets definable in second order arithmetic are definable with parameters in $\mathcal{D}_T$.
- The degrees below $\mathbf{0}^{(5)}$ form an automorphism base.
- Rigidity is equivalent to full biinterpretability.

Understanding the middle of the spectrum

Theorem (Slaman, Woodin)

The first order theory of $\mathcal{D}_e$ is computably isomorphic to the theory of second order arithmetic.

Theorem (S: Biinterpretability with parameters)

There is a way within $\mathcal{D}_e$ to represent the standard model of arithmetic $\langle \mathbb{N}, +, *, <, 0, 1 \rangle$ and each set of natural numbers X so that the relation

$$\vec{\mathbf{p}} \textit{ represents the set } X \textit{ and } \mathbf{x} \textit{ is the enumeration degree of } X.$$

can be defined using a parameter $\mathbf{g}$ in $\mathcal{D}_e$ as a property of $\vec{\mathbf{p}}$ and $\mathbf{x}$.

- There are at most countably many automorphisms of $\mathcal{D}_e$.
- Relations on degrees induced by a relations on sets definable in second order arithmetic are definable with parameters in $\mathcal{D}_e$.
- The degrees below $\mathbf{0}_e^{(8)}$ form an automorphism base.
- Rigidity is equivalent to full biinterpretability.

Local structures

Definition

$\mathcal{R}$ is the substructure consisting of all Turing degrees that contain c.e. sets.

$\mathcal{D}_T(\leq \mathbf{0}')$ is the substructure consisting of all Turing degrees that are bounded by $\mathbf{0}'_T$.

$\mathcal{D}_e(\leq \mathbf{0}'_e)$ is the substructure consisting of all enumeration degrees that are bounded by $\mathbf{0}'_e$.

Theorem (Harrington, Slaman; Shore; Ganchev, S)

The theory of each local structure is computably isomorphic to first order arithmetic.

Theorem (Slaman, S)

The local structure of the Turing degrees, $\mathcal{D}_T(\leq \mathbf{0}')$, is biinterpretable with first order arithmetic modulo the use of finitely many parameters.

Reducibilities

Reducibility	Oracle set B	Reduced set A
$A \leq_T B$	Complete information	Complete information
A c.e. in B	Complete information	Positive information
$A \leq_e B$	Positive information	Positive information

Definition

1. $A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \ \&\ D \subseteq B)\}.$$

2. A c.e. in B if there is a c.e. set W, such that

$$A = W^B = \left\{x \mid \exists D_1, D_2(\langle x, D_1, D_2\rangle \in W \ \&\ D_1 \subseteq B \& \ D_2 \subseteq \overline{B})\right\}.$$

3. $A \leq_T B$ if A c.e. in B and $\overline{A}$ c.e. in B.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$A \leq_T B \Leftrightarrow A \oplus \overline{A}$ is c.e. in $B \Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation.

$\mathcal{TOT} = \iota(\mathcal{D}_T)$ is the set of total enumeration degrees.

$$(\mathcal{D}_T, \leq_T, \vee, ', \mathbf{0}_T) \cong (\mathcal{TOT}, \leq_e, \vee, ', \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq_e, \vee, ', \mathbf{0}_e)$$

Theorem (Selman)

A is enumeration reducible to B if and only if
$\{\mathbf{x} \in \mathcal{TOT} \mid d_e(A) \leq \mathbf{x}\} \supseteq \{\mathbf{x} \in \mathcal{TOT} \mid d_e(B) \leq \mathbf{x}\}$.

$\mathcal{TOT}$ is an automorphism base for $\mathcal{D}_e$.

Definability in $\mathcal{D}_T$ and the local structures

Theorem (Shore, Slaman)

The Turing jump is first order definable in $\mathcal{D}_T$.

- A degree $\mathbf{a}$ is Low_n if $\mathbf{a}^{(n)} = \mathbf{0}_T^{(n)}$.
- A degree $\mathbf{a}$ is High_n if $\mathbf{a}^{(n)} = \mathbf{0}_T^{(n+1)}$.

Theorem (Nies, Shore, Slaman)

All jump classes apart from Low_1 are first order definable in $\mathcal{R}$ and in $\mathcal{D}_T(\leq \mathbf{0}')$.

Method: "Involves explicit translation of automorphism facts in definability facts via a coding of second order arithmetic."

Semi-computable sets

Definition (Jockusch)

A is semi-computable if there is a total computable function s_A, such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$.

Example:

- A *left cut* in a computable linear ordering is a semi-computable set.
- Every nonzero Turing degree contains a semi-computable set that is not c.e. or co-c.e.

Theorem (Arslanov, Cooper, Kalimullin)

If A is a semi-computable set then for every X:

$$(d_e(X) \vee d_e(A)) \wedge (d_e(X) \vee d_e(\overline{A})) = d_e(X).$$

Kalimullin pairs

Definition (Kalimullin)

A pair of sets A, B are called a $\mathcal{K}$-pair if there is a c.e. set W, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

Example:

1. A trivial example is $\{A, U\}$, where U is c.e: $W = \mathbb{N} \times U$.
2. If A is a semi-computable set, then $\{A, \overline{A}\}$ is a $\mathcal{K}$-pair: $W = \{(m, n) \mid s_A(m, n) = m\}$.

Theorem (Kalimullin)

A pair of sets A, B is a $\mathcal{K}$-pair if and only if their enumeration degrees $\mathbf{a}$ and $\mathbf{b}$ satisfy:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{D}_e)((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}).$$

Definability of the enumeration jump

Theorem (Kalimullin)

$\mathbf{0}'_e$ is the largest degree which can be represented as the least upper bound of a triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$, such that $\mathcal{K}(\mathbf{a}, \mathbf{b})$, $\mathcal{K}(\mathbf{b}, \mathbf{c})$ and $\mathcal{K}(\mathbf{c}, \mathbf{a})$.

Corollary (Kalimullin)

The enumeration jump is first order definable in $\mathcal{D}_e$.

Definability in the local structure of the enumeration degrees

Theorem (Ganchev, S)

The class of $\mathcal{K}$-pairs below $\mathbf{0}'_e$ is first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$...

Theorem (Cai, Lempp, Miller, S)

...by the same formula as in $\mathcal{D}_e$.

Theorem (Ganchev, S)

The low enumeration degrees are first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$: $\mathbf{a}$ is low if and only if every $\mathbf{b} \leq \mathbf{a}$ bounds a half of a $\mathcal{K}$-pair.

Maximal $\mathcal{K}$-pairs

Definition

A $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is maximal if for every $\mathcal{K}$-pair $\{\mathbf{c}, \mathbf{d}\}$ with $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{b} \leq \mathbf{d}$, we have that $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$.

Example: A semi-computable pair is a maximal $\mathcal{K}$-pair.
Total enumeration degrees are joins of maximal $\mathcal{K}$-pairs.

Theorem (Ganchev, S)

In $\mathcal{D}_e(\leq \mathbf{0}'_e)$ a nonzero degree is total if and only if it is the least upper bound of a maximal $\mathcal{K}$-pair.

The main definability question

Question (Rogers 1967)

Are the total enumeration degrees first order definable in $\mathcal{D}_e$?

1. The total degrees above $\mathbf{0}_e'$ are definable as the range of the jump operator.
2. The total degrees below $\mathbf{0}_e'$ are definable as joins of maximal $\mathcal{K}$-pairs.
3. The total degrees are definable with parameters in $\mathcal{D}_e$.

Every total degree is the join of a maximal $\mathcal{K}$-pair.

Question (Ganchev, S)

Is the the join of every maximal $\mathcal{K}$-pair total?

Defining totallity in $\mathcal{D}_e$

Theorem (Cai, Ganchev, Lempp, Miller, S)

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair in $\mathcal{D}_e$ then there is a semi-computable set C, such that $A \leq_e C$ and $B \leq_e \overline{C}$.

Proof flavor: Let W be a c.e. set witnessing that a pair of sets $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair.

1. The countable component: we use W to construct an effective labeling of the computable linear ordering $\mathbb{Q}$.
2. The uncountable component: C will be a left cut in this ordering.

Theorem (Cai, Ganchev, Lempp, Miller, S)

The set of total enumeration degrees is first order definable in $\mathcal{D}_e$.

The relation *c.e. in*

Definition

A Turing degree $\mathbf{a}$ is *c.e. in* a Turing degree $\mathbf{x}$ if some $A \in \mathbf{a}$ is c.e. in some $X \in \mathbf{x}$.

Recall that ι is the standard embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$.

Theorem (Cai, Ganchev, Lempp, Miller, S)

The set $\{\langle \iota(\mathbf{a}), \iota(\mathbf{x})\rangle \mid \mathbf{a} \text{ is c.e. in } \mathbf{x}\}$ is first order definable in $\mathcal{D}_e$.

1. Ganchev, S had observed that if $\mathcal{TOT}$ is definable by maximal $\mathcal{K}$-pairs then the image of the relation 'c.e. in' is definable for non-c.e. degrees.
2. A result by Cai and Shore allowed us to complete this definition.

The total degrees as an automorphism base

Theorem (Selman)

A is enumeration reducible to B if and only if $\{\mathbf{x} \in \mathcal{TOT} \mid d_e(A) \leq \mathbf{x}\} \supseteq \{\mathbf{x} \in \mathcal{TOT} \mid d_e(B) \leq \mathbf{x}\}$.

Corollary

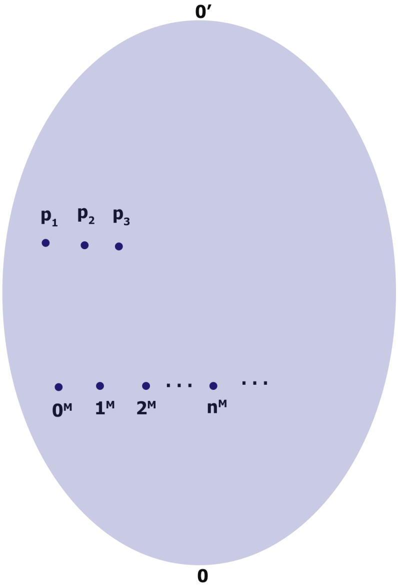
The total enumeration degrees form a definable automorphism base of the enumeration degrees.

- If $\mathcal{D}_T$ is rigid then $\mathcal{D}_e$ is rigid.
- The automorphism analysis for the enumeration degrees follows.
- The total degrees below $\mathbf{0}_e^{(5)}$ are an automorphism base of $\mathcal{D}_e$.

Question

Can we improve this bound further?

The local coding theorem of Slaman and Woodin



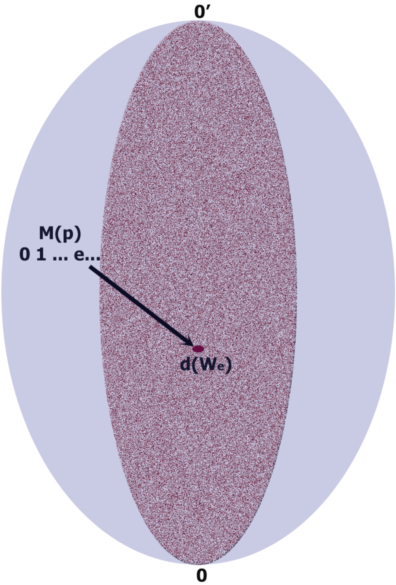
Using parameters we can code a model of arithmetic $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, 0^{\mathcal{M}}, s^{\mathcal{M}}, +^{\mathcal{M}}, \times^{\mathcal{M}}, \leq^{\mathcal{M}})$.

1. The set $\mathbb{N}^{\mathcal{M}}$ is definable with parameters $\vec{\mathbf{p}}$.
2. The graphs of s, $+$, $\times$ and the relation $\leq$ are definable with parameters $\vec{\mathbf{p}}$.
3. $\mathbb{N} \models \varphi$ iff $\mathcal{D}_T(\leq \mathbf{0}') \models \varphi_T(\vec{\mathbf{p}})$

An indexing of the c.e. degrees

Theorem (Slaman, Woodin)

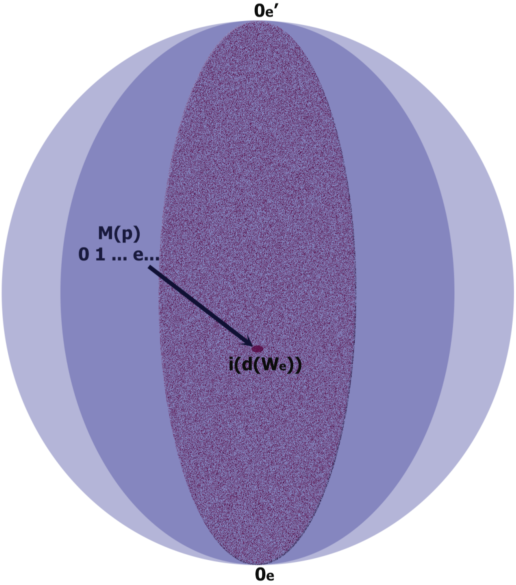
There are finitely many Δ^0_2 parameters which code a model of arithmetic $\mathcal{M}$ and an indexing of the c.e. degrees: a function $\psi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq \mathbf{0}')$ such that $\psi(e^{\mathcal{M}}) = d_T(W_e)$.



Towards a better automorphism base of $\mathcal{D}_e$

Theorem (Slaman, Woodin)

There are total Δ^0_2 parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the image of the c.e. Turing degrees.

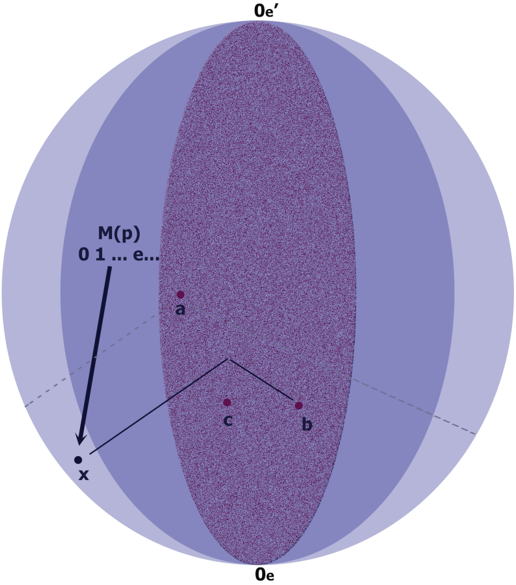


Towards a better automorphism base of $\mathcal{D}_e$

Theorem (Slaman, Woodin)

There are total Δ^0_2 parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the image of the c.e. Turing degrees.

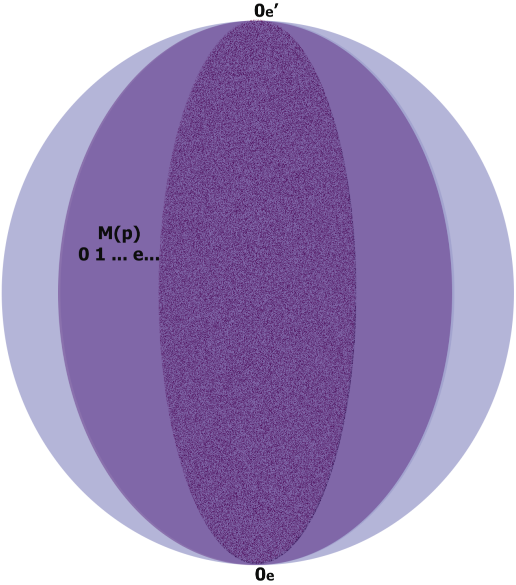
Idea: Can we extend this indexing to capture more elements in $\mathcal{D}_e$?



Towards a better automorphism base of $\mathcal{D}_e$

Theorem (Slaman, S)

If $\vec{p}$ defines a model of arithmetic $\mathcal{M}$ and an indexing of the image of the c.e. Turing degrees then $\vec{p}$ defines an indexing of the total Δ^0_2 enumeration degrees.



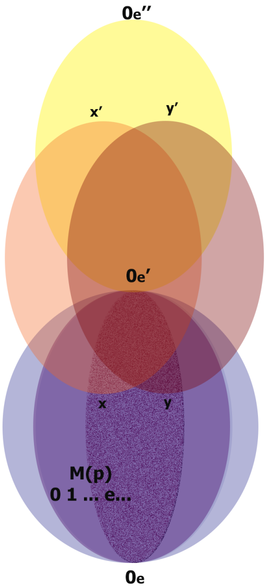
Proof flavour:

The image of the c.e. degrees
$\rightarrow$ The low co-d.c.e. e-degrees
$\rightarrow$ The low Δ^0_2 e-degrees
$\rightarrow$ The total Δ^0_2 e-degrees

Moving outside the local structure

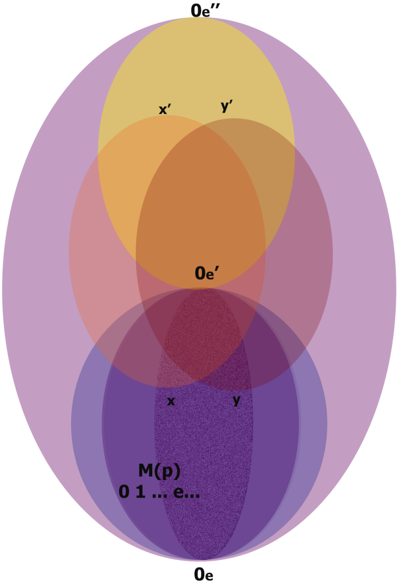
1. Extend to an indexing of all total degrees that are "c.e. in" and above some total Δ_2^0 enumeration degree.
   - The jump is definable.
   - The image of the relation "c.e. in" is definable.

2. Relativizing the previous theorem extend to an indexing of $\bigcup_{\mathbf{x}\leq\mathbf{0}'} \iota([\mathbf{x},\mathbf{x}'])$.

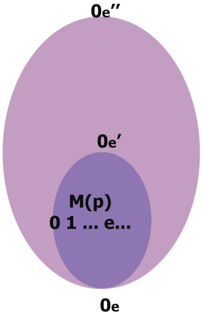


Moving outside the local structure

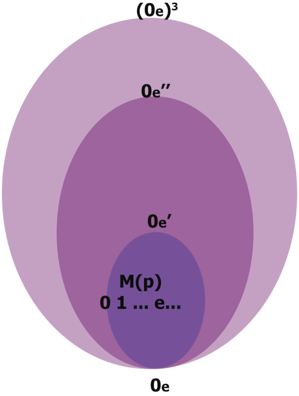
3. Extend to an indexing of all total degrees below $\mathbf{0}_e''$.



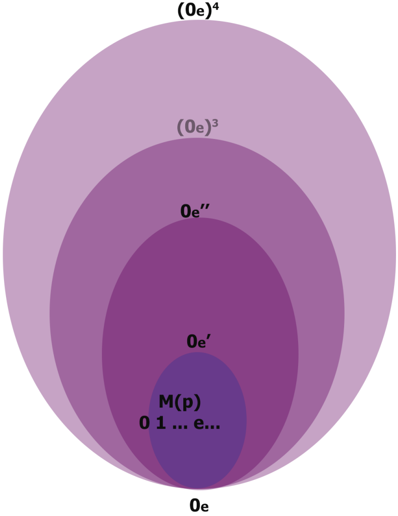
And now we iterate



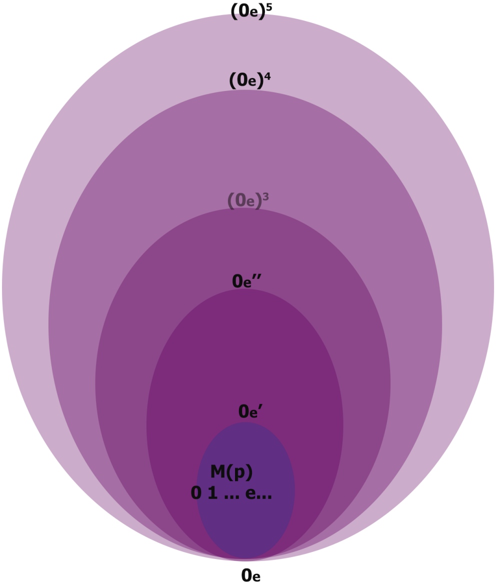
And now we iterate



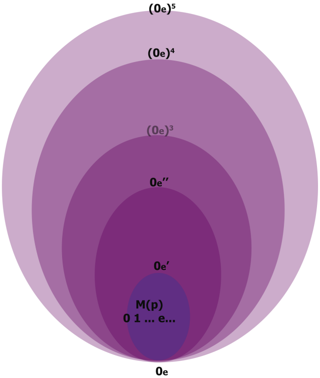
And now we iterate



And now we iterate



And now we iterate



Theorem (Slaman, S)

Let n be a natural number and $\vec{p}$ be parameters that index the image of the c.e. Turing degrees. There is a definable from $\vec{p}$ indexing of the total Δ^0_{n+1} degrees.

Consequences

Theorem (Slaman, S)

1. The enumeration degrees below $\mathbf{0}'_e$ are an automorphism base for $\mathcal{D}_e$.
2. The image of the c.e. Turing degrees is an automorphism base for $\mathcal{D}_e$.
3. If the structure of the c.e. Turing degrees is rigid then so is the structure of the enumeration degrees.

Question

1. Can we show that there is a similar interaction between the local and global structures of the Turing degrees?
2. Can we show that the local structure of the enumeration degrees is biinterpretable with first order arithmetic (with or without parameters)?

Thank you!